A SHORT COURSE ON CROSS-DIFFUSION PROBLEMS: EXISTENCE OF WEAK SOLUTIONS AND TURING BIFURCATION

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Contents

Chapter 1

Nonlinear analysis tools for proving existence of weak solutions of cross-diffusion problems

Our aim is proving the existence of weak solutions of evolution cross-diffusion problems of the Shigesada-Kawasaki-Teramoto (SKT) type, that is, cross-diffusion problems which admit a suitable entropy estimate.

To do this, we first analyze a linear heat equation in Section 1, and a nonlinear reaction-diffusion problem in Section 2, under the following rules:

- the maximum principle can not be applied, and
- the starting point to construct a solution is the Lax-Milgram's lemma.

The first rule is motivated by the fact that, in general, cross-diffusion problems do not enjoy the property of comparison of solutions, while the second is chosen because it involves approximating techniques which are also useful for computational porpouses.

The problems are set as deduced from standard population dynamics models, although the techniques we employ are easily applicable to other type of evolution reaction-convection-diffusion problems.

Along the way, we recall well known results of functional analysis that provide us with powerful tools to tackle these problems.

The contents of this review concerning to cross-diffusion problems has been partially extracted from the following articles:

1. G. Galiano, M. L. Garzón, A. Jüngel, Semi-discretization in time and numerical convergence of solutions of a nonlinear cross-diffusion population model, Numerische Mathematik 93 (2003) 655-673.
2. L. Chen, A. Jüngel, Analysis of a multidimensional parabolic population model with strong cross-diffusion, SIAM J. Mathematical Analysis, 36 (2004) 301-322.
3. G. Galiano, V. Selgas, On a cross-diffusion segregation problem arising from a model of interacting particles, Nonlinear Analysis: Real World Applications 18 (2014) 34-49.

1 A linear population model

In this section we start showing a proof of existence of weak solutions of an evolution problem with linear diffusion and linear reaction terms.

The problem is the following. Given a fixed $T > 0$ and a bounded set $\Omega \subset \mathbb{R}^N$, find (a non-negative) $u : (0,T) \times \Omega \to \mathbb{R}$ such that

$$\partial_t u - \Delta u = u \qquad \text{in } Q_T = (0,T) \times \Omega, \tag{1.1}$$
$$\nabla u \cdot n = 0 \qquad \text{on } \Gamma_T = \partial(0,T) \times \Omega, \tag{1.2}$$
$$u(\cdot,0) = u_0 \geq 0 \qquad \text{in } \Omega. \tag{1.3}$$

In terms of population dynamics, we are supposing that

- The population diffuses randomly.
- The newborns are proportional to the existent population, and there is no growth limit. The corresponding kinetics ($\partial_t u = u$) implies exponential growth.

The first ingredient for constructing a solution of (1.1)-(1.3) is an energy estimate which point us to a possible notion of weak solution. Suppose that the problem has a smooth solution, u. Multiplying (1.1) by u, integrating in Q_t, with $t \in (0,T)$, and then integrating by parts in Ω, we get the energy identity

$$\frac{1}{2}\int_\Omega u(t)^2 + \int_{Q_t} |\nabla u|^2 = \frac{1}{2}\int_\Omega u_0^2 + \int_{Q_t} u^2. \tag{1.4}$$

Lemma 1 (Gronwall's lemma) *Let $T > 0$, $a \in L^\infty(0,T)$, and $\lambda \in L^1(0,T)$, with $\lambda \geq 0$ in $(0,T)$. Suppose that, for $b \in C([0,T])$ increasing,*

$$a(t) \leq b(t) + \int_0^t \lambda(s) a(s) ds \quad \text{a.e. in } (0,T),$$

Let $\Lambda(t) = \int_0^t \lambda(s) ds$. Then

$$a(t) \leq e^{\Lambda(t)} b(t) \quad \text{a.e. in } (0,T).$$

Using Gronwall's lemma in (1.4), we deduce

$$\int_\Omega u(t)^2 \leq e^{2t} \int_\Omega u_0^2, \quad \text{which implies} \quad \int_{Q_T} u^2 \leq Te^{2T} \int_\Omega u_0^2.$$

Therefore, we get from (1.4)

$$\|u\|_{L^\infty(L^2)} + \|\nabla u\|_{L^2} \leq C, \tag{1.5}$$

and hence, $\|u\|_{L^2(H^1)} \leq C$. Here we have introduced the notation $L^p(X)$ for $L^p(0,T;X(\Omega))$. Thus, we may expect u and ∇u to be $L^2(Q_T)$ functions. However, since

$$\partial_t u = -\operatorname{div}(\nabla u) + u \in L^2(0,T;(H^1(\Omega))'),$$

we can not expect, in principle, to have $\partial_t u$ defined as an $L^p(Q_T)$ function. Therefore, we start considering it in a distributional sense and set a generic definition of weak solution as

$$\int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} \nabla u \cdot \nabla \varphi = \int_{Q_T} u\varphi, \quad \text{for all } \varphi \in V, \tag{1.6}$$

with $\langle \cdot, \cdot \rangle$ denoting a duality product, and V a space of test functions, both to be explicited later.

1.1 Formal arguments

Our method of proof consists on defining a sequence of approximating problems, let us say (P_n), where n denotes the approximating parameter, and such that $(P_n) \to (P)$ as $n \to \infty$ in some sense, being (P) the original problem (1.1)-(1.3).

Suppose that estimate (1.5) is also satisfied by the sequence of solutions, u_n, to approximated problems (P_n) of the form

$$\int_0^T \langle \partial_t u_n, \varphi \rangle + \int_{Q_T} \nabla u_n \cdot \nabla \varphi = \int_{Q_T} u_n\varphi, \quad \text{for all } \varphi \in V. \tag{1.7}$$

That is, suppose that $\|u_n\|_{L^\infty(L^2)} + \|u_n\|_{L^2(H^1)} \leq C$. Then, there exists a subsequence of u_n (that we do not relabel) and a function $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that

$$u_n \rightharpoonup u \quad \text{weakly*-weakly in } L^\infty(0,T;L^2(\Omega)), \tag{1.8}$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(Q_T). \tag{1.9}$$

Then, we already get from this convergences that, for all $\varphi \in V \subset L^2(0,T;H^1(\Omega))$,

$$\int_{Q_T} \nabla u_n \cdot \nabla \varphi \to \int_{Q_T} \nabla u \cdot \nabla \varphi,$$

$$\int_{Q_T} u_n \varphi \to \int_{Q_T} u\varphi.$$

We also need to establish the convergence of the time derivative. The idea is to use the definition of norm.

Definition 1 *Let V be a normed space, and $\psi : V \to \mathbb{R}$ be a linear functional. Then* the norm of ψ on the dual space V' of V *is defined by*

$$\|\psi\|_{V'} = \sup_{x\in V} \frac{\langle \psi, x \rangle_{V'\times V}}{\|x\|_V}.$$

We now fix the space of test functions as $V = L^2(0,T;H^1(\Omega))$, and write, using (1.7) and Hölder's inequality,

$$\begin{aligned}\int_0^T \langle \partial_t u_n, \varphi \rangle &\leq \int_{Q_T} |\nabla u_n||\nabla \varphi| + \int_{Q_T} |u_n||\varphi| \\ &\leq \|\nabla u_n\|_{L^2}\|\nabla \varphi\|_{L^2} + \|u_n\|_{L^2}\|\varphi\|_{L^2} \leq C\|\varphi\|_{L^2(H^1)},\end{aligned} \tag{1.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product in $(H^1(\Omega))' \times H^1(\Omega)$. Thus $\|\partial_t u_n\|_{L^2((H^1)')} \leq C$, so we again get the existence of a subsequence of $\partial_t u_n$ (not relabeled) and of an element $z \in L^2(0,T;(H^1(\Omega))')$ such that

$$\partial_t u_n \rightharpoonup z \quad \text{weakly in } L^2(0,T;(H^1(\Omega))'). \tag{1.11}$$

Let us now identify z as $\partial_t u$. We consider the space $C_c^\infty(0,T;H^1(\Omega))$, which is dense in $L^2(0,T;H^1(\Omega))$. Then, for $\psi \in C_c^\infty(0,T;H^1(\Omega))$, we have

$$\int_0^T \langle \partial_t u_n, \psi\rangle \to \int_0^T \langle z, \psi\rangle,$$

as well as, using the weak convergence (1.8),

$$\int_0^T \langle \partial_t u_n, \psi\rangle = -\int_0^T \langle u_n, \partial_t \psi\rangle = -\int_0^T \int_\Omega u_n \partial_t \psi \to -\int_0^T \int_\Omega u \partial_t \psi = \int_0^T \langle \partial_t u, \psi\rangle,$$

and, by the density and the uniqueness of the limit, we deduce

$$\int_0^T \langle z, \varphi\rangle = \int_0^T \langle \partial_t u, \varphi\rangle,$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$. That is, $z = \partial_t u$.

Therefore, taking into account the convergences (1.8), (1.9) and (1.11), and the above identification, we can pass to the limit in (1.7) to obtain a weak solution of (1.6).

There only rests to give a sense in which the initial data should be satisfied. We have, for $\psi \in C^\infty(Q_T)$

$$\int_0^T \langle \partial_t (u-u_0), \psi\rangle = -\int_{Q_T} (u-u_0)\partial_t \psi + \int_\Omega (u(T)-u_0)\psi(T) - \int_\Omega (u(0)-u_0)\psi(0).$$

Therefore, choosing $\psi \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $\psi(T) = 0$ (see Remark 1.1), we find that the initial condition is satisfied in the sense

$$\int_0^T \langle \partial_t u, \psi\rangle + \int_{Q_T} (u-u_0)\partial_t \psi = 0. \tag{1.12}$$

Theorem 1.1 (Sobolev's embedding theorem) *Let $\Omega \subset \mathbb{R}^N$ be bounded and of class C^1, and $1 \leq p \leq \infty$. The following injections are continuous:*

- $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, *with* $p^* = Np/(N-p)$, *if* $p < N$,
- $W^{1,p}(\Omega) \subset L^q(\Omega)$, *for all* $1 \leq q < \infty$, *if* $p = N$,
- $W^{1,p}(\Omega) \subset C(\bar{\Omega})$, *if* $p > N$.

Remark 1.1 *Sobolev's embedding theorem states that the injection $H^1(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))$ is continuous. That is , $\psi : [0,T] \to L^2(\Omega)$, is continuous, and hence it makes sense to set $\psi(T) = 0$ in Ω.*

Remark 1.2 *If the solution of (1.6) is more regular, say* $\partial_t u \in L^2(Q_T)$*, then integrating by parts in (1.12) we get*

$$0=\int_{Q_T}\partial_t u\psi+\int_{Q_T}(u-u_0)\partial_t\psi=-\int_\Omega u(0)\psi(0)-\int_{Q_T}u_0\partial_t\psi=\int_\Omega(u_0-u(0))\psi(0).$$

Since this identity holds for all $\psi(0)\in L^2(\Omega)$*, we deduce* $u(0)=u_0$ *a.e. in* Ω*.*

1.2 Time discretization

We introduce the following time discretization of problem (1.1)-(1.3). Let $K\in\mathbb{N}$, $\tau=T/K$, and consider the decomposition $(0,T]=\cup_{k=0}^K(t_{k-1},t_k]$, with $t_k=k\tau$. Then, in each time slice, we consider the following problem: Given $u^{k-1}\in L^2(\Omega)$, find $u^k:\Omega\to\mathbb{R}$ such that

$$\frac{1}{\tau}\int_\Omega(u^k-u^{k-1})\varphi+\int_\Omega\nabla u^k\cdot\nabla\varphi=\int_\Omega u^k\varphi\quad\text{for all }\varphi\in H^1(\Omega). \tag{1.13}$$

Lemma 2 (Lax-Milgram) *Let H be a Hilbert space and assume that* $A:H\times H\to\mathbb{R}$ *is a continuous coercive bilinear form. Then, given any* $F\in H'$*, there exists a unique element* $u\in H$ *such that* $A(u,\varphi)=\langle F,\varphi\rangle$ *for all* $\varphi\in H$*.*

We take $H=H^1(\Omega)$, and define $F=\frac{1}{\tau}u^{k-1}\in L^2(\Omega)\subset(H^1(\Omega))'$, and

$$A(u,\varphi)=\int_\Omega\nabla u\cdot\nabla\varphi+\frac{1}{\tau}\int_\Omega u\varphi.$$

The bilinear form A is clearly continuous and coercive in $H^1(\Omega)$. Then, Lax-Milgram's lemma provide us with a weak solution, $u^k\in H^1(\Omega)$ of (1.13). We can use $\varphi=u^k$ as test function in (1.13) to get

$$(1-\tau)\int_\Omega|u^k|^2+\tau\int_\Omega|\nabla u^k|^2=\int_\Omega u^{k-1}u^k.$$

Using Youngs' inequality, we get

$$\Big(\frac{1}{2}-\tau\Big)\int_\Omega|u^k|^2+\tau\int_\Omega|\nabla u^k|^2\le\frac{1}{2}\int_\Omega|u^{k-1}|^2. \tag{1.14}$$

Taking[1] $\tau<1/4$, and using the bound $(1-r)^{-1}\le\exp(r(1-r)^{-1})$ for all $r\in[0,1)$ (Exercise 1), from the inequality

$$(1-2\tau)\int_\Omega|u^k|^2\le\int_\Omega|u^{k-1}|^2, \tag{1.15}$$

we get,

$$\int_\Omega|u^k|^2\le e^{4T}\int_\Omega|u_0|^2\le C. \tag{1.16}$$

[1] Since we are interested in the limit $\tau\to0$, this restriction is irrelevant.

