Environmental Modeling & Assessment (2024) 29:365–373
https://doi.org/10.1007/s10666-023-09940-y

RESEARCH

New Economic Model of Eutrophication of Shallow Lakes and the Existence of Equivalent Policies

Luis Bayón[1] · **Pedro Fortuny Ayuso**[1] · **José M. Grau**[1] · **María M. Ruiz**[1]

Received: 1 June 2023 / Accepted: 27 October 2023 / Published online: 8 November 2023

Abstract
The search for Skiba points in infinite-time optimal control problems with several steady states is an active research area. In this paper, we apply the backward integration method and show its power and relative simplicity as a tool for this task. We apply the method for the classical model of the economy shallow lake based on the phosphorous dynamics and explicitly compute the optimal trajectories and the Skiba points. We also suggest a more natural welfare function, which takes into account the finite value of any economic resource, and show, again in an explicit manner, that it also possesses Skiba points.

Keywords Optimal control problem · Backward integration method · Skiba points · Eutrophication · Economy

MSC Classification 34H05 · 34K28 · 92E20

1 Introduction

In the last quarter of the previous century, a remarkable type of points in the phase-space of a dynamical control system was defined [1–4], called *indifference points* or otherwise *Skiba* or *DNSS* (from the initials of Dechert, Nishimura, Sethi, and Skiba) points. These points are defined by the property that an optimal control problem starting at them has two (or more) different optimal solutions *with the same final cost* (assuming, for simplicity, that the control problem is monetary). These Skiba points arise in problems with infinite-time horizon, even with only a single parameter and a single control, having at least three steady states. From the definition, it follows that, in order to control the system optimally, the "controller" must choose one of the solutions by means of a non-mathematical argument, which usually comes from distinguishing between "good" long-term steady states or a "bad" ones and choosing among the good states.

A classical well-known example is provided in [5] and [6]: counter-terror operations, which are divided into "fire" and "water" policies (requiring the use of force or not, respectively). The authors find two different steady states: one in which the terrorist organizations are practically eradicated and one in which there is a high number of terrorists. "Water" policies do not cause the recruitment of new terrorists, whereas "fire" ones do, and each has different costs and benefits. In the end, the authors find a Skiba point at which the long-term costs of either policy are the same, and it is the decision-maker who has to choose whether one outcome is better than the other from non-mathematical considerations.

Recent advances in environmental and resource economics [7] provide a realistic representation of the evolution of complex adaptive ecological systems. In them, the existence of Skiba points is also patent. We are going to focus on the *shallow lake system* [8–11], where the continuous dynamics of the concentration of phosphorus in a shallow lake is studied. This problem has also been studied from the discrete-stochastic point of view [12].

Our aim is twofold: on one hand, to give a detailed application of the backward integration method to this continuous-time problem with infinite horizon and discount factor [13]. Applying Pontryagin's maximum principle (PMP) [14], a dynamical system is constructed for which the solutions approaching the

✉ Luis Bayón
bayon@uniovi.es

Pedro Fortuny Ayuso
fortunypedro@uniovi.es

José M. Grau
grau.ribas@gmail.com

María M. Ruiz
mruiz@uniovi.es

[1] Department of Mathematics, University of Oviedo, EPI, Gijón 33203, Asturias, Spain

steady states must be computed [15]. We apply the backward integration method [16], which proves very effective for explicitly computing the optimal trajectories approaching the steady states and, as a consequence, for locating the Skiba points. On the other hand, we suggest changing the classical welfare function for a more realistic one and prove that the Skiba property still holds at least for some specific parameters.

The paper is organized as follows: In Section 2 we analyze the economic problem of eutrophication of shallow lakes, recalling the classical economic and chemical settings. Section 3 contains a summary review of the backward integration method, which is applied *in full detail* in Section 4 to the classical setting. Finally, in Section 5, we present the modification of the welfare function for this problem and apply (albeit in a less explicit way) the backward integration method to show that Skiba points can still appear. Section 6 contains the conclusions.

2 Economics of Eutrophication in Shallow Lakes

In biological systems, and especially in shallow lakes, the distinction between *oligotrophic* and *eutrophic* states is essential: in the former, there is a low concentration of nutrients, and waters are mostly clear and have little animal and plant life, whereas in the latter, nutrients abound and life (both animal and plant) is widespread.

The process of *eutrophication* is the increase of nutrients, which can be natural or artificial. Natural eutrophication leads, usually, to stable and well-balanced habitats, whereas man-made eutrophication tend to damage the underlying ecosystem [17]: they usually produce a fast increase in the algae and plant population, which may consume so much oxygen that fish and other animals suffocate and die. Even more, the shadow caused by algae and the pH increase boost the development of cyanobacteria which may even produce lethal toxins [18].

Among the main plant nutrients are nitrogen and phosphorus; however, most of eutrophication studies only use the latter as the basic parameter. We are going to briefly introduce the typical shallow lake model, following Carpenter et al. [19]. Shallow lakes are characterized by the larger amount of water in contact with the sediments, which causes a large recycling rate of nutrients from the sediments to water. Thus, nutrients are plenty, which favors the appearance of aquatic plants and algae. This is their main ecological difference with deep lakes.