Here, C is a constant which may change of value, but which is independent of k. Summing (1.14) for $k = 1,\dots,K$, we obtain

$$\frac{1}{2}\int_\Omega |u^K|^2 + \tau\sum_{k=1}^{K}\int_\Omega |\nabla u^k|^2 \leq \frac{1}{2}\int_\Omega |u_0|^2 + \tau\sum_{k=1}^{K}\int_\Omega |u^k|^2.$$

and thus, using (1.16) and $K\tau = T$,

$$\tau\sum_{k=1}^{K}\int_\Omega |\nabla u^k|^2 \leq \frac{1}{2}\int_\Omega |u_0|^2 + TC \leq C. \tag{1.17}$$

Gathering (1.16) and (1.17) yields

$$\max_{k=1,\dots,K}\int_\Omega |u^k|^2 + \tau\sum_{k=1}^{K}\int_\Omega |\nabla u^k|^2 \leq C. \tag{1.18}$$

1.3 Back to the evolution problem

Consider the piecewise constant and piecewise linear interpolators in time,

$$u^{(\tau)}(t,x) = u^k(x), \quad \tilde{u}^{(\tau)}(t,x) = u^k(x) + \frac{t_k - t}{\tau}(u^{k-1}(x) - u^k(x)),$$

for $(t,x) \in (t_{k-1},t_k] \times \Omega$, for $k = 1,\dots,K$. Then (1.18) implies

$$\max_{t\in(0,T)}\int_\Omega |u^{(\tau)}|^2 + \int_{Q_T} |\nabla u^{(\tau)}|^2 \leq C, \tag{1.19}$$

and taking into account that $t_k - t < \tau$, we also deduce

$$\max_{t\in(0,T)}\int_\Omega |\tilde{u}^{(\tau)}|^2 + \int_{Q_T} |\nabla \tilde{u}^{(\tau)}|^2 \leq C. \tag{1.20}$$

Replacing $u^{(\tau)}$ and $\tilde{u}^{(\tau)}$ in the weak formulation (1.13) we get

$$\int_{Q_T} \partial_t \tilde{u}^{(\tau)}\varphi + \int_{Q_T} \nabla u^{(\tau)} \cdot \nabla\varphi = \int_{Q_T} u^{(\tau)}\varphi \quad \text{for all } \varphi \in L^2(0,T;H^1(\Omega)). \tag{1.21}$$

From this identity and (1.19), we obtain, like in (1.10),

$$\|\partial_t \tilde{u}^{(\tau)}\|_{L^2((H^1)')} \leq C. \tag{1.22}$$

Therefore, from (1.19), (1.20), and (1.22) we deduce the existence of $u, z \in L^2(0,T;H^1(\Omega))$ and subsequences of $u^{(\tau)}$ and $\tilde{u}^{(\tau)}$ (not relabeled) such that

$$\begin{aligned}
u^{(\tau)} &\rightharpoonup u && \text{weakly in } L^2(0,T;H^1(\Omega)),\\
u^{(\tau)} &\rightharpoonup u && \text{weakly*-weakly in } L^\infty(0,T;L^2(\Omega))),\\
\tilde{u}^{(\tau)} &\rightharpoonup z && \text{weakly in } L^2(0,T;H^1(\Omega)),\\
\tilde{u}^{(\tau)} &\rightharpoonup z && \text{weakly*-weakly in } L^\infty(0,T;L^2(\Omega))),\\
\partial_t \tilde{u}^{(\tau)} &\rightharpoonup \partial_t z && \text{weakly in } L^2(0,T;((H^1(\Omega))').
\end{aligned}$$

Finally, let us obtain the identification $z = u$. Since, for $t \in (t_{k-1}, t_k]$,

$$|\tilde{u}^{(\tau)}(t,x) - u^{(\tau)}(t,x)| = |(t_k - t)\frac{u^{k-1}(x) - u^k(x)}{\tau}| \leq \tau|\partial_t \tilde{u}^{(\tau)}(t,x)|,$$

we deduce from (1.22)

$$\|\tilde{u}^{(\tau)} - u^{(\tau)}\|_{L^2((H^1)')} \leq \tau\|\partial_t \tilde{u}^{(\tau)}\|_{L^2((H^1)')} \to 0 \quad \text{as } \tau \to 0,$$

and hence $z = u$. Therefore, we may pass to the limit $\tau \to 0$ in (1.21) to deduce the existence of a weak solution of (1.1)-(1.3) in the sense of (1.6), with $V = L^2(0,T;H^1(\Omega))$.

Finally, we show lower and upper bounds of the solution. We use the Stampacchia truncature method, also useful for some systems of equations, although not for the cross-diffusion systems we have on mind.

We will show formal calculations, which are justified under enough regularity of the solutions. These computations can be done rigorously using similar arguments for the time discrete problem (1.13) (Exercise 3).

Assume that $\partial_t u \in L^2(Q_T)$. Let $T(u) = \min\{u - z, 0\}$, with $z = me^{-\lambda t}$, for some λ to be determined, and for $m \in \mathbb{R}$ such that $u_0 \geq m$ a.e. in Ω. Using $T(u)$ as a test function in the weak formulation of problem (1.1)-(1.3), we obtain

$$\int_{Q_T} T(u)\partial_t u + \int_{Q_T} \nabla u \cdot \nabla T(u) = \int_{Q_T} uT(u).$$

Then, since $\nabla u \cdot \nabla T(u) \geq 0$,

$$\int_{Q_T} T(u)\partial_t(u-z) - \lambda\int_{Q_T} zT(u) \leq \int_{Q_T} (u-z)T(u) + \int_{Q_T} z|T(u)|.$$

Since $zT(u) \leq 0$, we find that

$$\frac{1}{2}\int_{Q_T} \partial_t T(u)^2 + (\lambda - 1)\int_{Q_T} z|T(u)| \leq \int_{Q_T} T(u)^2.$$

Therefore, taking $\lambda \geq 1$, and using Gronwall's lemma we obtain

$$\int_{\Omega} T(u(t,\cdot))^2 \leq e^{2t}\int_{\Omega} T(u_0)^2 = 0,$$

yielding $u(t,\cdot) \geq me^{-\lambda t}$ a.e. in Ω.

For the upper bound we use as test funtion $T(u) = \max\{Z - u, 0\}$, with $Z = Me^{\lambda t}$, for some λ to be determined, and for $M \in \mathbb{R}$ such that $u_0 \leq M$ a.e. in Ω. We obtain, since $\nabla u \cdot \nabla T(u) \geq 0$,

$$\int_{Q_T} T(u)\partial_t(u-z) + \lambda\int_{Q_T} zT(u) \leq \int_{Q_T} (u-z)T(u) + \int_{Q_T} T(u).$$

Now, $zT(u) \geq 0$, so taking $\lambda > 1$ we find

$$\frac{1}{2}\int_{Q_T} \partial_t T(u)^2 \leq \int_{Q_T} T(u)^2,$$

and Gronwall's lemma yields $u(t,\cdot) \leq Me^{-\lambda t}$ a.e. in Ω.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz continuous boundary, and let $T > 0$. Suppose that $u_0 \in L^2(\Omega)$, and that there exist constants $m, M \in \mathbb{R}$ such that*

$$m \leq u_0 \leq M \quad \text{a.e. in } \Omega.$$

Then, problem (1.23)-(1.25) has a weak solution $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$, in the sense that for all $\varphi \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T < \partial_t u, \varphi > + \int_{Q_T} \nabla u \cdot \nabla \varphi = \int_{Q_T} u \varphi,$$

with $< \cdot, \cdot >$ denoting the duality product between $H^1(\Omega)$ and its dual $(H^1(\Omega))'$. In addition, for $t \in (0,T)$, u satisfies

$$me^{\lambda t} \leq u(t,\cdot) \geq Me^{\lambda t} \quad \text{a.e. in } \Omega,$$

and the initial data is satisfied in the sense

$$\int_0^T < \partial_t u, \psi > + \int_{Q_T} (u - u_0) \partial_t \psi = 0,$$

for all $\psi \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $\psi(T) = 0$ a.e. in Ω.

2 A nonlinear population model

In this section we introduce nonlinearities in the diffusion and reaction terms of the partial differential equation (1.1).

The problem is the following. Given a fixed $T > 0$ and a bounded set $\Omega \subset \mathbb{R}^N$, find (a non-negative) $u : (0,T) \times \Omega \to \mathbb{R}$ such that

$$\partial_t u - \operatorname{div}(u\nabla u) = f(u) \qquad \text{in } Q_T, \tag{1.23}$$

$$u\nabla u \cdot n = 0 \qquad \text{on } \Gamma_T, \tag{1.24}$$

$$u(\cdot,0) = u_0 \geq 0 \qquad \text{in } \Omega, \tag{1.25}$$

where $f(u) = u(\alpha - \beta u)$ is a logistic reaction term ($\alpha, \beta \geq 0$). In terms of population dynamics, we are supposing that

- The population diffuses to avoid overcrowding (maxima of u).
- The newborns are proportional to the existent population, but there is a growth limit given in terms of the so-called *carrying capacity* of the habitat. The corresponding kinetics ($\partial_t u = f(u)$) has a stable equilibrium at $u = \alpha/\beta$.

The generic form of weak solution we shall deal with is

$$\int_{Q_T} \langle \partial_t u, \varphi \rangle + \int_{Q_T} u\nabla u \cdot \nabla\varphi = \int_{Q_T} f(u)\varphi, \quad \text{for all } \varphi \in V, \tag{1.26}$$

with V to be explicited later.

2.1 Formal arguments

For problem (1.23)-(1.25), we have the following formal estimates:

- Using $\varphi = \ln(u)$ in (1.26) we get, for $F(s) = s(\ln(s) - 1) + 1 \geq 0$,

$$\int_\Omega F(u(T)) + \int_{Q_T} |\nabla u|^2 = \int_\Omega F(u_0) + \int_{Q_T} f(u)\ln(u). \tag{1.27}$$

  The term $E(t) = \int_\Omega F(u(t))$ is called the *entropy* of the system, since it is related to the physical entropy defined in thermodynamics. Observe that this identity only makes sense if $u > 0$.

- Using $\varphi = 1$ in (1.26) we get (if $u \geq 0$)

$$\int_\Omega u(T) \leq \int_\Omega u_0 + \alpha \int_{Q_T} u,$$

  and then Gronwall's lemma implies

$$\int_\Omega u(T) \leq e^{\alpha T} \int_\Omega u_0 \leq C.$$

Suppose that the right hand side of (1.27) may be controled in terms of the left hand side. We then deduce

$$\max_{t\in(0,T)} \int_\Omega F(u(t)) + \|u\|_{L^\infty(L^1)} + \|\nabla u\|_{L^2} \leq C, \tag{1.28}$$

and therefore, $\|u\|_{L^2(H^1)} \leq C$ (we shall see later why).

Now, suppose that estimate (1.28) is also satisfied by the sequence of solutions, u_n, to approximated problems (P_n) of the form

$$\int_{Q_T} \partial_t u_n \, \varphi + \int_{Q_T} \psi_n(u_n)\nabla u_n \cdot \nabla\varphi = \int_{Q_T} f(u_n)\varphi, \quad \text{for all } \varphi \in V,$$

with $\psi_n \to id$. That is, suppose that $\|u_n\|_{L^2(H^1)} \leq C$. Then, there exists $u \in L^2(0,T;H^1(\Omega))$ such that

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(Q_T).$$

The gradient estimate is the first ingredient to prove the (relative) strong compactness of the sequence u_n in some L^p space, which provides strong convergence in L^p, and a.e. convergence in Q_T. Both of these convergences are necessary to pass to the limit in the nonlinear terms. Clearly, the limit (if it does exist) is a candidate to solution of (1.23)-(1.24).

The second ingredient to prove the compactness is an estimate for the time derivative.

Lemma 3 (Simon, Aubin-Lions) *Let X, B, and Y be Banach spaces with $X \subset B \subset Y$ such that*

- *X is compactly embedded in B.*
- *B is continuously embedded in Y.*

Suppose that the sequence u_n satisfies:

- *u_n is bounded in $L^q(0,T;X)\cap L^1_{loc}(0,T;X)$, for $1 < q \leq \infty$.*
- *$\partial_t u_n$ is bounded in $L^1_{loc}(0,T;Y)$.*

Then, for all $p < q$, there exists a subsequence of u_n (not relabeled) and an element $u \in L^p(0,T;B)$ such that

$$u_n \to u \quad \text{strongly in } L^p(0,T;B) \text{ and a.e. in } Q_T.$$

A usual situation is that of taking $X = H^1(\Omega)$, and $B = L^2(\Omega)$. Indeed,

Theorem 1.3 (Rellich-Kondrachov) *Let $\Omega \subset \mathbb{R}^N$ be bounded and of class C^1, and $1 \leq p \leq \infty$. The following injections are compact:*

- *$W^{1,p}(\Omega) \subset L^q(\Omega)$, for all $1 \le q < p^*$, with $p^* = Np/(N-p)$, if $p < N$,*
- *$W^{1,p}(\Omega) \subset L^q(\Omega)$, for all $p \le q < \infty$, if $p = N$,*
- *$W^{1,p}(\Omega) \subset C(\bar{\Omega})$, if $p > N$.*

Then we get that

$$\partial_t u_n \text{ bounded in } L^1(0,T;Y) \Longrightarrow u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.$$

Summarizing, if the right hand side of (1.27) may be absorbed by the letf hand side, and the time derivative estimate is available, we have

$$\begin{aligned}
&\nabla u_n \rightharpoonup \nabla u && \text{weakly in } L^2(Q_T),\\
&\partial_t u_n \rightharpoonup \partial_t u && \text{weakly in } L^1(0,T;Y),\\
&u_n \to u && \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.
\end{aligned}$$

Observe that with these kind of estimates (and others), we have to justify the following limits in the weak formulation

$$\begin{aligned}
&\int_{Q_T} \langle \partial_t u_n, \varphi \rangle \to \int_{Q_T} \langle \partial_t u, \varphi \rangle,\\
&\int_{Q_T} \psi_n(u_n) \nabla u_n \cdot \nabla \varphi \to \int_{Q_T} u \nabla u \cdot \nabla \varphi,\\
&\int_{Q_T} (\alpha u_n - \beta u_n^2)\varphi = \int_{Q_T} f(u_n)\varphi \to \int_{Q_T} f(u)\varphi.
\end{aligned}$$

2.2 Time discretization

Like in the linear case, we introduce the following time discretization of problem (1.23)-(1.25). Consider a Banach space, V, defined on Ω, to be fixed later. Let $K \in \mathbb{N}$, $\tau = T/K$, and consider the decomposition $(0,T] = \cup_{k=0}^{K}(t_{k-1}, t_k]$, with $t_k = k\tau$.

First (non-successful) attempt. In each time slice, we consider the following nonlinear problem: Given $u^{k-1} \in V$, find $u^k : \Omega \to \mathbb{R}$ such that

$$\frac{1}{\tau}(u^k - u^{k-1}) - \operatorname{div}(u^k \nabla u^k) = f(u^k) \quad \text{in } \Omega, \tag{1.29}$$

$$u^k \nabla u^k \cdot n = 0 \quad \text{on } \partial\Omega. \tag{1.30}$$

We, further, linearize problem (1.29)-(1.30) in order to apply Lax-Milgram's lemma (Lemma 2): Given u^{k-1}, $v \in V$, find $u^k : \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\frac{1}{\tau}(u^k - u^{k-1}) - \operatorname{div}(v \nabla u^k) = f(v) && \text{in } \Omega,\\
&v \nabla u^k \cdot n = 0 && \text{on } \partial\Omega.
\end{aligned}$$

Like in the linear case, we would like to take $H = H^1(\Omega)$, and define

$$A(u,\varphi) = \int_\Omega v\nabla u \cdot \nabla\varphi + \frac{1}{\tau}\int_\Omega u\varphi.$$

However, $A(u,\varphi)$ is not coercive in $H^1(\Omega)$ since v might vanish. Moreover, since our proof is based in using $\ln(u)$ as a test function, we also need to avoid the singularity arising when $u = 0$. We adopt the following approximation.

Approximation of the linear problem

Let $\varepsilon > 0$. The regularized problem reads as follows: Given u_ε^{k-1}, $v \in V$, find $u_\varepsilon^k : \Omega \to \mathbb{R}$ such that

$$\frac{1}{\tau}(u_\varepsilon^k - u_\varepsilon^{k-1}) - \text{div}(a_\varepsilon(v)\nabla u_\varepsilon^k)) = f_\varepsilon(v) \qquad \text{in } \Omega, \tag{1.31}$$

$$a_\varepsilon(v)\nabla u_\varepsilon^k \cdot n = 0 \qquad \text{on } \partial\Omega, \tag{1.32}$$

with $f_\varepsilon(s) = \alpha s - \beta a_\varepsilon(s)^2$. Here, a_ε, must be an approximation to the identity function, to which we shall impose $\varepsilon^{-1} \geq a_\varepsilon(s) \geq \varepsilon$ for all $s \in \mathbb{R}$. In weak form, we write:

$$\frac{1}{\tau}\int_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})\varphi + \int_\Omega a_\varepsilon(v)\nabla u_\varepsilon^k \cdot \nabla\varphi = \int_\Omega f_\varepsilon(v)\varphi, \quad \text{for all } \varphi \in H^1(\Omega). \tag{1.33}$$

Now we can take $H = H^1(\Omega)$, $V = L^2(\Omega)$ in the Lax-Milgram's lemma, and define

$$A_\varepsilon(u,\varphi) = \int_\Omega a_\varepsilon(v)\nabla u \cdot \nabla\varphi + \frac{1}{\tau}\int_\Omega u\varphi, \qquad F = f_\varepsilon(v) + \frac{1}{\tau}u_\varepsilon^{k-1} \in L^2(\Omega) \subset (H^1(\Omega))'.$$

The bilinear form A_ε is clearly continuous and coercive in $H^1(\Omega)$, and therefore there exists a weak solution, $u_\varepsilon^k \in H^1(\Omega)$, of (1.31)-(1.32).

At this point, we go back to the formulation of the nonlinear time-discrete problem and add the perturbation introduced in the linear problem. That is, we replace problem (1.29)-(1.30) by the following: Given $u_\varepsilon^{k-1} \in L^2(\Omega)$, find $u^k : \Omega \to \mathbb{R}$ such that

$$\frac{1}{\tau}(u_\varepsilon^k - u_\varepsilon^{k-1}) - \text{div}(a_\varepsilon(u_\varepsilon^k)\nabla u_\varepsilon^k)) = f_\varepsilon(u_\varepsilon^k) \qquad \text{in } \Omega, \tag{1.34}$$

$$a_\varepsilon(u_\varepsilon^k)\nabla u_\varepsilon^k \cdot n = 0 \qquad \text{on } \partial\Omega, \tag{1.35}$$

or, in weak form,

$$\frac{1}{\tau}\int_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})\varphi + \int_\Omega a_\varepsilon(u_\varepsilon^k)\nabla u_\varepsilon^k \cdot \nabla\varphi = \int_\Omega f_\varepsilon(u_\varepsilon^k)\varphi, \quad \text{for all } \varphi \in H^1(\Omega). \tag{1.36}$$

Now, observe that we have the following inconvenient: Assuming that we may use $\varphi = F'(u_\varepsilon^k) = \ln(u_\varepsilon^k)$ as test function, we obtain in the diffusion term

$$\int_\Omega a_\varepsilon(u_\varepsilon^k)F''(u_\varepsilon^k)|\nabla u_\varepsilon^k|^2 = \int_\Omega \frac{a_\varepsilon(u_\varepsilon^k)}{u_\varepsilon^k}|\nabla u_\varepsilon^k|^2,$$

instead of the original formal identity

$$\int_\Omega uF''(u)|\nabla u|^2 = \int_\Omega |\nabla u|^2.$$

Thus, we also need to approximate F by a suitable sequence F_ε which allows us to obtain an L^2 estimate of ∇u_ε.

The definition of a_ε and F_ε

For $\varepsilon > 0$, we want to produce approximations:

- a_ε such that $a_\varepsilon(s) \to s$ as $\varepsilon \to 0$, with $\varepsilon^{-1} \geq a_\varepsilon(s) \geq \varepsilon$ for all $s \in \mathbb{R}$.
- F_ε non-negative and smooth such that $F_\varepsilon(s) \to F(s) = s(\ln(s) - 1) + 1$, as $\varepsilon \to 0$,
- $a_\varepsilon(s) F_\varepsilon''(s) = 1$ for all $s \in \mathbb{R}$.

Let $a_\varepsilon : \mathbb{R} \to [\varepsilon, \varepsilon^{-1}]$ be given by the truncature function

$$a_\varepsilon(s) := \begin{cases} \varepsilon & \text{if } s \leq \varepsilon, \\ s & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}, \\ \varepsilon^{-1} & \text{if } \varepsilon^{-1} \leq s. \end{cases}$$

Using the third condition, we set $F_\varepsilon''(s) = 1/a_\varepsilon(s)$. Integrating and adjusting the integration constants for continuity, we get $F_\varepsilon \in C^{2,1}(\mathbb{R}, \mathbb{R}_+)$ given by

$$F_\varepsilon(s) := \begin{cases} \dfrac{s^2 - \varepsilon^2}{2\varepsilon} + s(\ln \varepsilon - 1) + 1 & \text{if } s \leq \varepsilon, \\ s(\ln s - 1) + 1 & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}, \\ \dfrac{\varepsilon(s^2 - \varepsilon^{-2})}{2} + s(\ln \varepsilon^{-1} - 1) + 1 & \text{if } \varepsilon^{-1} \leq s, \end{cases}$$

with

$$F_\varepsilon'(s) := \begin{cases} \dfrac{s}{\varepsilon} + \ln \varepsilon - 1 & \text{if } s \leq \varepsilon, \\ \ln s & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}, \\ \varepsilon s + \ln \varepsilon^{-1} - 1 & \text{if } \varepsilon^{-1} \leq s. \end{cases}$$

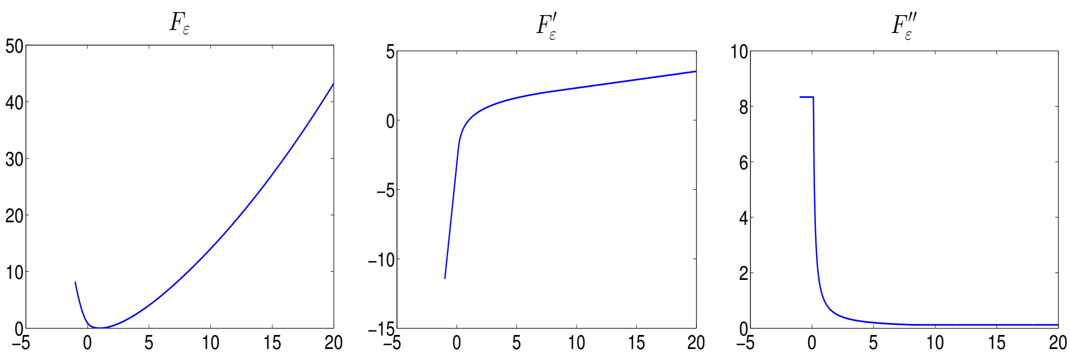


Figure 1.1: The convex function F_ε and its derivatives.

Fixed point method to couple the nonlinearities

Theorem 1.4 (Leray-Schauder fixed point theorem) *Let V be a Banach space and let $S : V \times [0,1] \to V$ be a continuous and compact map such that*

- $S(v,0) = 0$ *for all* $v \in V$.
- *For each pair* $(v,\sigma) \in V \times [0,1]$ *satisfying* $v = S(v,\sigma)$*, there exists a positive constant* C*, such that* $\|v\|_V \le C$.

Then there exist a fixed point, $w \in V$*, of the map* $S(v,1)$*, i.e.* $w = S(w,1)$.

To solve the nonlinear time-discrete problem (1.34)-(1.35) we define the operator $S : L^2(\Omega) \times [0,1] \to L^2(\Omega)$ such that, for $u_\varepsilon^{k-1} \in L^2(\Omega)$ given, applies $(v,\sigma) \in L^2(\Omega) \times [0,1]$ into the solution $u_\varepsilon^{k,\sigma}$, of the following linear problem (mind σ at the right hand side). Find $u_\varepsilon^{k,\sigma} : \Omega \to \mathbb{R}$ such that

$$\frac{1}{\tau} u_\varepsilon^{k,\sigma} - \operatorname{div}(a_\varepsilon(v)\nabla u_\varepsilon^{k,\sigma})) = \sigma\big(f_\varepsilon(v) + \frac{1}{\tau} u_\varepsilon^{k-1}\big) \qquad \text{in } \Omega, \tag{1.37}$$

$$a_\varepsilon(v)\nabla u_\varepsilon^{k,\sigma} \cdot n = 0 \qquad \text{on } \partial\Omega. \tag{1.38}$$

A straightforward application of Lax-Milgram's lemma, mimicking that of Subsection 2.2, shows that there exists a unique solution $u_\varepsilon^{k,\sigma} \in H^1(\Omega)$ of problem (1.37)-(1.38). Thus, S is well defined.

To apply the Leray-Schauder's theorem, we have to check the following:

1. Continuity: Let $v_n \in L^2(\Omega)$, $\sigma_n \in [0,1]$ be given sequences, with $v_n \to v$ strongly in $L^2(\Omega)$, and $\sigma_n \to \sigma$. Let us denote by $u_{\varepsilon,n}^k$ to the solution of the linear problem (1.37)-(1.38) corresponding to (v_n,σ_n), that is $S(v_n,\sigma_n)$. We must check that $u_{\varepsilon,n}^k \to u_\varepsilon^{k,\sigma}$ strongly in $L^2(\Omega)$, as $n \to \infty$.
2. Compactness: Since we start with $v \in L^2(\Omega)$ and finish in $S(v,\sigma) = u_\varepsilon^{k,\sigma} \in H^1(\Omega)$, and by Theorem 1.3 the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact, we deduce that S is compact.
3. $S(v,0) = 0$, which is inmmediate, after using $\varphi = u_\varepsilon^{k,\sigma} \in H^1(\Omega)$ as test function in the weak formulation of (1.37)-(1.38).
4. If $v = S(v,\sigma)(= u_\varepsilon^{k,\sigma})$ for $(v,\sigma) \in L^2(\Omega) \times [0,1]$ then $\|u_\varepsilon^{k,\sigma}\|_{L^2} \le C$.

We start proving the continuity. Using $\varphi = u_{\varepsilon,n}^k \in H^1(\Omega)$ as test function in the weak formulation of (1.37)-(1.38) (with (v,σ) replaced by (v_n,σ_n), and $u_\varepsilon^{k,\sigma}$ replaced by $u_{\varepsilon,n}^k$) we get

$$\frac{1}{\tau}\int_\Omega |u_{\varepsilon,n}^k|^2 + \int_\Omega a_\varepsilon(v_n)|\nabla u_{\varepsilon,n}^k|^2 = \sigma_n \int_\Omega f_\varepsilon(v_n) u_{\varepsilon,n}^k + \frac{\sigma_n}{\tau}\int_\Omega u_\varepsilon^{k-1} u_{\varepsilon,n}^k.$$

Since $a_\varepsilon(s) \ge \varepsilon$ for all $s \in \mathbb{R}$, we have

$$\int_\Omega |u_{\varepsilon,n}^k|^2 + \tau\varepsilon \int_\Omega |\nabla u_{\varepsilon,n}^k|^2 \le \tau\alpha\sigma_n \int_\Omega v_n u_{\varepsilon,n}^k - \tau\beta\sigma_n \int_\Omega a_\varepsilon(v_n)^2 u_{\varepsilon,n}^k + \sigma_n \int_\Omega u_\varepsilon^{k-1} u_{\varepsilon,n}^k.$$

Using Young's inequality in the form $ab \le \gamma a^2 + \frac{b^2}{\gamma}$, and $\sigma_n \le 1$, we get,

$$\frac{1}{4}\int_\Omega |u^k_{\varepsilon,n}|^2 + \tau\varepsilon \int_\Omega |\nabla u^k_{\varepsilon,n}|^2 \le 4\tau^2\alpha^2 \int_\Omega v_n^2 + 4\tau^2\beta^2 \int_\Omega |a_\varepsilon(v_n)|^4 + 4\int_\Omega |u^{k-1}_\varepsilon|^2.$$

Thus, since $v_n,\ u^{k-1}_\varepsilon \in L^2(\Omega)$, and $a_\varepsilon(s) \le \varepsilon^{-1}$, we obtain

$$\int_\Omega |u^k_{\varepsilon,n}|^2 + \tau\varepsilon \int_\Omega |\nabla u^k_{\varepsilon,n}|^2 \le C(1+\tau^2\varepsilon^{-4}), \tag{1.39}$$

implying that $\|u^k_{\varepsilon,n}\|_{H^1(\Omega)}$ is bounded. Then, using Theorem 1.3, the compact embedding $L^2(\Omega) \subset H^1(\Omega)$, implies the existence of $z \in H^1(\Omega)$ such that, up to a subsequence (not relabeled),

$$\begin{aligned} u^k_{\varepsilon,n} &\rightharpoonup z \quad \text{weakly in } H^1(\Omega), \\ u^k_{\varepsilon,n} &\to z \quad \text{strongly in } L^2(\Omega), \text{ and a.e. in } \Omega. \end{aligned} \tag{1.40}$$

Finally, the continuity will be proven if we identify z as $S(v,\sigma)(= u^{k,\sigma}_\varepsilon)$. We take the limit $n \to \infty$ in the weak formulation of problem (1.37)-(1.38), that is, in the identity

$$\frac{1}{\tau}\int_\Omega u^k_{\varepsilon,n}\varphi + \int_\Omega a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n}\cdot\nabla\varphi = \sigma_n \int_\Omega f_\varepsilon(v_n)\varphi + \frac{\sigma_n}{\tau}\int_\Omega u^{k-1}_\varepsilon \varphi. \tag{1.41}$$

By assumption, $v_n \to v$ strongly in $L^2(\Omega)$. Since a_ε is Lipschitz continuous (uniform constant equal to one), we have

$$\|a_\varepsilon(v_n) - a_\varepsilon(v)\|_{L^2} \le \|v_n - v\|_{L^2},$$

and thus $a_\varepsilon(v_n) \to a_\varepsilon(v)$ strongly in $L^2(\Omega)$ and a.e. in Ω, as $n \to \infty$.

Theorem 1.5 (Dominated convergence theorem) *Let f_n be a sequence of functions of $L^1(\Omega)$ satisfying*

- *$f_n(x) \to f(x)$ a.e. in Ω,*
- *there is a function $g \in L^p(\Omega)$, with $1 \le p < \infty$, such that, for all n, $|f_n(x)| \le g(x)$ a.e. in Ω.*

Then $f \in L^p(\Omega)$ and $f_n \to f$ strongly in $L^p(\Omega)$.