2.1 The Lake Phosphorous Dynamics

Let $P(t)$ be the stock of phosphorous at time t, and $L(t)$ be the total external contributions of phosphorous to the water (e.g., due to agricultural activities or sewage). Phosphorous is also spontaneously eliminated by sedimentation and binding to biomass, mostly. These losses are usually modeled as linear functions with a constant rate s. Finally, inside the lake, phosphorous is also recycled from the sediments. In the formulation of internal phosphorus recycling by Carpenter et al. [19], it is assumed that (i) sediment is the major source of internal phosphorus recycling and (ii) the increase in internal phosphorus recycling follows a sigmoid function

$$\frac{rP^q(t)}{m^q + P^q(t)}, \tag{1}$$

where r is the maximum rate of recycling, m is the value of P at which recycling reaches $r/2$, and q determines the steepness of the curve at the point of inflection. The parameter for steepness of sigmoid function, q, is reported to be in the range of $(2 \le q \le 20)$ [19]. The exponent q affects the steepness of the sigmoid curve at the point of inflection. Larger values of q give a steeper curve. The value of 20 is suggested for a shallow and warm lake while 2 is suggested for a deep and cold lake.

When phosphorus concentration is low, internal phosphorus recycling is limited by low sediment re-suspension and low algal decomposition. However, a high concentration of phosphorus leads to the formation of algal blooms. And since the decomposition of algae produces an anoxic condition at the sediment-water interface, this helps the release of phosphorus from the sediments into the water column, causing the internal recycling rate of phosphorus to reach almost the maximum rate. This is the reason for considering the sigmoid function. It has been verified [20] that the phosphorus can be released from sediment as deep as 20 cm and that internal phosphorus recycling can persist for 5–15 years after reduction in external phosphorus loading [21].

Merging all the models into one, the dynamics of the stock of phosphorous is described by the differential equation

$$\dot{P}(t) = L(t) - sP(t) + \frac{rP^q(t)}{m^q + P^q(t)}. \tag{2}$$

As in most studies of shallow lakes [8, 9, 22], the steepness parameter q is set to 2, and we are going to conform to that use, so that the phosphorous dynamics is

$$\dot{P}(t) = L(t) - sP(t) + r\frac{P^2(t)}{P^2(t) + m^2}, \; P(0) = P_0. \tag{3}$$

Thus, the lake eutrophication dynamics is based on the phosphorous stock, $P(t)$, as the state variable, and the phosphorous input, $L(t)$, as the control variable.

It is convenient to perform the following substitutions:

$$P(t) = \frac{x(t)}{m}, \; L(t) = ra(t), \; s = \frac{rb}{m}, \tag{4}$$

and to change the time scale to tr/m in Eq. (3) to obtain

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1},\ x(0) = x_0. \tag{5}$$

where $a(t)$ can be interpreted as the external loading of phosphorous, and $x(t)$ the phosphorous stock.

2.2 The Economics of Eutrophication

We present now the first economic analysis we are going to study, which is based on one integrating social interactions with the dynamics of shallow lakes explained above. Mäler et al. [9] and Salerno [11] assume the lake is shared by n communities, each having the same welfare function. In each, agricultural activities require fertilizers, whose use causes an input of phosphorous in the lake. At the same time, the use of fertilizers serves as proxy for the benefits of each community.

On the other hand, there are other benefits provided by the lake (for instance, drinkable water or even ecological policies), which are incompatible with high phosphorous levels. The value of these depends on the total stock of phosphorous in the water. Thus, there is a *social* cost of the stock, which has to be taken into account. With this in mind, the welfare function of community i used by the authors above is

$$W_i(t) = \ln a_i(t) - cx(t)^2,\ \text{for } i = 1, \dots, n, \tag{6}$$

where $c > 0$ is the coefficient of relative preference between the agricultural and non-agricultural benefits of the lake.

The authors above consider two types of economic analysis, *static* and *dynamic*, and also discuss socially optimal equilibrium and private equilibrium, when more than a single community exists. The former is achieved by a single entity which tries to obtain a Pareto optimal distribution of the benefits, and in it, all the communities play exactly the same role. Private equilibria, however, depend on the individual use of the lake by each community, and each wishes to maximize its long-term welfare.

In Mäler et al. [9], the problem is studied from the point of view of the strategies in an interactive economy, and they prove that if n communities share the resources of the lake, and all have the same welfare function, then an open-loop Nash equilibrium exists *which reduces the problem to one single community* (albeit with a different welfare function, obviously). Wagener [8] study takes advantage of that result and studies the parameter space of the problem to construct the different sets in which the problem has or does not have Skiba points, or where there is a single attractive steady state, etc.

Our focus will be in explicitly applying the backward integration method to the shallow lake problem, in full detail, and, on the other hand, to suggest a different welfare function, which looks more natural to us, and using the backward integration method again shows that there are parameters for which Skiba points exist and compute these.

Thus, instead of working with a collection of functions $W_i(t)$, we start with the optimization problem

$$\max \int_0^\infty e^{-\rho t}\left[h(a(t)) - cx(t)^2\right]dt, \tag{7}$$

where $h(t)$ is a suitable concave function, subject to the dynamic phosphorous equation of the lake:

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1},\ x(0) = x_0. \tag{8}$$

As stated above, we shall first set $h(a(t)) = \ln a(t)$ following [8] and [9]. Later on, we shall replace $\ln a(t)$ with a different function (namely $k - \exp(-a(t))$, for some $k \in \mathbb{R}$), which is also concave but sets a maximum asymptotic value k to the lake resources.

In (7), $x(t)$ is the state variable, and the control variable is the phosphorous loading $a(t)$.