Being $a_\varepsilon(v_n) \le \varepsilon^{-1}$ for all n, we may use the dominated convergence theorem to deduce

$$a_\varepsilon(v_n) \to a_\varepsilon(v) \quad \text{strongly in } L^p(\Omega), \text{ for all } p < \infty.$$

Thus,

$$a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n} \rightharpoonup a_\varepsilon(v)\nabla z \quad \text{weakly in } L^q(\Omega), \text{ for } q = \frac{2p}{p+2} < 2, \text{ and } 2 < p < \infty.$$

Since $\nabla\varphi \in L^2(\Omega)$, the above convergence is not enough to pass to the limit in the diffusion term of (1.41). However, having the bound

$$\|a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n}\|_{L^2} \le \|a_\varepsilon(v_n)\|_{L^\infty}\|\nabla u^k_{\varepsilon,n}\|_{L^2} \le C,$$

we deduce that, in fact, up to a subsequence,

$$a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n} \rightharpoonup a_\varepsilon(v)\nabla z \quad \text{weakly in } L^2(\Omega). \tag{1.42}$$

Finally, f_ε is also Lipschitz continuous (constant equal to $\alpha + 2\beta\varepsilon^{-1}$), and a similar argument to that used for the sequence $a_\varepsilon(v_n)$ shows that

$$f_\varepsilon(v_n) \to f_\varepsilon(v) \quad \text{strongly in } L^2(\Omega). \tag{1.43}$$

Thus, gathering (1.40), (1.42), and (1.43), we get from (1.41), as $n \to \infty$,

$$\frac{1}{\tau}\int_\Omega z\varphi + \int_\Omega a_\varepsilon(v)\nabla z \cdot \nabla\varphi = \sigma\int_\Omega f_\varepsilon(v)\varphi + \frac{\sigma}{\tau}\int_\Omega u_\varepsilon^{k-1}\varphi,$$

so z is a weak solution of (1.37)-(1.38) corresponding to v. Moreover, the limit z is unique because the solution of the limit problem may be obtained by Lax-Milgram's lemma. Therefore, we deduce that the whole sequence converges, this is, $z = S(v,\sigma)$.

Finally, we prove point 4, this is, the uniform bound of the fixed points of S. Assume $v = u_\varepsilon^{k,\sigma}$ and let us prove that $\|u_\varepsilon^{k,\sigma}\|_{L^2} \leq C$, for all $\sigma \in [0,1]$. In this case, $u_\varepsilon^{k,\sigma}$ satisfies

$$\frac{1}{\tau}u_\varepsilon^{k,\sigma} - \operatorname{div}(a_\varepsilon(u_\varepsilon^{k,\sigma})\nabla u_\varepsilon^{k,\sigma})) = \sigma\big(f_\varepsilon(u_\varepsilon^{k,\sigma}) + \frac{1}{\tau}u_\varepsilon^{k-1}\big) \qquad \text{in } \Omega, \tag{1.44}$$

$$a_\varepsilon(u_\varepsilon^{k,\sigma})\nabla u_\varepsilon^{k,\sigma} \cdot n = 0 \qquad \text{on } \partial\Omega. \tag{1.45}$$

Using $\varphi = u_\varepsilon^{k,\sigma} \in H^1(\Omega)$ as test function in the weak formulation of (1.44)-(1.45), we obtain, like we did before for $u^k_{\varepsilon,n}$, an estimate similar to (1.39)

$$\int_\Omega |u_\varepsilon^{k,\sigma}|^2 + \tau\varepsilon\int_\Omega |\nabla u_\varepsilon^{k,\sigma}|^2 \leq C(1+\tau^2\varepsilon^{-4}\sigma^2) \leq C(1+\tau^2\varepsilon^{-4}),$$

implying that $\|u_\varepsilon^{k,\sigma}\|_{H^1(\Omega)}$ is bounded uniformly with respect to σ.

Therefore, we deduce the existence of a fixed point of $S(v,1)$, which we denote by u_ε^k, and that satisfies the nonlinear time-discrete problem (1.36).

Further estimates for the nonlinear time-dicrete problem

Until now, we have shown the existence of a weak solution u_ε^k to the nonlinear time-discrete problem

$$\frac{1}{\tau}\int_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})\varphi + \int_\Omega a_\varepsilon(u_\varepsilon^k)\nabla u_\varepsilon^k \cdot \nabla\varphi = \int_\Omega f_\varepsilon(u_\varepsilon^k)\varphi, \quad \text{for all } \varphi \in H^1(\Omega).$$

Now, we shall deduce some uniform estimates with respect to ε. Taking $\varphi = F'_\varepsilon(u_\varepsilon^k)$ and recalling that $F''_\varepsilon = 1/a_\varepsilon$, we get

$$\frac{1}{\tau}\int_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})F'_\varepsilon(u_\varepsilon^k) + \int_\Omega |\nabla u_\varepsilon^k|^2 = \int_\Omega f_\varepsilon(u_\varepsilon^k)F'_\varepsilon(u_\varepsilon^k). \tag{1.46}$$

For the first term of the left hand side, we use the convexity estimate (Exercise 2)

$$(s-t)F'_\varepsilon(s) \geq F_\varepsilon(s) - F_\varepsilon(t), \quad \text{for all} \quad s,t \in \mathbb{R}. \tag{1.47}$$

For the term at the right hand side, we use (Exercise 2)

$$F_\varepsilon(s) \geq \frac{\varepsilon}{2}s^2 - 2 \quad \text{for all } s \geq 0, \quad F_\varepsilon(s) \geq \frac{s^2}{2\varepsilon} \quad \text{for all } s \leq 0, \tag{1.48}$$

$$\max\{a_\varepsilon(s), sF_\varepsilon'(s)\} \leq 2F_\varepsilon(s) + 1 \quad \text{for all } s \in \mathbb{R}, \tag{1.49}$$

$$a_\varepsilon(s)F_\varepsilon'(s) \geq s - 1 \quad \text{for all } s \in \mathbb{R}, \tag{1.50}$$

$$F_\varepsilon(a_\varepsilon(s)) \leq F_\varepsilon(s) \quad \text{for all } s \in \mathbb{R}. \tag{1.51}$$

From (1.49), (1.50), and (1.51), and noting that[2] $[1-s]_+ \leq 1 + [s]_-$, we deduce

$$\begin{aligned} f_\varepsilon(s)F_\varepsilon'(s) &= \alpha sF_\varepsilon'(s) - \beta a_\varepsilon(s)^2 F_\varepsilon'(s) \leq \alpha(2F_\varepsilon(s)+1) + \beta a_\varepsilon(s)[1-s]_+ \\ &\leq (\alpha+\beta)(2F_\varepsilon(s)+1) + \beta a_\varepsilon(s)[s]_- \\ &\leq (\alpha+\beta)(2F_\varepsilon(s)+1) + \frac{\beta}{2\varepsilon}([s]_-)^2 + \frac{\beta\varepsilon}{2}a_\varepsilon(s)^2 \\ &\leq (\alpha+\beta)(2F_\varepsilon(s)+1) + \beta F_\varepsilon(s) + \beta(2 + F_\varepsilon(a_\varepsilon(s))) \\ &\leq (\alpha+\beta)(2F_\varepsilon(s)+1) + \beta F_\varepsilon(s) + \beta(2 + F_\varepsilon(s)) \\ &= (2\alpha+4\beta)F_\varepsilon(s) + \alpha + 3\beta. \end{aligned} \tag{1.52}$$

Using (1.47) and (1.52) in (1.46), we obtain

$$\int_\Omega F_\varepsilon(u_\varepsilon^k) + \tau \int_\Omega |\nabla u_\varepsilon^k|^2 \leq C\tau + \int_\Omega F_\varepsilon(u_\varepsilon^{k-1}) + 2\tau(\alpha+\beta)\int_\Omega F_\varepsilon(u_\varepsilon^k),$$

and thus,

$$(1-\omega\tau)\int_\Omega F_\varepsilon(u_\varepsilon^k) + \tau \int_\Omega |\nabla u_\varepsilon^k|^2 \leq C\tau + \int_\Omega F_\varepsilon(u_\varepsilon^{k-1}), \tag{1.53}$$

with $\omega = 2(\alpha+\beta)$. Here, we impose $\tau < \omega^{-1}$.

Estimate of the entropy. From (1.53) and reasoning as in (1.15)-(1.16), we get

$$\max_{k=1,\ldots,K} \int_\Omega F_\varepsilon(u_\varepsilon^k) \leq e^{\omega T/(1-\omega\tau)}\Big(CT + \int_\Omega F_\varepsilon(u_0)\Big) \leq C. \tag{1.54}$$

Estimate of the gradient. Summing (1.53) in k and recalling that $K\tau = T$, we get

$$\begin{aligned} \int_\Omega F_\varepsilon(u_\varepsilon^k) + \tau\sum_{k=1}^K \int_\Omega |\nabla u_\varepsilon^k|^2 &\leq C\tau K + \int_\Omega F_\varepsilon(u_0) + \omega\tau\sum_{k=1}^K \int_\Omega F_\varepsilon(u_\varepsilon^k) \\ &\leq CT + \int_\Omega F_\varepsilon(u_0) + \omega T \max_{k=1,\ldots,K}\int_\Omega F_\varepsilon(u_\varepsilon^k), \end{aligned}$$

and thus, by (1.54),

$$\tau\sum_{k=1}^K \int_\Omega |\nabla u_\varepsilon^k|^2 \leq C.$$

[2] We define $[s]_+ = \max\{0,s\}$, and $[s]_- = -\min\{0,s\} \geq 0$. Thus, $s = [s]_+ - [s]_-$.

Other estimates. From (1.48), we have

$$\frac{1}{2\varepsilon}\int_\Omega |[u_\varepsilon^k]_-|^2 \le \int_\Omega F_\varepsilon([u_\varepsilon^k]_-) = \int_{u_\varepsilon^k\le 0} F_\varepsilon(u_\varepsilon^k)$$
$$\le \int_{u_\varepsilon^k\le 0} F_\varepsilon(u_\varepsilon^k) + \int_{u_\varepsilon^k\ge 0} F_\varepsilon(u_\varepsilon^k) = \int_\Omega F_\varepsilon(u_\varepsilon^k) \le C.$$

Thus, from (1.54), we obtain the following bound for the extenct of negativity of u_ε^k:

$$\max_{k=1,\ldots,K}\int_\Omega |[u_\varepsilon^k]_-|^2 \le C\varepsilon. \tag{1.55}$$

Using the test function $\varphi = 1$ in the weak formulation (1.36) we get

$$(1-\alpha\tau)\int_\Omega u_\varepsilon^k \le \int_\Omega u_\varepsilon^{k-1}, \quad \text{implying (Gronwall's lemma)} \quad \max_{k=1,\ldots,K}\int_\Omega u_\varepsilon^k \le C.$$

Using this estimate and Young's inequality, we obtain

$$\int_\Omega |u_\varepsilon^k| = \int_\Omega ([u_\varepsilon^k]_+ + [u_\varepsilon^k]_-) = \int_\Omega u_\varepsilon^k + 2\int_\Omega [u_\varepsilon^k]_- \le C\Big(1+\int_\Omega |[u_\varepsilon^k]_-|^2\Big),$$

and then, from (1.55)

$$\max_{k=1,\ldots,K}\int_\Omega |u_\varepsilon^k| \le C.$$

Summarizing, we have obtained the bound

$$\max_{k=1,\ldots,K}\Big(\int_\Omega F_\varepsilon(u_\varepsilon^k) + \int_\Omega |u_\varepsilon^k| + \frac{1}{\varepsilon}\int_\Omega ([u_\varepsilon^k]_-)^2\Big) + \tau\sum_{k=1}^K \int_\Omega |\nabla u_\varepsilon^k|^2 \le C. \tag{1.56}$$

2.3 Back to the evolution problem

Consider the piecewise constant and piecewise linear interpolators in time,

$$u_\varepsilon^{(\tau)}(t,x) = u_\varepsilon^k(x), \quad \tilde{u}_\varepsilon^{(\tau)}(t,x) = u_\varepsilon^k(x) + \frac{t_k - t}{\tau}(u_\varepsilon^{k-1}(x) - u_\varepsilon^k(x)),$$

for $(t,x)\in(t_{k-1},t_k]\times\Omega$, for $k=1,\ldots,K$, with $t_k = k\tau$ and $\tau = T/K$. Replacing these functions in (1.41), we obtain the identity

$$\int_0^T \partial_t \tilde{u}_\varepsilon^{(\tau)}\varphi + \int_{Q_T} a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)}\cdot\nabla\varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon^{(\tau)})\varphi, \tag{1.57}$$

for all $\varphi\in V$, where V is to be chosen such that $V\subset L^2(0,T;H^1(\Omega))$. For passing to the limits $\tau\to 0$ and $\varepsilon\to 0$ in the identity (1.57) we need:

1. For the time derivative: weak convergence of $\partial_t\tilde{u}_\varepsilon^{(\tau)}$ in some *large* space (of distributions).
2. For the diffusive term: strong convergence of $a_\varepsilon(u_\varepsilon^{(\tau)})$, and weak convergence of $\nabla u_\varepsilon^{(\tau)}$. Since the latter will be in $L^2(Q_T)$, we need to investigate the larger space in which $a_\varepsilon(u_\varepsilon^{(\tau)})$ converges strongly to fix the space of test functions.
3. For the reaction term f_ε: strong convergence of $u_\varepsilon^{(\tau)}$ in some $L^p(Q_T)$.
4. We also need to check that the limits of $u_\varepsilon^{(\tau)}$ and $\tilde{u}_\varepsilon^{(\tau)}$ are the same function.

Uniform estimates in ε and τ

The estimates deduced in the sequences of time-independent problems give us, directly, the following uniform estimates for $u_\varepsilon^{(\tau)}$. From (1.56) we get

$$\max_{t\in(0,T)}\Big(\int_\Omega F_\varepsilon(u_\varepsilon^{(\tau)}(t)) + \int_\Omega |u_\varepsilon^{(\tau)}(t)| + \frac{1}{\varepsilon}\int_\Omega ([u_\varepsilon^{(\tau)}(t)]_-)^2\Big) + \int_{Q_T} |\nabla u_\varepsilon^{(\tau)}|^2 \leq C. \tag{1.58}$$

Theorem 1.6 (Poincaré-Wirtinger's inequality) *Let Ω be a connected open set of class C^1 and let $1 \leq p \leq \infty$. Then, for all $u \in W^{1,p}(\Omega)$, there exists a constant C such that*

$$\|u - u_\Omega\|_{L^p} \leq C\|\nabla u\|_{L^p}, \quad \text{where } u_\Omega = \frac{1}{|\Omega|}\int_\Omega u.$$

From (1.58) and the Poincaré-Wirtinger's inequality we easliy get (Exercise 4)

$$\|u_\varepsilon^{(\tau)}\|_{L^2}^2 \leq \frac{1}{|\Omega|}\|u_\varepsilon^{(\tau)}\|_{L^1}^2 + C\|\nabla u_\varepsilon^{(\tau)}\|_{L^2}^2 \leq C, \tag{1.59}$$

and thus

$$\|u_\varepsilon^{(\tau)}\|_{L^2(H^1)} \leq C. \tag{1.60}$$

We also have, for $\sigma_\tau u_\varepsilon^{(\tau)}(t) = u_\varepsilon^{k-1}$ if $t \in (t_{k-1}, t_k]$,

$$\|\tilde{u}_\varepsilon^{(\tau)}\|_{L^2} \leq 2\|u_\varepsilon^{(\tau)}\|_{L^2} + \|\sigma_\tau u_\varepsilon^{(\tau)}\|_{L^2} \leq C, \tag{1.61}$$

$$\|\tilde{u}_\varepsilon^{(\tau)}\|_{L^2(H^1)} \leq 2\|u_\varepsilon^{(\tau)}\|_{L^2(H^1)} + \|\sigma_\tau u_\varepsilon^{(\tau)}\|_{L^2(H^1)} \leq C. \tag{1.62}$$