3 Backward Integration Method

The concept of backward integration entails first converting a boundary value problem (BVP) ("reaching the steady state") into an initial value problem (IVP) ("starting near the steady state") and then making use of the desirable property of the unstable manifold of a hyperbolic saddle: trajectories near the saddle tend to approach it exponentially. This enables the application of common numerical techniques. Backward integration flips the time variable and, as a result, switches the roles of the stable and unstable manifolds in order to make use of the latter attribute. Assume we are given an optimization problem (7), (8) with a single company (i.e., $i = 1$). Applying Pontryagin's maximum principle (PMP), and using the fact that the integrand in (7) has a negative-exponential dependence on time, we obtain an autonomous system of equations in $x(t)$ and $a(t)$, of the form

$$\begin{cases} \dot{x}(t) = f(x, a), \\ \dot{a}(t) = g(x, a). \end{cases} \tag{9}$$

The infinite-time problem (in a sufficiently general setting) implies that the only trajectories of (9) which can give rise to a solution of the optimal control problem are those approaching the steady states of (9) when $t \to \infty$ [5]. In the autonomous case (with discount), the only steady states to which trajectories can approach asymptotically are saddles or saddle-nodes [5].

In [16], the *backward integration method* is introduced for optimal control methods with saddle-type steady states: the reverse-time stability of the unstable manifold of a saddle allows them to compute, with great precision, the trajectories approaching these saddles, using the associated dynamical system (9) *backwards*, hence the name of their method. Their statement is somewhat rough, and some precision by means of the eigenvectors of the linear part of the vector field is required (this can be even generalized to optimal control problems with degenerate saddles) [23]. It is important to notice (as indicated in [9]) that, in general, the hard problem of optimal control is not the computation of the optimal cost but the computation of the *initial condition* for the control $a(t)$ given the initial condition $x(0)$. This is what backward integration achieves and why a careful implementation is essential.

We shall apply the backward integration method to several instances of the shallow lake model. Our aim is to explicitly compute the Skiba points reported in [8] and [9] in different cases and for (single community) lake models with different welfare functions.

4 Shallow Lake: Base Case

The "base case" of the shallow lake model, as presented in [8] and [9], with a single community is described by the following optimal control problem

$$\max_{a(t)} W = \int_0^\infty e^{-\rho t}\left(\log(a(t) - cx(t)^2\right) dt, \qquad \dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1},\ x(0) = x_0, \tag{10}$$

where there are no conditions on either the control $a(t)$ or the state variable $x(t)$, and all the constants ρ, b, c are positive (as described in Section 2). We are going to apply the backward integration method in full detail to an example, with $\rho = 0.035$, $b = 0.65$, and $c = 0.52$. Thus, our control problem is, specifically, removing the dependencies on t for simplicity:

$$\max_{a(t)} W = \int_0^\infty e^{-0.035t}\left(\log(a) - 0.52x^2\right) dt, \qquad \dot{x} = a - 0.65x + \frac{x^2(t)}{x^2(t)+1},\ x(0) = x_0, \tag{11}$$

where x_0 will depend on the initial state of the lake. From now on, we apply PMP. The current-value Hamiltonian of (11) with multiplier m is

$$H(x,a,m) = (-0.52x^2 + \log(a)) + m(a - 0.65x + \frac{x^2}{1+x^2}). \tag{12}$$

The necessary condition given by PMP for an optimal solution is

$$\frac{\partial H}{\partial a} = 0 \Leftrightarrow \frac{1}{a} + m = 0, \tag{13}$$

so that, on any optimal solution, we must have $m = -1/a$. On the other hand, the associated dynamical system (also given by PMP) is, in x and m, taking into account that $m = -1/a$, we obtain the ordinary differential equation

$$\begin{cases} \dot{x}(t) = \dfrac{-1}{m} - 0.65x + \dfrac{x^2}{1+x^2}, \\ \dot{m}(t) = 1.04x + m\left(0.65 - \dfrac{2x}{(1+x^2)^2}\right), \end{cases} \tag{14}$$

which seems to have a singularity at $m = 0$. However, this value is unreachable, as it corresponds to $a = \infty$, and it poses no restriction. From this dynamical system, we shall obtain the candidate solutions using the backwards integration method, as it is known that solutions of (11) correspond to trajectories approaching a steady state of (14) asymptotically [5].

The steady states (equilibria) of (14) are the points where $\dot{x}(t) = \dot{m}(t) = 0$. Figure 1 shows the curves $\dot{x}(t) = 0$ and $\dot{m}(t) = 0$, which meet at the three points P_1, P_2 and P_3:

$$\begin{aligned} P_1 &\simeq (0.4578, -8.0448), \\ P_2 &\simeq (0.8722, -7.4138), \\ P_3 &\simeq (1.4122, -3.9695). \end{aligned} \tag{15}$$

4.1 Backward Integration

We need to know the type of equilibrium of each P_i. The values of the eigenvalues of the Jacobian matrices are, respectively (all our values are approximate), as follows:

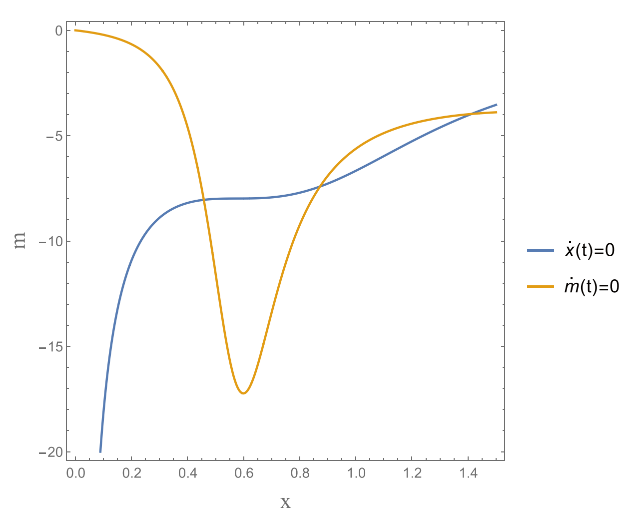


Fig. 1 Plots of $\dot{x} = 0, \dot{m} = 0$, and the steady states

$$M_1 : \lambda_1 = 0.27811, \lambda_2 = -0.24311 \quad (16)$$

$$M_2 : \lambda_1 = 0.25180, \lambda_2 = -0.21680 \quad (17)$$

$$M_3 : \lambda_1 = 0.32891, \lambda_2 = -0.29391 \quad (18)$$

which, after the usual computations, show that P_1 and P_3 are saddle points, whereas P_2 is a center-focus which is actually unstable. We are, therefore, in a convex-concave situation with the possibility of having a Skiba point. Mäler et al. [9] argument is enough to show that indeed there exists one, but our purpose is to compute it explicitly.

The structure of the singularities and the control problem forces the situation so that any solution of (11) starting with initial value of the state $x = x_0$ between $P_{1,x} = 0.4578$ and $P_{3,x} = 1.4122$ either converges to P_1 or to P_3. In order to properly integrate backwards, we compute the eigenvectors at P_1, P_3 corresponding to the unstable manifolds at each of them (i.e., the eigenvector corresponding to the negative eigenvalue). Calling these vectors, respectively, v_1 and v_3, we get, after normalization,

$$v_1 = (0.0691, -0.9976), \; v_3 = (-0.8372, -0.5468), \quad (19)$$

where (and this is important) we need them to point "backwards," and we know that the initial points P_1 and P_3 are reached in the forward direction, thus the need to choose a direction pointing to the right for P_1 (so v_1 has positive first coordinate) and to the left for P_3 (so v_3 has negative first coordinate). We have chosen their orientations so that each points to the inside of the interval $x \in [P_{1,x}, P_{3,x}]$.

The method now uses an initial position near each equilibrium with a small displacement in the direction of the corresponding eigenvector. We choose a threshold of $\varepsilon = 10^{-6}$, so that the initial conditions $(x_1(0), m_1(0)), (x_3(0), m_3(0))$ are, respectively, as follows:

$$\begin{aligned}(x_1(0), m_1(0)) &= (0.457831, -8.04483),\\ (x_3(0), m_3(0)) &= (1.41227, -3.96952).\end{aligned} \quad (20)$$

We now integrate (using Runge–Kutta, for instance) the differential equation given by (14) *with a negative sign* (backwards) for each initial condition. The respective solutions will be called $\overline{\gamma}_1(t) = (\overline{x}_1(t), \overline{m}_1(t))$ and $\overline{\gamma}_3(t) = (\overline{x}_3(t), \overline{m}_3(t))$. The time interval must be large enough to cover all the possibilities (in this case, we might stop when $\dot{\overline{x}}_1(t) = 0$ and when $\dot{\overline{x}}_3(t) = 0$, respectively). Figure 2 shows a plot of the respective curves $\overline{\gamma}_1(t)$ and $\overline{\gamma}_3(t)$. Notice how they spiral towards the focus point and how this point is (in the original system, not the negative one) an unstable focus. The fact that both solutions spiral around P_2 already *suggests* (does not prove) the existence of a Skiba point.

At this point, the first value of t where $\dot{\overline{x}}_1(t) = 0$ (and the same for $\overline{x}_3(t) = 0$) needs to be found in order to start the *forward trajectory* there (that is, that value must be set as $t = 0$ for the corresponding trajectory). These are the most extreme possible starting points for the candidate trajectories approaching P_1 and P_3, as it is well know (see [5], for instance) that in single-state control problems with infinite horizon, the trajectories are monotonic [24]. These extremal points are as follows:

$$I_1 = (1.12538, -5.79002), \; I_3 = (0.631168, -7.97601), \quad (21)$$

which are also the extremal initial conditions for the *true* candidate solutions $\gamma_1(t) = (x_1(t), m_1(t)), \gamma_3(t) = (x_3(t), m_3(t))$ approaching P_1 and P_3 respectively. Figure 3 shows these curves (notice how they both start with vertical slopes, as $\dot{x}_1(0) = \dot{x}_3(0) = 0$). The plots in Fig. 4 are the corresponding curves depending on t (i.e., two coordinates for each curve).

4.2 Costs

We study, thus, the specific case of a general maximization problem:

$$\max \int_0^\infty e^{-\rho t} F(x, a, t), \quad (22)$$

$$\dot{x}(t) = f(x, a, t), \; x(0) = x_0, \quad (23)$$

where

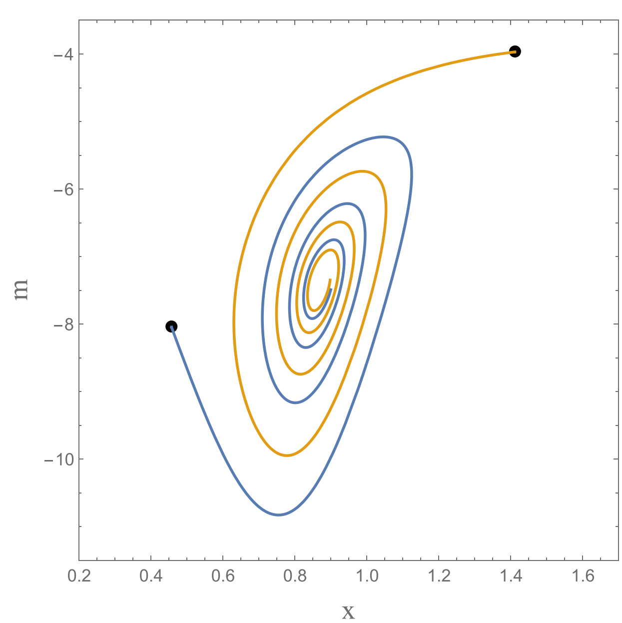


Fig. 2 The candidate (backwards) trajectories spiraling around the focus

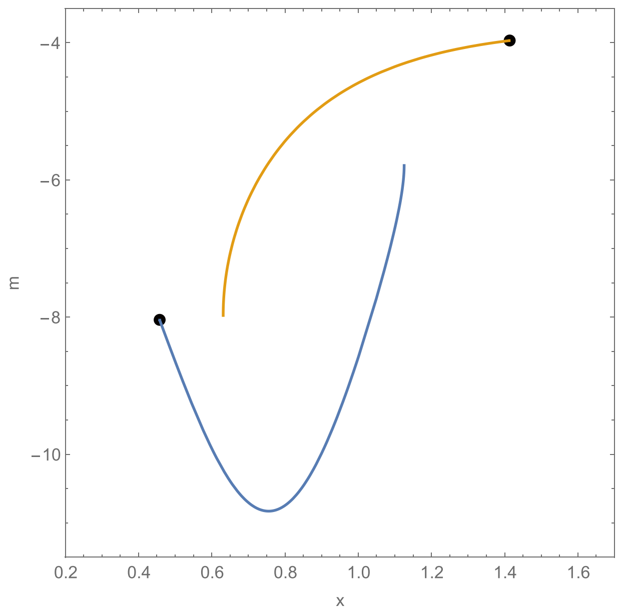


Fig. 3 The two candidate trajectories and their corresponding saddles

$$F(x, a, t) = \log(a(t) - cx(t)^2, \tag{24}$$

$$f(x, a, t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1}. \tag{25}$$

The curves $\gamma_1(t)$ and $\gamma_3(t)$ realize all the possible initial conditions $(x_1(0), m_1(0))$ and $(x_3(0), m_3(0))$ belonging to the candidate solutions of the problem (11), as well as, certainly, the corresponding trajectory. In order to compute the value (cost) of each trajectory, one takes into account Skiba's (1978) result which states that, if $\gamma_{x_0}(t)$ is a candidate solution starting at x_0 (and the corresponding m_0, which belongs to the trace of either γ_1 or γ_3), then the following equality holds:

$$C(x_0) = \int_0^\infty e^{-\rho t} F(x, a, t)\, dt = \frac{1}{\rho} H(x_0, a_0, \phi(x_0, a_0)), \tag{26}$$

where $H(x, a, m)$ is the current-value Hamiltonian, and

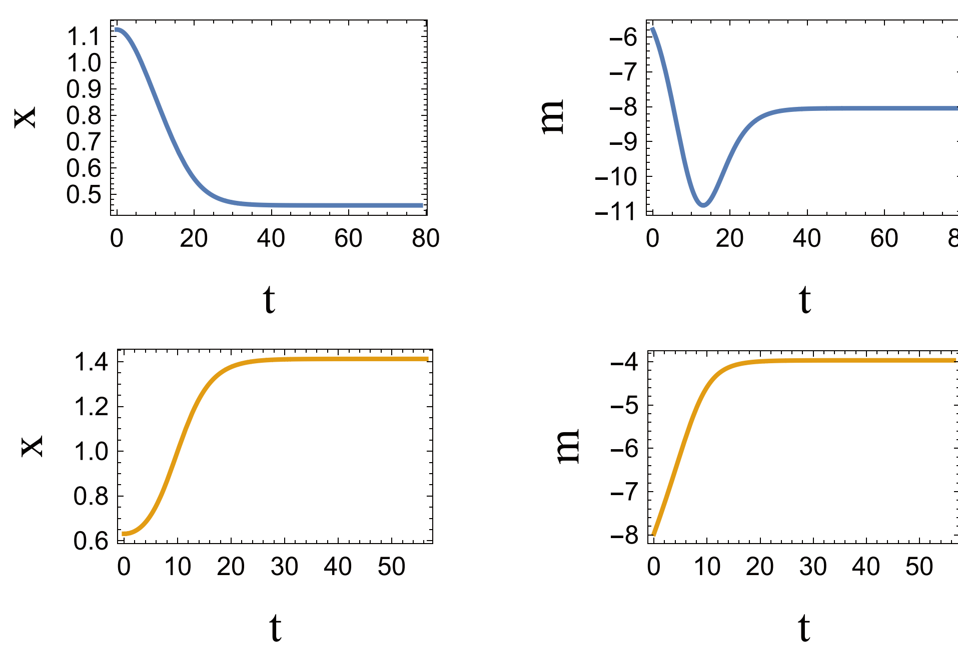
$$\phi(x, a) = -\frac{\partial F/\partial a}{\partial f/\partial a}(x, a). \tag{27}$$

In Fig. 5, we have plotted the points $(x_0, C(x_0))$ for each initial condition x_0, where $C(x_0)$ is, as above, the costs of the trajectory starting at x_0 and ending at the relevant steady state. The intersection point $x \simeq 0.839$ is the Skiba point, where the decision-maker can choose whether to tend long-term to the eutrophic state or the oligotrophic one, as the costs are equal.