Time derivative estimate

Like in the linear case, we obtain this estimate using the definition of norm, see Definition 1. We have, using (1.57),

$$\begin{aligned}\int_0^T \langle \partial_t \tilde{u}_\varepsilon^{(\tau)}, \varphi\rangle &\leq \int_{Q_T} |a_\varepsilon(u_\varepsilon^{(\tau)})||\nabla u_\varepsilon^{(\tau)}||\nabla\varphi| + \int_{Q_T} |f_\varepsilon(u_\varepsilon^{(\tau)})||\varphi| \\ &\leq \|a_\varepsilon(u_\varepsilon^{(\tau)})\|_{L^\infty}\|\nabla u_\varepsilon^{(\tau)}\|_{L^2}\|\nabla\varphi\|_{L^2} + \alpha\|u_\varepsilon^{(\tau)}\|_{L^2}\|\varphi\|_{L^2} \\ &\qquad + \beta\|a_\varepsilon(u_\varepsilon^{(\tau)})\|_{L^\infty}^2\|\varphi\|_{L^1},\end{aligned} \tag{1.63}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product in $(H^1(\Omega))' \times H^1(\Omega)$. Then, we take $L^2(0,T;H^1(\Omega))$ as the (provisional) space of test funtions. Therefore, noting the uniform estimates (1.58), (1.59), and that $\varepsilon \leq a_\varepsilon(s) \leq \varepsilon^{-1}$, we find

$$\int_0^T \langle \partial_t \tilde{u}_\varepsilon^{(\tau)}, \varphi\rangle \leq C\varepsilon^{-1}\|\nabla\varphi\|_{L^2} + \big(C + \varepsilon^{-2}\big)\|\varphi\|_{L^2} \leq C\varepsilon^{-2}\|\varphi\|_{L^2(H^1)},$$

and thus,

$$\|\partial_t \tilde{u}_\varepsilon^{(\tau)}\|_{L^2((H^1)')} \leq C\varepsilon^{-2}. \tag{1.64}$$

2.4 The limit $\tau \to 0$

From the bounds (1.58), (1.60), (1.61), (1.62) and (1.64) we deduce the existence of $u_\varepsilon, z_\varepsilon \in L^2(0,T;H^1(\Omega))$ and of subsequences of $u_\varepsilon^{(\tau)}$ and $\tilde{u}_\varepsilon^{(\tau)}$ (not relabeled) such that, as $\tau \to 0$,

$$u_\varepsilon^{(\tau)} \rightharpoonup u_\varepsilon \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{1.65}$$
$$u_\varepsilon^{(\tau)} \rightharpoonup u_\varepsilon \qquad \text{weakly in } L^2(Q_T),$$
$$\tilde{u}_\varepsilon^{(\tau)} \rightharpoonup z_\varepsilon \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{1.66}$$
$$\tilde{u}_\varepsilon^{(\tau)} \rightharpoonup z_\varepsilon \qquad \text{weakly in } L^2(Q_T),$$
$$\partial_t \tilde{u}_\varepsilon^{(\tau)} \rightharpoonup \partial_t z_\varepsilon \qquad \text{weakly in } L^2(0,T;(H^1(\Omega))'). \tag{1.67}$$

The identification $\mathbf{z} = \mathbf{u}$

Since, for $t \in (t_{k-1}, t_k]$,

$$|\tilde{u}_\varepsilon^{(\tau)}(t,x) - u_\varepsilon^{(\tau)}(t,x)| = |(t_{k+1} - t)\frac{u_\varepsilon^{k-1}(x) - u_\varepsilon^k(x)}{\tau}| \leq \tau |\partial_t \tilde{u}_\varepsilon^{(\tau)}(t,x)|,$$

we deduce from (1.67)

$$\|\tilde{u}_\varepsilon^{(\tau)} - u_\varepsilon^{(\tau)}\|_{L^2((H^1)')} \leq \tau \|\partial_t \tilde{u}_\varepsilon^{(\tau)}\|_{L^2((H^1)')} \to 0 \quad \text{as } \tau \to 0, \tag{1.68}$$

and hence $z_\varepsilon = u_\varepsilon$.

Compactness and strong convergences

Once we obtained a time derivative uniform estimate, we use the compactness Aubin-Lions lemma, Lemma 3, to get strong convergence. We get the existence of a subsequence (not relabeled) such that

$$\tilde{u}_\varepsilon^{(\tau)} \to u_\varepsilon \quad \text{strongly in } L^2(Q_T), \quad \text{and a.e. in } Q_T. \tag{1.69}$$

Lemma 4 *Let $(H, \|\cdot\|_H)$ be a Hilbert space and let $V \subset H$ be a proper linear subspace dense in H. Assume that $(V, \|\cdot\|_V)$ is a Banach space and, under the identification $H = H'$, consider the triplet $V \subset H \subset V'$. Then*

$$\langle f, v \rangle_{V' \times V} = (f, v)_H, \quad \textit{for all } f \in H, v \in V.$$

In particular, for all $v \in V$,

$$\|v\|_H^2 = \langle v, v \rangle_{V' \times V} \leq \|v\|_{V'} \|v\|_V.$$

Setting $V = L^2(0,T;H^1(\Omega))$, $H = L^2(Q_T)$, and noticing that $u_\varepsilon^{(\tau)} - \tilde{u}_\varepsilon^{(\tau)} \in L^2(0,T;H^1(\Omega))$, we also deduce strong convergence for $u_\varepsilon^{(\tau)}$ using Lemma 4. Indeed,

$$\begin{aligned}\|u_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2} &\leq \|u_\varepsilon^{(\tau)} - \tilde{u}_\varepsilon^{(\tau)}\|_{L^2} + \|\tilde{u}_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2} \\ &\leq \|u_\varepsilon^{(\tau)} - \tilde{u}_\varepsilon^{(\tau)}\|_{L^2((H^1)')}^{1/2} \|u_\varepsilon^{(\tau)} - \tilde{u}_\varepsilon^{(\tau)}\|_{L^2(H^1)}^{1/2} + \|\tilde{u}_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2},\end{aligned} \tag{1.70}$$

and since $\|u_\varepsilon^{(\tau)}-\tilde{u}_\varepsilon^{(\tau)}\|_{L^2(H^1)} \le \|u_\varepsilon^{(\tau)}\|_{L^2(H^1)} + \|\tilde{u}_\varepsilon^{(\tau)}\|_{L^2(H^1)} \le C$, we get from (1.70),

$$\|u_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2} \le C\|u_\varepsilon^{(\tau)} - \tilde{u}_\varepsilon^{(\tau)}\|^{1/2}_{L^2((H^1)')} + \|\tilde{u}_\varepsilon^{(\tau)} - u\|_{L^2} \to 0 \quad \text{as } \tau \to 0,$$

in view of (1.68) and (1.69). Thus,

$$u_\varepsilon^{(\tau)} \to u_\varepsilon \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{1.71}$$

Convergence

We have to pass to the limit $\tau \to 0$ in the expression

$$\int_0^T \langle \partial_t \tilde{u}_\varepsilon^{(\tau)}, \varphi\rangle + \int_{Q_T} a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)} \cdot \nabla\varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon^{(\tau)})\varphi, \quad \text{for all } \varphi \in L^2(0,T;H^1(\Omega)). \tag{1.72}$$

The time derivative term, recalling $z_\varepsilon = u_\varepsilon$, passes to the limit without any additional reasoning. The linear part of the reaction term, also passes to the limit, thanks to, e.g., the strong convergence (1.71).

For the convergence of the sequence $a_\varepsilon(u_\varepsilon^{(\tau)})$ we use the dominated convergence theorem. By (1.71) and the continuity of a_ε we deduce that $a_\varepsilon(u_\varepsilon^{(\tau)}) \to a_\varepsilon(u_\varepsilon)$ a.e. in Q_T as $\tau \to 0$. Observing that $\|a_\varepsilon(u_\varepsilon^{(\tau)})\|_{L^\infty} \le \varepsilon^{-1}$, we deduce $a_\varepsilon(u_\varepsilon) \in L^p(Q_T)$ and, as $\tau \to 0$,

$$a_\varepsilon(u_\varepsilon^{(\tau)}) \to a_\varepsilon(u_\varepsilon) \quad \text{strongly in } L^p(Q_T) \text{ for any } 1 \le p < \infty. \tag{1.73}$$

Then, by Hölder's inequality,

$$\begin{aligned}\int_{Q_T} \left|a_\varepsilon(u_\varepsilon^{(\tau)})^2 - a_\varepsilon(u_\varepsilon)^2\right|^2 &= \int_{Q_T} \left|a_\varepsilon(u_\varepsilon^{(\tau)}) - a_\varepsilon(u_\varepsilon)\right|^2 \left|a_\varepsilon(u_\varepsilon^{(\tau)}) + a_\varepsilon(u_\varepsilon)\right|^2 \\ &\le \|a_\varepsilon(u_\varepsilon^{(\tau)}) - a_\varepsilon(u_\varepsilon)\|^2_{L^4} \|a_\varepsilon(u_\varepsilon^{(\tau)}) + a_\varepsilon(u_\varepsilon)\|^2_{L^4},\end{aligned}$$

and therefore, (1.73) leads to

$$a_\varepsilon(u_\varepsilon^{(\tau)})^2 \to a_\varepsilon(u_\varepsilon)^2 \quad \text{strongly in } L^2(Q_T).$$

For the diffusion term, we have that (1.73) and $\nabla u_\varepsilon^{(\tau)} \rightharpoonup \nabla u_\varepsilon$ weakly in $L^2(Q_T)$, imply

$$a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)} \rightharpoonup u_\varepsilon \nabla u_\varepsilon \quad \text{weakly in } L^q(Q_T) \text{ for any } q < 2.$$

However, we also have

$$\|a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)}\|_{L^2} \le \|a_\varepsilon(u_\varepsilon^{(\tau)})\|_{L^\infty} \|\nabla u_\varepsilon^{(\tau)}\|_{L^2} \le C\varepsilon^{-1},$$

implying

$$a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)} \rightharpoonup u_\varepsilon \nabla u_\varepsilon \quad \text{weakly in } L^2(Q_T).$$

Therefore, we may pass to the limit in (1.72) to obtain that $u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$ satisfies

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi\rangle + \int_{Q_T} a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon)\varphi, \quad \text{for all } \varphi \in L^2(0,T;H^1(\Omega)). \tag{1.74}$$

2.5 The limit $\varepsilon \to 0$

Since the uniform boundedness of a_ε is lost in the limit $\varepsilon \to 0$, we can not expect

$$\lim_{\varepsilon\to 0}\int_{Q_T} a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)}\cdot\nabla\varphi$$

to be well defined for test functions $\varphi\in L^2(0,T;H^1(\Omega))$. Thus, first we have to investigate in which L^p space may $a_\varepsilon(u_\varepsilon^{(\tau)})$ converge strongly, and then seek for a correct space of test functions in which this limit may be performed.

In addition, observe that the time derivative bounds we obtained are dependent of the regularity of the other terms (through the argument for the duality $\langle\partial_t\tilde{u},\varphi\rangle$, see (1.63)). Thus, if the other terms are less regular, the time derivative will be less regular too, and we shall therefore need to impose more regularity of φ in both the space and the time variables.

Uniform estimates in ε and weak convergences

Taking the limit $\tau\to 0$ in (1.58), (1.60) we get

$$\max_{t\in(0,T)}\Big(\int_\Omega F_\varepsilon(u_\varepsilon(t))+\int_\Omega|u_\varepsilon(t)|+\frac{1}{\varepsilon}\int_\Omega([u_\varepsilon(t)]_-)^2\Big)+\int_{Q_T}|\nabla u_\varepsilon|^2\le C. \tag{1.75}$$

and then

$$\|u_\varepsilon\|_{L^2(H^1)}\le C. \tag{1.76}$$

Theorem 1.7 (Gagliardo-Niremberg's interpolation inequality) *Let $\Omega\subset\mathbb{R}^N$ be a regular open bounded set, and let $u\in L^q(\Omega)\cap W^{m,r}(\Omega)$, with $1\le p,q\le\infty$, and $m\in\mathbb{N}$. Then $u\in W^{j,p}(\Omega)$, and*

$$\|D^j u\|_{L^p}\le C\|D^m u\|_{L^r}^\theta\|u\|_{L^q}^{1-\theta},$$

where

$$\frac{1}{p}=\frac{j}{N}+\Big(\frac{1}{r}-\frac{m}{N}\Big)\theta+\frac{1-\theta}{q},\quad \textit{and}\quad \frac{j}{m}\le\theta\le 1.$$

Using Gagliardo-Nirenberg inequality with $p=(2N+2)/N$, $\theta=2N(p-1)/(p(N+2))$, and thus $\theta p=2$, yields (Exercise 5)

$$\|u_\varepsilon\|_{L^p}\le\Big(\int_0^T\|u_\varepsilon\|_{L^1(\Omega)}^{(1-\theta)p}\|u_\varepsilon\|_{H^1(\Omega)}^{\theta p}\Big)^{1/p}\le\|u_\varepsilon\|_{L^\infty(L^1)}^{1-\theta}\|u_\varepsilon\|_{L^2(H^1)}^{\theta}\le C. \tag{1.77}$$

For the time derivative estimate, let $r'=r/(r-1)$ to be determined, and write, using (1.74) and $p>2$,

$$\begin{aligned}\int_0^T\langle\partial_t u_\varepsilon,\varphi\rangle &\le\int_{Q_T}|a_\varepsilon(u_\varepsilon)||\nabla u_\varepsilon||\nabla\varphi|+\int_{Q_T}|f_\varepsilon(u_\varepsilon)||\varphi|\\ &\le\|a_\varepsilon(u_\varepsilon)\|_{L^p}\|\nabla u_\varepsilon\|_{L^2}\|\nabla\varphi\|_{L^{r'}}+\alpha\|u_\varepsilon\|_{L^p}\|\varphi\|_{L^{p'}}\\ &\qquad+\beta\|a_\varepsilon(u_\varepsilon)\|_{L^p}^2\|\varphi\|_{L^{(p/2)'}},\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product in $(W^{1,r'}(\Omega))'\times W^{1,r'}(\Omega)$. Here, r' is such that

$$1=\frac{1}{p}+\frac{1}{2}+\frac{1}{r'}\implies r'=2(N+1).$$

Then, we take $L^{r'}(0,T;W^{1,r'}(\Omega))$ as the new (smaller, more regular) space of test funtions. In addition, notice that $r'\geq\max\{p',(p/2)'\}$, and thus the norms of the reaction term are also well defined (Exercise 7). Therefore, noting that $a_\varepsilon(s)\leq\varepsilon+s$, we find

$$\int_0^T\langle\partial_t u_\varepsilon,\varphi\rangle\leq(C+\|u_\varepsilon\|_{L^p})\|\nabla u_\varepsilon\|_{L^2}\|\nabla\varphi\|_{L^{r'}}+\big(C+\|u_\varepsilon\|_{L^p}+\|u_\varepsilon\|^2_{L^p}\big)\|\varphi\|_{L^{r'}}$$
$$\leq C\|\varphi\|_{L^{r'}(W^{1,r'})},$$

and thus, for $r=(2N+2)/(2N+1)$,

$$\|\partial_t u_\varepsilon\|_{L^r((W^{1,r'})')}\leq C. \tag{1.78}$$

Finally, from (1.75) we also deduce

$$\|[u_\varepsilon]_-\|_{L^\infty(L^2)}\leq C\sqrt{\varepsilon}. \tag{1.79}$$

From the bounds (1.76), (1.77), (1.78), and (1.79) we deduce the existence of $u,z\in L^2(0,T;H^1(\Omega))$ and of subsequences of u_ε (not relabeled) such that

$$\begin{aligned}
&u_\varepsilon\rightharpoonup u && \text{weakly in } L^2(0,T;H^1(\Omega)),\\
&u_\varepsilon\rightharpoonup u && \text{weakly in } L^p(Q_T),\\
&\partial_t u_\varepsilon\rightharpoonup\partial_t u && \text{weakly in } L^r(0,T;(W^{1,r'}(\Omega))'),\\
&[u_\varepsilon]_-\rightharpoonup 0 && \text{weakly*-weakly in } L^\infty(0,T;L^2(\Omega))
\end{aligned} \tag{1.80}$$

Compactness and strong convergences

We again use the compactness Aubin-Lions lemma, Lemma 3, to get the existence of a subsequence (not relabeled) such that

$$u_\varepsilon\to u\quad\text{strongly in } L^\gamma(0,T;L^2(\Omega)),\ \text{for any }\gamma<2,\quad\text{and a.e. in } Q_T.$$

Lemma 5 *Let $\Omega\subset\mathbb{R}^N$ be an open set, and let f_n be a sequence in $L^p(\Omega)\cap L^\gamma(\Omega)$, with $p>\gamma$, and $f\in L^\gamma(\Omega)$. Assume that*

$$f_n\to f\quad\textit{strongly in } L^\gamma(\Omega)\textit{ and }\|f_n\|_{L^p}\leq C.$$

Then $f\in L^q(\Omega)$ and $f_n\to f$ strongly in $L^q(\Omega)$ for all $\gamma\leq q<p$.

Thus, using the bound $\|u_\varepsilon\|_{L^p}\leq C$, see (1.77), we get

$$u_\varepsilon\to u\quad\text{strongly in } L^q(Q_T),\ \text{for any}^3\gamma\leq q<p. \tag{1.81}$$

Observe that, in particular, we may choose $2 \le q < p = (2N+2)/N$. This convergence together with (1.79) further implies, using $\|[u_\varepsilon]_-\|_{L^p} \le \|u_\varepsilon\|_{L^p}$,

$$[u_\varepsilon]_- \to 0 \quad \text{strongly in } L^q(Q_T) \text{ and a.e. in } Q_T, \text{ that is } u \ge 0 \text{ a.e. in } Q_T. \tag{1.82}$$

Finally, for the convergence of the sequence $a_\varepsilon(u_\varepsilon)$, let us write

$$\|u - a_\varepsilon(u_\varepsilon)\|_{L^q} \le \|u - \tilde{a}_\varepsilon(u)\|_{L^q} + \|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)\|_{L^q} + \|\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)\|_{L^q},$$

where

$$\tilde{a}_\varepsilon(s) := \begin{cases} s & \text{if } s \le \varepsilon^{-1}, \\ \varepsilon^{-1} & \text{if } s \ge \varepsilon^{-1}. \end{cases}$$

Theorem 1.8 (Monotone convergence theorem) *Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $f_n \in L^1(\Omega)$ be a sequence of functions satisfying*

1. *$f_1 \le f_2 \le \cdots$ a.e. in Ω,*
2. *$\sup_n \int_\Omega f_n < \infty$.*

Then there exists $f \in L^1(\Omega)$ such that $f_n \to f$ strongly in $L^1(\Omega)$ and a.e. in Ω.

Since $\tilde{a}_\varepsilon(s)$ is monotone increasing, $\tilde{a}_\varepsilon(s) \le s$ for all $s \in \mathbb{R}$, and $u \in L^1(Q_T)$ by (1.81), we have, first, that $\tilde{a}_\varepsilon(u) \to a(u)$ strongly in $L^1(Q_T)$ (by the monotone convergence theorem), and then, using the uniform bound (1.77), we deduce $\|u - \tilde{a}_\varepsilon(u)\|_{L^q} \to 0$ as $\varepsilon \to 0$.

Since $\tilde{a}_\varepsilon$ is Lipschitz continuous (with Lipschitz constant equal to one), we get

$$|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)| \le |u - u_\varepsilon|$$

and then (1.81) implies $\|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)\|_{L^q} \to 0$ as $\varepsilon \to 0$.

Finally,

$$|\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)| = |u_\varepsilon - \varepsilon| 1_{u_\varepsilon \le \varepsilon} = (\varepsilon - u_\varepsilon) 1_{0 \le u_\varepsilon \le \varepsilon} + (|u_\varepsilon| + \varepsilon) 1_{u_\varepsilon < 0}.$$

The first term of the right hand side is bounded by $\varepsilon |Q_T|$, while the second is equal to $[u_\varepsilon]_- + \varepsilon 1_{u_\varepsilon < 0}$. Thus

$$\int_{Q_T} |\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)|^q \le C\Big(\varepsilon^q + \int_{Q_T} |[u_\varepsilon]_-|^q\Big) \to 0$$

as $\varepsilon \to 0$, in view of (1.82). Therefore

$$a_\varepsilon(u_\varepsilon) \to u \quad \text{strongly in } L^q(Q_T). \tag{1.83}$$

[3] In fact, we may take $1 \le q < p$, since Q_T is bounded.

Convergence

We have to pass to the limit in the expression

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle + \int_{Q_T} a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon)\varphi, \quad \text{for all } \varphi \in L^{r'}(0,T;W^{1,r'}(\Omega)).$$

The time derivative term passes to the limit without any additional reasoning, due to (1.80). For the reaction term, we directly have

$$\int_{Q_T} u_\varepsilon \varphi \to \int_{Q_T} u\varphi,$$

since $u_\varepsilon \to u$ strongly in, e.g., $L^2(Q_T)$, by (1.81). Using (1.83), we deduce that $a_\varepsilon(u_\varepsilon)^2 \to u^2$ strongly in $L^{q/2}(Q_T)$. Indeed, Hölder's inequality implies

$$\begin{aligned}\int_{Q_T} \left|a_\varepsilon(u_\varepsilon)^2 - u^2\right|^{q/2} &\le \Big(\int_{Q_T} \left|a_\varepsilon(u_\varepsilon) - u\right|^q\Big)^{1/2}\Big(\int_{Q_T} \left|a_\varepsilon(u_\varepsilon) + u\right|^q\Big)^{1/2} \\ &\le \|a_\varepsilon(u_\varepsilon) - u\|_{L^q}^{q/2}\|a_\varepsilon(u_\varepsilon) + u\|_{L^q}^{q/2} \to 0,\end{aligned}$$

as $\varepsilon \to 0$, in view of (1.83). Thus,

$$\int_{Q_T} a_\varepsilon(u_\varepsilon)^2\varphi \to \int_{Q_T} u^2\varphi,$$

since

$$\frac{2}{q} + \frac{1}{r'} \le 1 \quad \text{if we choose} \quad q \ge \frac{4(N+1)}{2N+1},$$

which is possible due to (1.81). For the diffusion term, we have that since $a_\varepsilon(u_\varepsilon) \to u$ strongly in $L^q(Q_T)$ and $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(Q_T)$, the product

$$a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \rightharpoonup u\nabla u \quad \text{weakly in } L^\gamma(Q_T),$$

with $\gamma = 2q/(2+q)$, which is smaller than r. However, we also have

$$\|a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\|_{L^{r'}} \le \|a_\varepsilon(u_\varepsilon)\|_{L^p}\|\nabla u_\varepsilon\|_{L^2} \le C,$$

implying

$$a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \rightharpoonup u\nabla u \quad \text{weakly in } L^r(Q_T).$$

Finally, observe that due to the convergence of $[u_\varepsilon]_- \to 0$ in $L^q(Q_T)$, see (1.82), we deduce $u \ge 0$ a.e. in Q_T. In fact, we may use a similar argument to that employed at the end of Section 1 to get upper and lower bounds for u in terms of the initial data (Exercise 6). However, as already mentioned, this technique will not work for the cross-diffusion system.

Theorem 1.9 *Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz continuous boundary, and let $T > 0$. Suppose that $u_0 \in L^2(\Omega)$. Then, problem (1.23)-(1.25) has a weak solution u satisfying $u \geq 0$ in Q_T and*

$$u \in L^2(0,T;H^1(\Omega)) \cap L^p(Q_T) \cap W^{1,r}(0,T;(W^{1,r'}(\Omega))'),$$

where $p = 2(N+1)/N$, $r = 2(N+1)/(2N+1)$, and $r' = 2(N+1)$, in the sense that for all $\varphi \in L^{r'}(0,T;W^{1,r'}(\Omega))$,

$$\int_0^T < \partial_t u, \varphi > + \int_{Q_T} u \nabla u \cdot \nabla \varphi = \int_{Q_T} f(u)\varphi,$$

with $< \cdot, \cdot >$ denoting the duality product between $W^{1,r'}(\Omega)$ and its dual $(W^{1,r'}(\Omega))'$, being the initial data satisfied in the sense

$$\int_0^T < \partial_t u, \psi > + \int_{Q_T} (u - u_0)\partial_t \psi = 0,$$

for all $\psi \in L^{r'}(0,T;W^{1,r'}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $\psi(T) = 0$ a.e. in Ω.

3 A cross-diffusion population model

In this section we finally deal with cross-diffusion systems of equations. The problem is the following. Given a fixed $T>0$ and a bounded set $\Omega \subset \mathbb{R}^N$, find (non-negative) functions $u_1, u_2 : (0,T)\times\Omega \to \mathbb{R}$ such that, using the notation $\mathbf{u} = (u_1, u_2)$,

$$\partial_t u_1 - \operatorname{div} J_1(\mathbf{u}) = f_1(\mathbf{u}) \qquad \text{in } Q_T, \tag{1.84}$$

$$\partial_t u_2 - \operatorname{div} J_2(\mathbf{u}) = f_2(\mathbf{u}) \qquad \text{in } Q_T, \tag{1.85}$$

$$J_1(\mathbf{u})\cdot n = J_2(\mathbf{u})\cdot n = 0 \qquad \text{on } \Gamma_T, \tag{1.86}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{1.87}$$

Here, the reaction terms are of the competitive Lotka-Volterra type

$$f_i(\mathbf{u}) = u_i\big(\alpha_i - (\beta_{i1}u_1 + \beta_{i2}u_2)\big), \quad \text{for } i = 1,2, \tag{1.88}$$

with $\alpha_i, \beta_{ij} \geq 0$, for $i,j = 1,2$. We shall deal with diffusion terms given by the flows of the Bousenberg-Travis (BT) model

$$J_i(\mathbf{u}) = a_{i0}\nabla u_i + u_i(a_{i1}\nabla u_1 + a_{i2}\nabla u_2) - b_i u_i \nabla\Phi, \quad \text{for } i = 1,2,$$

with $a_{ij} \geq 0$, $b_i \geq 0$, for $i,j = 1,2$, being Φ the *environmental potential*. The numbers a_{ii} are called *self-diffusion* coefficients, while a_{12} and a_{21} are refered to as to the *cross-diffusion coefficients*. Let us remark here that the Shigesada-Kawasaki-Teramoto (SKT) model, for which

$$J_i^{SKT}(\mathbf{u}) = \nabla\big(u_i(a_{i0} + a_{i1}u_1 + a_{i2}u_2)\big) - b_i u_i \nabla\Phi, \quad \text{for } i = 1,2,$$

may be treated in a similarly way to what we shall follow for the BT model, see Subsection 3.8.

In terms of population dynamics, we are supposing that

- The populations diffuses partly randomly, and partly to avoid overcrowding caused by both populations.
- The populations are drifted to the minima of the environmental potential Φ, representing the best environmental locations.
- The newborns are proportional to the existent population, but there is a growth limit given in terms of the intra- and inter-specific competence between populations. The corresponding kinetics ($\partial_t u_i = f_i(\mathbf{u})$) has stable equilibria at

$$\big(\frac{\alpha_1}{\beta_{11}},0\big), \quad \big(0,\frac{\alpha_2}{\beta_{22}}\big), \quad \big(\frac{\alpha_1\beta_{22}-\alpha_2\beta_{12}}{\beta_{11}\beta_{22}-\beta_{12}\beta_{21}}, \frac{\alpha_2\beta_{11}-\alpha_1\beta_{21}}{\beta_{11}\beta_{22}-\beta_{12}\beta_{21}}\big),$$

  depending on the relationship between the coefficients. However, due to the cross-diffusion, these equilibria are not always the steady state solutions of problem (1.84)-(1.87).

Introducing the rescaling[4] $U_1 = a_{21}u_1$ and $U_2 = a_{12}u_2$, the new cross-diffusion coefficients are the unity, while the other coefficients remain with the same sign. Thus, from now on, we shall use the flows

$$J_i(\mathbf{u}) = a_{i0}\nabla u_i + u_i(a_i\nabla u_i + \nabla u_j) - b_i u_i \nabla\Phi, \quad \text{for } i,j = 1,2, \text{ with } j \neq i. \tag{1.89}$$

We shall follow the line of the proof of existence of weak solutions developed in Section 2 for an scalar equation, to prove the corresponding result for the problem (1.84)-(1.87) with reaction and convection-diffusion terms given by (1.88) and (1.89), respectively.

[4]Here, we assume $a_{12} \neq 0$ and $a_{21} \neq 0$. Otherwise, the system is triangular (instead of full), and the problem is simpler.

3.1 Formal estimates

For problem (1.84)-(1.87), we have the following formal estimates:

- Multiplying (1.84) by $\ln(u_1)$, (1.85) by $\ln(u_2)$, integrating and adding the resulting identities, we get, for $F(s) = s(\ln(s)-1)+1 \geq 0$,

$$\sum_{i=1}^{2}\int_{\Omega} F(u_i(T)) + \sum_{i=1}^{2}\int_{Q_T} \frac{a_{0i}}{u_i}|\nabla u_i|^2 + \int_{Q_T}\Big(a_1|\nabla u_1|^2 + a_2|\nabla u_2|^2 + 2\nabla u_1\cdot\nabla u_2\Big)$$
$$= \sum_{i=1}^{2}\int_{\Omega} F(u_{i0}) + \sum_{i=1}^{2}\int_{Q_T} f_i(\mathbf{u})\ln(u_i) + \sum_{i=1}^{2}\int_{Q_T}\nabla\Phi\cdot\nabla u_i.$$

  We have $\frac{1}{u_i}|\nabla u_i|^2 = 4|\nabla\sqrt{u_i}|^2$, and, if[5] $a_1a_2 > 1$,

$$a_1|\nabla u_1|^2 + a_2|\nabla u_2|^2 + 2\nabla u_1\cdot\nabla u_2 \geq a_0(|\nabla u_1|^2 + |\nabla u_2|^2),$$

  for some $a_0 > 0$. Thus, if the right hand side may be controled by the left hand side, we get

$$\sum_{i=1}^{2}\int_{\Omega} F(u_i(T)) + \int_{Q_T}(|\nabla u_1|^2 + |\nabla u_2|^2) \leq C.$$

- Integrating the equations (1.84) and (1.85), and using the boundary conditions, we get (if $u_i \geq 0$)

$$\int_{\Omega}(u_1(T)+u_2(T)) \leq \int_{\Omega}(u_{10}+u_{20}) + \tilde{\alpha}\int_{Q_T}(u_1+u_2),$$

  with $\tilde{\alpha} = \max\{\alpha_1, \alpha_2\}$, and then Gronwall's lemma implies

$$\int_{\Omega}(u_1(T)+u_2(T)) \leq e^{\tilde{\alpha}T}\int_{\Omega}(u_{10}+u_{20}) \leq C.$$

We then deduce from these two estimates that $\|u_i\|_{L^2(H^1)} \leq C$, like in the scalar case.