5 Changing the Welfare Function

We propose, in this section, a new welfare function, specifically the term related to the benefit from agricultural use of the lake. We suggest one which seems a bit more adequate than the standard $\ln(a)$, used in [8, 9], and [11]: as the resources of the lake are limited, it seems natural to set a maximum value for them. Logarithmic functions are unsuitable for this task, as they are unbounded (from below and from above). In [25], the reader can find a summary of the pros and cons that various authors have presented of the logarithmic function and specifically of its unbounded character. We suggest using $k - \exp(-a)$

Fig. 4 The forward trajectories as functions of time. Above $(x_1(t), m_1(t))$, below $(x_3(t), m_3(t))$



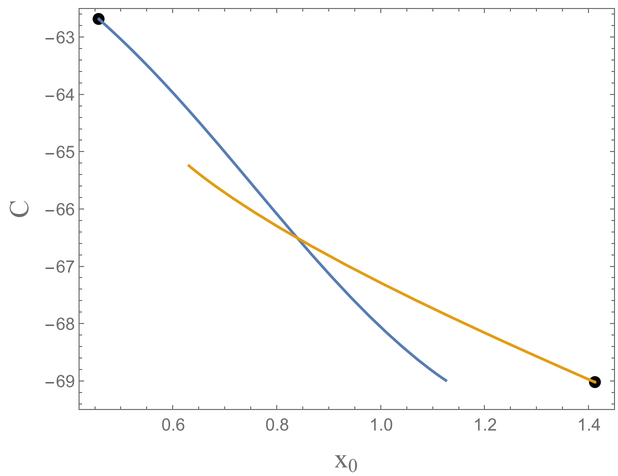


Fig. 5 Optimal costs and the Skiba point

as an alternative. If $a = 0$ (no economic activity), then the lake has an intrinsic value $k - 1$, whereas as a increases (as a proxy to the economic production), the value is greater up to the limit k. In contrast, the $\log(a)$ function has $-\infty$ value under no economic activity ($a \to 0$) (which makes no real sense) and is unbounded as the economic activity increases, also an unrealistic situation. Figure 6 shows the differences.

Our intent here is to show how this alternative model also fits in the Skiba setting (obviously, for some parameters). Unlike [9], we are not going to try and find the values for which Skiba points are, but only to compute the trajectories and the Skiba point using the backward integration method in this new model.

Specifically, we are going to study the optimal control problem:

$$\max \int_0^\infty e^{-\rho t} \left[(k - \exp(-a(t))) - cx(t)^2\right] dt, \tag{28}$$

where $h(t)$ is a suitable concave (welfare) function, subject to the dynamic phosphorous equation of the lake, which is

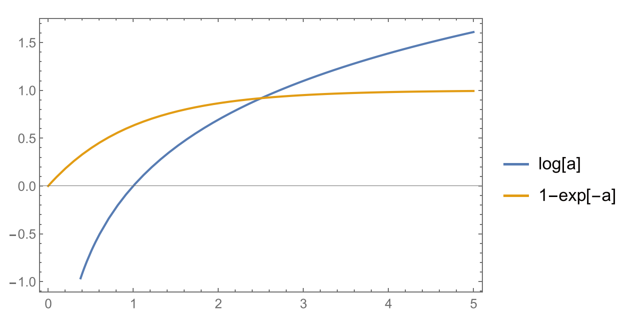


Fig. 6 The welfare functions $\log(a)$ and $1 - \exp(-a)$. Notice: the positive value of $1 - \exp(-a)$ and its horizontal asymptote, as opposed to the infinite limits for $\log(a)$ at $a \to 0$ and $a \to \infty$

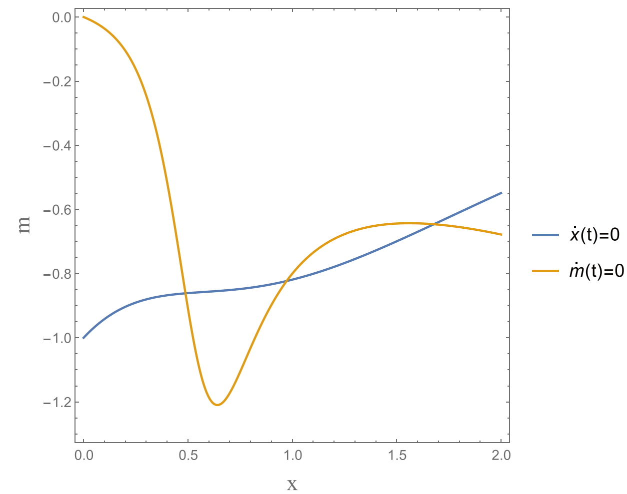


Fig. 7 Plots of $\dot{x} = 0, \dot{m} = 0$, and the steady states for the second model

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t) + 1}, \; x(0) = x_0, \tag{29}$$

for the following values of the parameters: $c = 0.1$, $k = 15$, $b = 0.7$, $\rho = 0.05$.