3.2 Symmetrization

Since the treatment of the linear diffusion and convection terms is straightforward, we shall assume in what follows $a_{i0} = 0$ and $\Phi = 0$. See Subsection 3.8 for the details of how to handle these terms.

Equations (1.84)-(1.85) may be written as

$$\partial_t\mathbf{u} - \operatorname{div}(a(\mathbf{u})\nabla\mathbf{u}) = \mathbf{f}(\mathbf{u}), \tag{1.90}$$

with $\mathbf{f} = (f_1, f_2)$, and $a(\mathbf{u})$ is the non-symmetric matrix given by

$$a(\mathbf{u}) = \begin{pmatrix} a_1u_1 & u_1 \\ u_2 & a_2u_2 \end{pmatrix}.$$

[5]Before rescaling, the condition is $\det(A) > 0$.

In (1.90) and in what follows, we use the notation

$$\operatorname{div}(a(\mathbf{u})\nabla\mathbf{u}) = \begin{pmatrix} \operatorname{div}(a_1u_1\nabla u_1 + u_1\nabla u_2) \\ \operatorname{div}(u_2\nabla u_1 + a_2u_2\nabla u_2) \end{pmatrix}.$$

Following the line of the previous sections, we first discretize the problem in time,

$$\frac{1}{\tau}(\mathbf{u}^k - \mathbf{u}^{k-1}) - \operatorname{div}(a(\mathbf{u}^k)\nabla\mathbf{u}^k) = \mathbf{f}(\mathbf{u}^k),$$

and then (approximate) and linearize to use Lax-Milgram's lemma

$$\frac{1}{\tau}(\mathbf{u}^k - \mathbf{u}^{k-1}) - \operatorname{div}(a(\mathbf{v})\nabla\mathbf{u}^k) = \mathbf{f}(\mathbf{v}).$$

Since a is non-symmetric, the corresponding bilinear form

$$A(\mathbf{u},\mathbf{u}) = \int_\Omega \big(a_1v_1|\nabla u_1|^2 + a_2v_2|\nabla u_2|^2 + (v_1+v_2)\nabla u_1\cdot\nabla u_2\big),$$

is not, in general, coercive since the condition for this form to be coercive is that the matrix

$$\begin{pmatrix} a_1v_1 & \frac{1}{2}(v_1+v_2) \\ \frac{1}{2}(v_1+v_2) & a_2v_2 \end{pmatrix}$$

is positive definite, that is, $4a_1a_2v_1v_2 > (v_1+v_2)^2$, which is not true in general.

The problem with this approach is that the entropy estimate of the nonlinear problem is not inherited by the linear approximation, as it happened for the scalar problem. This is a common issue when approximating nonlinear systems.

Fortunately, the existence of an entropy estimate is usually accompanied by a change of unknowns which symmetrizes the problem. In our case, defining $w_i = F'(u_i) = \ln(u_i)$, we get that $\mathbf{w}$ satisifies

$$\partial_t \begin{pmatrix} e^{w_1} \\ e^{w_2} \end{pmatrix} - \operatorname{div}(b(\mathbf{w})\nabla\mathbf{w}) = \mathbf{f}(e^{w_1}, e^{w_2}), \tag{1.91}$$

being $b(\mathbf{w})$ the symmetric matrix

$$b(\mathbf{w}) = \begin{pmatrix} a_1e^{2w_1} & e^{w_1+w_2} \\ e^{w_1+w_2} & a_2e^{2w_2} \end{pmatrix}.$$

Thus, our strategy will be to solve the problem in terms of the unknown $\mathbf{w}$ and the equation (1.91), and then to justify the equivalence with a solution of (1.90). Observe that this is not straightforward. For instance, since

$$\nabla u_i = \nabla e^{w_i} = e^{w_i}\nabla w_i,$$

if we obtain, as expected from equation (1.91), $\nabla w_i \in L^2$, this regularity does not inmmediately translates to ∇u_i, unless $w_i \in L^\infty$, which is not expected, in general, from a system like (1.91).

3.3 Solving a time discrete approximated symmetric problem

The formal calculations of the previous section may be also done in terms of the approximation to the logarithm given by F'_ε. Since F'_ε is increasing in $\mathbb{R}$, its inverse is well defined. We introduce the notation

$$g_\varepsilon = (F'_\varepsilon)^{-1}, \quad \text{satisfying} \quad g'_\varepsilon = a_\varepsilon \circ g_\varepsilon. \tag{1.92}$$

Then, for $\sigma \in [0,1]$, we set the problem: Given $\mathbf{w}_\varepsilon^{k-1} \in L^2(\Omega)^2$, with $F_\varepsilon(g_\varepsilon(w_{i,\varepsilon}^{k-1})) \in L^1(\Omega)$, find $\mathbf{w}_\varepsilon^k : \Omega \to \mathbb{R}^2$ such that

$$\frac{\sigma}{\tau}(g_\varepsilon(w_{i,\varepsilon}^k) - g_\varepsilon(w_{i,\varepsilon}^{k-1})) - \operatorname{div} G_i^\varepsilon(\mathbf{w}_\varepsilon^k) + \varepsilon w_{i,\varepsilon}^k = \sigma h_i^\varepsilon(\mathbf{w}_\varepsilon^k) \quad \text{in } \Omega, \tag{1.93}$$

$$G_i^\varepsilon(\mathbf{w}_\varepsilon^k) \cdot n = 0 \quad \text{on } \partial\Omega, \tag{1.94}$$

with, for $i, j = 1,2$ and $i \neq j$,

$$G_i^\varepsilon(\mathbf{w}) = g'_\varepsilon(w_i)(a_i g'_\varepsilon(w_i)\nabla w_i + g'_\varepsilon(w_j)\nabla w_j),$$
$$h_i^\varepsilon(\mathbf{w}) = \alpha_i g_\varepsilon(w_i) - g'_\varepsilon(w_i)\big(\beta_{i1} g'_\varepsilon(w_1) + \beta_{i2} g'_\varepsilon(w_2)\big).$$

Lax-Milgram. Let us consider the operators $A : H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{R}$ and $F : L^2(\Omega)^2 \times L^2(\Omega)^2 \to \mathbb{R}$ defined by, for $\mathbf{v} \in L^2(\Omega)^2$ and $\sigma \in [0,1]$,

$$A(\mathbf{w}, \varphi) = \sum_{i=1}^2 \Big(\int_\Omega \varepsilon w_i \varphi_i + \sum_{\substack{i,j=1 \\ j\neq i}}^2 \int_\Omega g'_\varepsilon(v_i)\big(a_i g'_\varepsilon(v_i)\nabla w_i + g'_\varepsilon(v_j)\nabla w_j\big) \cdot \nabla \varphi_i \Big), \tag{1.95}$$

$$F(\varphi) = \sigma \sum_{i=1}^2 \Big(\int_\Omega \big(\alpha_i g_\varepsilon(v_i) - g'_\varepsilon(v_i)\big(\beta_{i1} g'_\varepsilon(v_1) + \beta_{i2} g'_\varepsilon(v_2)\big)\big)\varphi_i - \frac{1}{\tau}\int_\Omega (g_\varepsilon(v_i) - g_\varepsilon(w_{i,\varepsilon}^{k-1}))\varphi_i \Big). \tag{1.96}$$

with $\varphi = (\varphi_1, \varphi_2) \in H^1(\Omega)^2$. We have, using $a_1 a_2 > 1$,

$$A(\mathbf{w}, \mathbf{w}) \geq \sum_{i=1}^2 \Big(\varepsilon \int_\Omega w_i^2 + a_0 c(\varepsilon) \int_\Omega |\nabla w_i|^2 \Big),$$

with $c(\varepsilon) = \min_{s\in\mathbb{R}} (g'_\varepsilon(s))^2 > \varepsilon^2$, in view of (1.92). Thus, A is coercive. Both A and F are clearly continuous. Then, Lax-Milgram's lemma ensures the existence of a unique weak solution, $\mathbf{w}_{\varepsilon,\sigma}^k \in H^1(\Omega)^2$, of

$$\frac{\sigma}{\tau}(g_\varepsilon(v_i) - g_\varepsilon(w_{i,\varepsilon}^{k-1})) - \operatorname{div} G_i^\varepsilon(\mathbf{w}_{\varepsilon,\sigma}^k, \mathbf{v}) + \varepsilon w_{i,\varepsilon,\sigma}^k = \sigma h_i^\varepsilon(\mathbf{w}_{\varepsilon,\sigma}^k, \mathbf{v}) \quad \text{in } \Omega,$$
$$G_i^\varepsilon(\mathbf{w}_{\varepsilon,\sigma}^k, \mathbf{v}) \cdot n = 0 \quad \text{on } \partial\Omega,$$

with (abusing on the notation by splitting the arguments)

$$G_i^\varepsilon(\mathbf{w}, \mathbf{v}) = g'_\varepsilon(v_i)\big(a_i g'_\varepsilon(v_i)\nabla w_i + g'_\varepsilon(v_j)\nabla w_j\big),$$
$$h_i^\varepsilon(\mathbf{w}, \mathbf{v}) = \alpha_i g_\varepsilon(v_i) - g'_\varepsilon(v_i)\big(\beta_{i1} g'_\varepsilon(v_1) + \beta_{i2} g'_\varepsilon(v_2)\big).$$

Fixed point. Define the map $S : L^2(\Omega)^2 \times [0,1] \to L^2(\Omega)^2$ given by $S(\mathbf{v}, \sigma) = \mathbf{w}_{\varepsilon,\sigma}^k$. To apply the Leray-Schauder's theorem, we have to check the following:

1. Continuity and compactness of S. The arguments are similar to the case of a scalar equation, see Subsection 2.2.
2. $S(\mathbf{v},0)=0$, which is inmmediate.
3. If $\mathbf{v}=S(\mathbf{v},\sigma)$ for $(\mathbf{v},\sigma)\in L^2(\Omega)^2\times[0,1]$ then $\|\mathbf{v}\|_{L^2}\leq C$.

Let us prove the last point. Thus, we assume that $\mathbf{v}=\mathbf{w}^k_{\varepsilon,\sigma}$, and we have to show an uniform bound, with respect to $\sigma\in[0,1]$, of $\|\mathbf{w}^k_{\varepsilon,\sigma}\|_{L^2}$. For clarity in the notation, we replace $\mathbf{w}^k_{\varepsilon,\sigma}$ by $\mathbf{w}$, and $\mathbf{w}^{k-1}_{\varepsilon}$ by $\tilde{\mathbf{w}}$ in what follows. We have that, by assumption, $\mathbf{w}$ solves

$$\begin{aligned}
&\frac{\sigma}{\tau}(g_\varepsilon(w_i)-g_\varepsilon(\tilde{w}_i))-\operatorname{div}G_i^\varepsilon(\mathbf{w})+\varepsilon w_i=\sigma h_i^\varepsilon(\mathbf{w}) && \text{in } \Omega,\\
&G_i^\varepsilon(\mathbf{w})\cdot n=0 && \text{on } \partial\Omega.
\end{aligned}$$

Using $\varphi=w_i$ as a test function, for $i=1,2$, and summing the resulting identities, we get, similarly to the deduction of the coercivity of A,

$$\sum_{i=1}^{2}\Big(\sigma\int_\Omega(g_\varepsilon(w_i)-g_\varepsilon(\tilde{w}_i))w_i+\tau\varepsilon\int_\Omega w_i^2+\tau a_0c(\varepsilon)\int_\Omega|\nabla w_i|^2\Big)\leq\tau\sigma\sum_{i=1}^{2}\int_\Omega h_i^\varepsilon(\mathbf{w})w_i. \tag{1.97}$$

The convexity of F_ε implies $F_\varepsilon(x)-F_\varepsilon(y)\leq F'_\varepsilon(x)(x-y)$. Choosing $x=g_\varepsilon(w_i)$ and $y=g_\varepsilon(\tilde{w}_i)$, and noticing that g_ε is the inverse of F'_ε, we deduce

$$\int_\Omega(g_\varepsilon(w_i)-g_\varepsilon(\tilde{w}_i))w_i\geq\int_\Omega(F_\varepsilon(g_\varepsilon(w_i))-F_\varepsilon(g_\varepsilon(\tilde{w}_i))).$$

For the right hand side term, we claim that, for $i=1,2$,

$$f_i^\varepsilon(s_1,s_2)F'_\varepsilon(s_i)\leq C(1+F_\varepsilon(s_1)+F_\varepsilon(s_2))\quad\text{for all } s_1,s_2\in\mathbb{R}, \tag{1.98}$$

with $f_i^\varepsilon(s_1,s_2))=\alpha_is_i-a_\varepsilon(s_i)\big(\beta_{i1}a_\varepsilon(s_1)+\beta_{i2}a_\varepsilon(s_2)\big)$. Taking $s_i=g_\varepsilon(w_i)$, from (1.98) we infer

$$\sum_{i=1}^{2}\int_\Omega h_i^\varepsilon(\mathbf{w})w_i\leq C\sum_{i=1}^{2}\int_\Omega(1+F_\varepsilon(g_\varepsilon(w_i))).$$

Therefore, we obtain from (1.97), under the assumption $\tau<1/C$,

$$\begin{aligned}
\sum_{i=1}^{2}\Big(\sigma(1-C\tau)\int_\Omega F_\varepsilon(g_\varepsilon(w_i))+\tau\varepsilon\int_\Omega w_i^2+\tau a_0c(\varepsilon)\int_\Omega|\nabla w_i|^2\Big)&\\
\leq C\sigma\tau+\sigma\sum_{i=1}^{2}\int_\Omega\int_\Omega F_\varepsilon(g_\varepsilon(\tilde{w}_i)).&
\end{aligned} \tag{1.99}$$

Since, by assumption, $F_\varepsilon(g_\varepsilon(w^{k-1}_{i,\varepsilon}))\in L^1(\Omega)$, we deduced the required uniform estimate with respect to σ for $\mathbf{w}^k_{\varepsilon,\sigma}$.