5.1 Backward Integration

The associated dynamical system for the above optimal control problem with the specified values is, after substituting m for a, as follows:

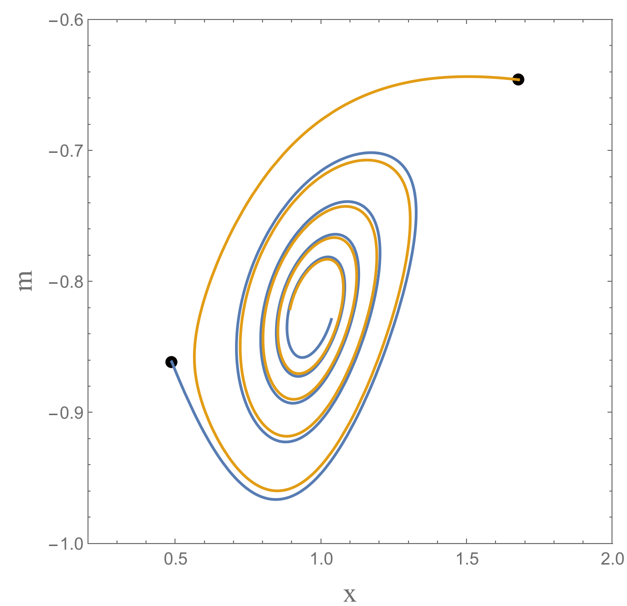


Fig. 8 The candidate (backwards) trajectories for the new model

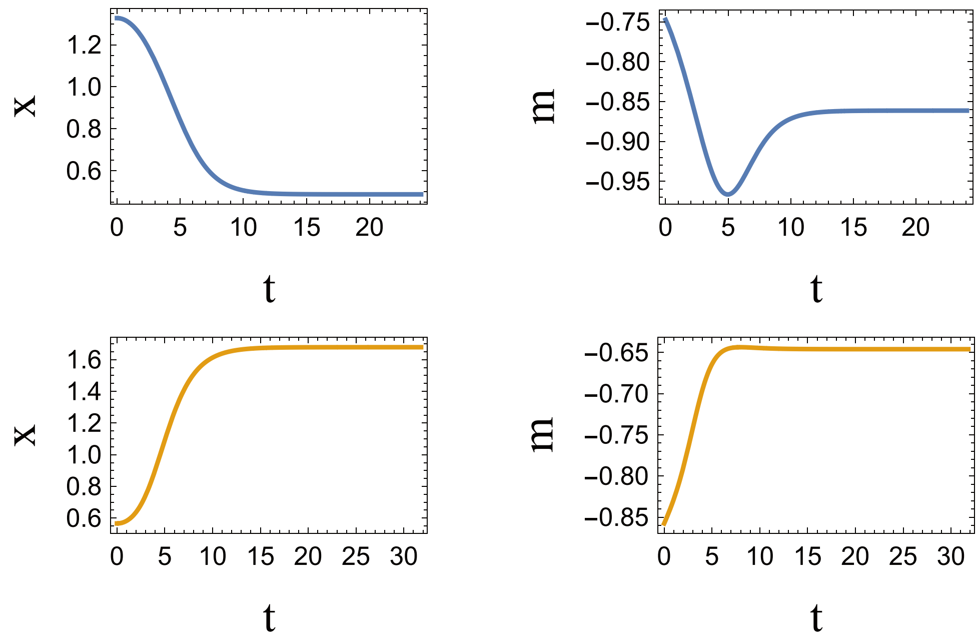


Fig. 9 The forward candidate trajectories depending on time

$$\dot{x} = -1.\log(-1.m) + \frac{x^2}{x^2+1} - 0.7x,$$
$$\dot{m} = m\left(0.7 - \frac{2x}{(x^2+1)^2}\right) + 0.2x, \tag{30}$$

whose set of steady states is, following the notation of (15):

$$P_1 \simeq (0.4880, -0.8613),$$
$$P_2 \simeq (0.9727, -0.8231), \tag{31}$$
$$P_3 \simeq (1.6785, -0.6460),$$

and the corresponding zeroes of $\dot{x}$ and $\dot{m}$ are shown in Fig. 7.

At P_1, the eigenvector corresponding to the unstable manifold is $v_1 = (0.8735, -0.4868)$, once the orientation has been chosen "backwards," whereas for P_3, it is $v_3 = (-0.9997, 0.0261)$, which is almost horizontal. The backward trajectories starting at $P_1 + \epsilon v_1$ and $P_3 + \epsilon v_3$ for $\epsilon = 10^{-6}$ are plotted in Fig. 8.

The part of those trajectories which correspond to true candidate solutions to (28)–(29), depending on time, is shown in Fig. 9.

Finally, the cost plot is given in Fig. 10. Notice the Skiba point at $x \simeq 0.718$.

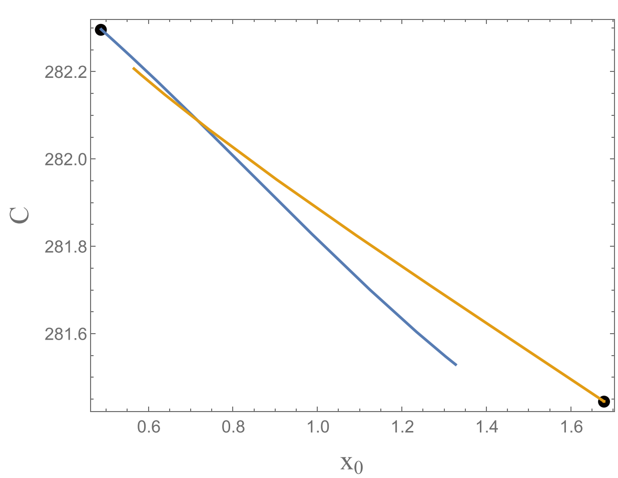


Fig. 10 The Skiba point of the second model

6 Conclusions

We have applied, in full detail, the backward integration method [16, 23] to the economics of the eutrophication of shallow lakes, in the case of the existence of Skiba points, as presented in [8, 9], and [11]: in those references, Skiba points are proved to exist, but no method for computing them is provided.

The backward integration method, as presented above, can be implemented straightforwardly in any symbolic algebra system (we have used, specifically, Mathematica©). The ability to easily compute trajectories converging to steady states of an optimal control problem makes the task of finding Skiba points [1, 3, 4] relatively simple.

In another section, we have changed the standard welfare function $\ln a$ in the classical model for what we deem is the more suitable (of course, concave) function $k - \exp(-a)$, which possesses a finite limit both when $t \to 0$ and when $t \to \infty$: these limits reflect the (natural) fact that any resource has a finite value, whether it be economically exploited or not. We have proved that there are parameters for which Skiba points arise in this model and have computed them explicitly again using the backward integration method.

Author Contribution L.B.: conceptualization, methodology, revision. P.F.A.: software, methodology, writing—original draft preparation. J.G.: conceptualization, methodology. M.R.: software, methodology, revision.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethics Approval Not applicable.

Consent to Participate Not applicable.

Consent for Publication Not applicable.

Conflict of Interest The authors declare no competing interests.

References

1. Sethi, S. P. (1977). Nearest feasible paths in optimal control problems: Theory, examples and counterexamples. *Journal of Optimization Theory and Applications, 23*(4), 563–579.
2. Sethi, S. P. (1979). Optimal advertising policy with the contagion model. *Journal of Optimization Theory and Applications, 29*(4), 615–627.
3. Skiba, A. K. (1978). Optimal growth with a convex-concave production function. *Econometrica, 46*(3), 527–539.
4. Dechert, W. D., & Nishimura, K. (1983). A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Journal of Economic Theory, 31*(2), 332–354.
5. Grass, D., Caulkins, J. P., Feichtinger, G., Tragler, G., & Behrens, D. A. (2008). *Optimal control of nonlinear processes*. Berlino: Springer.
6. Caulkins, J. P., Feichtinger, G., Grass, D., Hartl, R. F., Kort, P. M., & Seidl, A. (2015). Skiba points in free end-time problems. *Journal of Economic Dynamics and Control, 51*, 404–419.
7. Xepapadeas, A. (2010). Modeling complex systems. *Agricultural Economics, 41*, 181–191.
8. Wagener, F. (2003). Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *Journal of Economic Dynamics & Control, 27*, 1533–1561.
9. Mäler, K. G., Xepapadeas, A., & Zeeuw, A. D. (2003). The economics of shallow lakes. *Environmental and Resource Economics, 26*, 603–624.
10. Kossioris, G., Plexousakis, M., Xepapadeas, A., de Zeeuw, A., & Mäler, K. G. (2008). Feedback Nash equilibria for non-linear differential games in pollution control. *Journal of Economic Dynamics and Control, 32*(4), 1312–1331.
11. Salerno, G. (2009). *The political economy of shallow lakes*. Research Project.
12. Dechert, W. D., & O'Donnell, S. L. (2006). The stochastic lake game: A numerical solution. *Journal of Economic Dynamics and Control, 30*(9–10), 1569–1587.
13. Chiang, A. (2000). *Elements of dynamic optimization*. Waveland Press.
14. Pontryagin, L. S. (1987). *Mathematical theory of optimal processes (classics of Soviet mathematics)*. CRC Press.
15. Tsur, Y., & Zemel, A. (2001). The infinite horizon dynamic optimization problem revisited: A simple method to determine equilibrium states. *European Journal of Operational Research, 131*(3), 482–490.
16. Brunner, M., & Strulik, H. (2002). Solution of perfect foresight saddlepoint problems: A simple method and applications. *Journal of Economic Dynamics and Control, 26*(5), 737–753.
17. Ricklefs, R. E. (1979). *Ecology*. Chiron Press, 2nd edition, 1979.
18. Scheffer, M. (1998). *Ecology of shallow lakes*. Chapman & Hall.
19. Carpenter, S., Ludwig, D., & Brock, W. (1999). Management of eutrophication for lakes subject to potentially irreversible change. *Ecological Applications, 9*(3), 751–771.
20. Søndergaard, M., Jensen, J. P., & Jeppesen, E. (2003). Role of sediment and internal loading of phosphorus in shallow lakes. *Hydrobiologia, 506–509*(1–3), 135–145.
21. Søndergaard, M., Bjerring, R., & Jeppesen, E. (2013). Persistent internal phosphorus loading during summer in shallow eutrophic lakes. *Hydrobiologia, 710*(1), 95–107.
22. Grüne, L., Kato, M., & Semmler, W. (2005). Solving ecological management problems using dynamic programming. *Journal of Economic Behavior & Organization, 57*(4), 448–473.
23. Fortuny Ayuso, P., Bayón, L., Grau, J. M., Ruiz, M., & Suárez, P. M. (2021). A modified backward integration method for optimal control problems with degenerate equilibrium points. *Optimal Control Applications and Methods, 42*(4), 927–942.
24. Leonard, D., Van Long, N., & Ngo, V. L. (1992). *Optimal control theory and static optimization in economics*. Cambridge University Press.
25. Van den Bosch, K. (2017). Identifying the poor: Using subjective and consensual measures. Routledge.