We finally prove our claim (1.98) using the properties (1.48)-(1.51). We have

$$\begin{aligned}
f_i^\varepsilon(s_1,s_2)F'_\varepsilon(s_i)&=\alpha_is_iF'_\varepsilon(s_i)-\big(\beta_{ii}a_\varepsilon(s_i)+\beta_{ij}a_\varepsilon(s_j)\big)a_\varepsilon(s_i)F'_\varepsilon(s_i)\\
&\leq\alpha_i(2F_\varepsilon(s_i)+1)+\big(\beta_{ii}a_\varepsilon(s_i)+\beta_{ij}a_\varepsilon(s_j)\big)[1-s_i]_+\\
&\leq(\alpha_i+\beta_{ii})(2F_\varepsilon(s_i)+1)+\beta_{ij}(2F_\varepsilon(s_j)+1)+\beta_{ii}a_\varepsilon(s_i)[s_i]_-+\beta_{ij}a_\varepsilon(s_j)[s_i]_-,
\end{aligned} \tag{1.100}$$

and since

$$
\begin{aligned}
\beta_{ii}a_\varepsilon(s_i)[s_i]_- + \beta_{ij}a_\varepsilon(s_j)[s_i]_- &\le \frac{1}{2\varepsilon}(\beta_{ii}+\beta_{ij})([s_i]_-)^2 + \frac{\varepsilon}{2}(\beta_{ii}a_\varepsilon(s_i)^2 + \beta_{ij}a_\varepsilon(s_j)^2) \\
&\le (\beta_{ii}+\beta_{ij})F_\varepsilon(s_i) + \beta_{ii}(2+F_\varepsilon(a_\varepsilon(s_i))) + \beta_{ij}(2+F_\varepsilon(a_\varepsilon(s_j))) \\
&\le (\beta_{ii}+\beta_{ij})F_\varepsilon(s_i) + \beta_{ii}(2+F_\varepsilon(s_i)) + \beta_{ij}(2+F_\varepsilon(s_j)) \\
&= (2\beta_{ii}+\beta_{ij})F_\varepsilon(s_i) + \beta_{ij}F_\varepsilon(s_j) + 2(\beta_{ii}+\beta_{ij}),
\end{aligned}
$$

we deduce (1.98) from (1.100):

$$
f_i^\varepsilon(s_1,s_2)F_\varepsilon'(s_i) \le C + (2\alpha_i + 4\beta_{ii} + \beta_{ij})F_\varepsilon(s_i) + 3\beta_{ij}F_\varepsilon(s_j).
$$

Therefore, a fixed point of the operator $S(\mathbf{v},1)$ does exist, which is a solution $\mathbf{w}_\varepsilon^k \in H^1(\Omega)^2$ of problem (1.93)-(1.94), with $\sigma = 1$.

3.4 Back to the original unknowns

We define $u_{i,\varepsilon}^k = g_\varepsilon(w_{i,\varepsilon}^k)$ and notice that $u_{i,\varepsilon}^k \in H^1(\Omega)$, since

$$
\nabla u_{i,\varepsilon}^k = g_\varepsilon'(w_{i,\varepsilon}^k)\nabla w_{i,\varepsilon}^k = a_\varepsilon(g_\varepsilon(w_{i,\varepsilon}^k))\nabla w_{i,\varepsilon}^k,
$$

and $\varepsilon \le a_\varepsilon \le \varepsilon^{-1}$. Introducing this change of unknowns in (1.93)-(1.94), with $\sigma = 1$, we see that $\mathbf{u}_\varepsilon^k$ satisfies, for given $\mathbf{u}_\varepsilon^{k-1} \in L^2(\Omega)^2$ with $F_\varepsilon(u_{i,\varepsilon}^{k-1}) \in L^1(\Omega)$,

$$
\frac{1}{\tau}(u_{i,\varepsilon}^k - u_{i,\varepsilon}^{k-1}) - \operatorname{div} J_i^\varepsilon(\mathbf{u}_\varepsilon^k) + \varepsilon F_\varepsilon'(u_{i,\varepsilon}^k) = f_i^\varepsilon(\mathbf{u}_\varepsilon^k) \qquad \text{in } \Omega, \tag{1.101}
$$

$$
J_i^\varepsilon(\mathbf{u}_\varepsilon^k)\cdot n = 0 \qquad \text{on } \partial\Omega, \tag{1.102}
$$

with, for $i,j = 1,2$ and $i \neq j$,

$$
\begin{aligned}
J_i^\varepsilon(\mathbf{u}) &= a_\varepsilon(u_i)(a_i\nabla u_i + \nabla u_j), \\
f_i^\varepsilon(\mathbf{u}) &= \alpha_i u_i - a_\varepsilon(u_i)\big(\beta_{i1}a_\varepsilon(u_1) + \beta_{i2}a_\varepsilon(u_2)\big).
\end{aligned}
$$

Moreover, using the test function $\varphi = F_\varepsilon'(u_{i,\varepsilon}^k)$ in the weak formulation of (1.101)-(1.102) we obtain, similaly to what we did to deduce (1.99),

$$
\begin{aligned}
\sum_{i=1}^2 \Big((1-C\tau)\int_\Omega F_\varepsilon(u_{i,\varepsilon}^k) + \tau\varepsilon\int_\Omega |F_\varepsilon'(u_{i,\varepsilon}^k)|^2 + \tau a_0\int_\Omega |\nabla u_{i,\varepsilon}^k|^2 \Big) \\
\le C\tau + \sum_{i=1}^2 \int_\Omega\int_\Omega F_\varepsilon(u_{i,\varepsilon}^{k-1}).
\end{aligned} \tag{1.103}
$$

Notice that in this estimate, the gradient bound is independent of ε due to the property $a_\varepsilon(s) = 1/F_\varepsilon''(s)$. Estimate (1.103) is similar to (1.53). From here, it is easy to deduce an estimate for $\mathbf{u}_\varepsilon^k$ similar to what we found for u_ε^k in (1.56), i.e.

$$
\begin{aligned}
\sum_{i=1}^2 \max_{k=1,\ldots,K} \Big(\int_\Omega F_\varepsilon(u_{i,\varepsilon}^k) + \int_\Omega |u_{i,\varepsilon}^k| + \frac{1}{\varepsilon}\int_\Omega ([u_{i,\varepsilon}^k]_-)^2 + \tau\varepsilon\int_\Omega |F_\varepsilon'(u_{i,\varepsilon}^k)|^2 \Big) + \\
\tau\sum_{i=1}^2\sum_{k=1}^K \int_\Omega |\nabla u_{i,\varepsilon}^k|^2 \le C.
\end{aligned} \tag{1.104}
$$

3.5 Back to the evolution problem

Like in previous sections, we consider piecewise constant and piecewise linear functions in time. For $(t,x) \in (t_{k-1},t_k] \times \Omega$, and for $k = 1,\dots,K$, with $t_k = k\tau$ and $\tau = T/K$, we define

$$u_{i,\varepsilon}^{(\tau)}(t,x) = u_{i,\varepsilon}^k(x), \quad \tilde{u}_{i,\varepsilon}^{(\tau)}(t,x) = u_{i,\varepsilon}^k(x) + \frac{t_k - t}{\tau}(u_{i,\varepsilon}^{k-1}(x) - u_{i,\varepsilon}^k(x)).$$

Replacing these functions in the weak formulation of (1.101)-(1.102), we obtain the identity

$$\int_0^T \partial_t \tilde{u}_{i,\varepsilon}^{(\tau)} \varphi + \int_{Q_T} J_i^\varepsilon(\mathbf{u}_\varepsilon^{(\tau)}) \cdot \nabla\varphi + \varepsilon \int_{Q_T} F_\varepsilon'(u_{i,\varepsilon}^{(\tau)})\varphi = \int_{Q_T} f_i^\varepsilon(\mathbf{u}_\varepsilon^{(\tau)})\varphi, \tag{1.105}$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$.

Uniform estimates in ε and τ

From (1.104) we get

$$\max_{t\in(0,T)} \Big(\int_\Omega F_\varepsilon(u_{i,\varepsilon}^{(\tau)}(t)) + \int_\Omega |u_{i,\varepsilon}^{(\tau)}(t)| + \frac{1}{\varepsilon}\int_\Omega ([u_{i,\varepsilon}^{(\tau)}(t)]_-)^2 \Big) + \varepsilon \int_{Q_T} |F_\varepsilon'(u_{i,\varepsilon}^{(\tau)})|^2 + \int_{Q_T} |\nabla u_{i,\varepsilon}^{(\tau)}|^2 \le C. \tag{1.106}$$

From (1.106) and the Poincaré-Wirtinger's inequality we deduce

$$\|u_{i,\varepsilon}^{(\tau)}\|_{L^2(H^1)} \le C, \quad \|\tilde{u}_{i,\varepsilon}^{(\tau)}\|_{L^2(H^1)} \le C. \tag{1.107}$$

Time derivative estimate

We have, using (1.105) and $\varphi \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T \langle \partial_t \tilde{u}_{i,\varepsilon}^{(\tau)}, \varphi \rangle \le a_i \int_{Q_T} |a_\varepsilon(u_{i,\varepsilon}^{(\tau)})| \big(|\nabla u_{i,\varepsilon}^{(\tau)}| + |\nabla u_{j,\varepsilon}^{(\tau)}|\big)|\nabla\varphi| + \int_{Q_T} |f_i^\varepsilon(\mathbf{u}_\varepsilon^{(\tau)})||\varphi| + \varepsilon \int_{Q_T} |F_\varepsilon'(u_{i,\varepsilon}^{(\tau)})||\varphi| \le C\varepsilon^{-2}\|\varphi\|_{L^2(H^1)},$$

and thus, from (1.106),

$$\|\partial_t \tilde{u}_{i,\varepsilon}^{(\tau)}\|_{L^2((H^1)')} \le C\varepsilon^{-2}. \tag{1.108}$$

3.6 The limit τ → 0

From the bounds (1.106), (1.107), and (1.108) we deduce the existence of $\mathbf{u}_\varepsilon, \mathbf{z}_\varepsilon \in L^2(0,T;H^1(\Omega))$ and of subsequences of $\mathbf{u}_\varepsilon^{(\tau)}$ and $\tilde{\mathbf{u}}_\varepsilon^{(\tau)}$ (not relabeled) such that, as $\tau \to 0$,

$$\begin{aligned}
&\mathbf{u}_\varepsilon^{(\tau)} \rightharpoonup \mathbf{u}_\varepsilon && \text{weakly in } L^2(0,T;H^1(\Omega))^2,\\
&\mathbf{u}_\varepsilon^{(\tau)} \rightharpoonup \mathbf{u}_\varepsilon && \text{weakly in } L^2(Q_T)^2,\\
&\tilde{\mathbf{u}}_\varepsilon^{(\tau)} \rightharpoonup \mathbf{z}_\varepsilon && \text{weakly in } L^2(0,T;H^1(\Omega))^2,\\
&\tilde{\mathbf{u}}_\varepsilon^{(\tau)} \rightharpoonup \mathbf{z}_\varepsilon && \text{weakly in } L^2(Q_T)^2,\\
&\partial_t \tilde{\mathbf{u}}_\varepsilon^{(\tau)} \rightharpoonup \partial_t \mathbf{z}_\varepsilon && \text{weakly in } L^2(0,T;(H^1(\Omega))')^2,
\end{aligned}$$

the identification $\mathbf{z}_\varepsilon = \mathbf{u}_\varepsilon$ being deduced like in Subsection 2.4.

Compactness and strong convergences

We use the compactness Aubin-Lions lemma, Lemma 3, to get strong convergence. We get the existence of a subsequence (not relabeled) such that

$$\tilde{\mathbf{u}}_\varepsilon^{(\tau)} \to \mathbf{u}_\varepsilon \quad \text{strongly in } L^2(Q_T)^2, \text{ and a.e. in } Q_T.$$

In particular, like in (1.71), we also obtain

$$\mathbf{u}_\varepsilon^{(\tau)} \to \mathbf{u}_\varepsilon \quad \text{strongly in } L^2(Q_T), \text{ and a.e. in } Q_T.$$

Convergence

Passing to the limit $\tau \to 0$ is justified like in Subsection 2.4. We obtain that $\mathbf{u}_\varepsilon \in L^2(0,T;H^1(\Omega))^2 \cap H^1(0,T;(H^1(\Omega))')^2$ satisfies, for all $\varphi \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T \langle \partial_t u_{i,\varepsilon}, \varphi\rangle + \int_{Q_T} J_i^\varepsilon(\mathbf{u}_\varepsilon)\cdot\nabla\varphi + \varepsilon\int_{Q_T} F_\varepsilon'(u_{i,\varepsilon})\varphi = \int_{Q_T} f_i^\varepsilon(\mathbf{u}_\varepsilon)\varphi. \tag{1.109}$$

3.7 The limit $\varepsilon \to 0$

Uniform estimates in ε and weak convergences

Taking the limit $\tau \to 0$ in (1.106), (1.107) we get

$$\max_{t\in(0,T)} \Big(\int_\Omega F_\varepsilon(u_\varepsilon(t)) + \int_\Omega |u_\varepsilon(t)| + \frac{1}{\varepsilon}\int_\Omega ([u_\varepsilon(t)]_-)^2 \Big) + \varepsilon \int_{Q_T} |F_\varepsilon'(u_{i,\varepsilon})|^2 + \int_{Q_T} |\nabla u_\varepsilon|^2 \le C. \tag{1.110}$$

and then

$$\|u_\varepsilon\|_{L^2(H^1)} \le C. \tag{1.111}$$

Using Gagliardo-Nirenberg inequality like in (1.77) yields, for $p=(2N+2)/N$,

$$\|u_\varepsilon\|_{L^p} \le C. \tag{1.112}$$

For the time derivative estimate, let $r=(2N+2)/(2N+1)$, and then $r'=2(N+1)$, and write, using (1.109) and $p>2$,

$$\begin{aligned}\int_0^T \langle \partial_t u_{i,\varepsilon}, \varphi\rangle \le& \int_{Q_T} |a_\varepsilon(u_{i,\varepsilon})|\big(a_i|\nabla u_{i,\varepsilon}| + |\nabla u_{j,\varepsilon}|\big)|\nabla\varphi| + \int_{Q_T} |f_i^\varepsilon(\mathbf{u}_\varepsilon)||\varphi| \\ &+ \varepsilon\int_{Q_T} |F_\varepsilon'(u_{i,\varepsilon})||\varphi| \le \|a_\varepsilon(u_{i,\varepsilon})\|_{L^p}\big(a_i\|\nabla u_{i,\varepsilon}\|_{L^2} + \|\nabla u_{j,\varepsilon}\|_{L^2}\big)\|\nabla\varphi\|_{L^{r'}} \\ &+ \alpha_i\|u_{i,\varepsilon}\|_{L^p}\|\varphi\|_{L^{p'}} + \big(\beta_{ii}\|a_\varepsilon(u_{i,\varepsilon})\|_{L^p}^2 + \beta_{ij}\|a_\varepsilon(u_{i,\varepsilon})\|_{L^p}\|a_\varepsilon(u_{j,\varepsilon})\|_{L^p}\big)\|\varphi\|_{L^{(p/2)'}} \\ &+ \varepsilon\|F_\varepsilon'(u_{i,\varepsilon})\|_{L^2} \le C\|\varphi\|_{L^{r'}(W^{1,r'})},\end{aligned}$$

and thus

$$\|\partial_t u_\varepsilon\|_{L^r((W^{1,r'})')} \le C. \tag{1.113}$$

Finally, from (1.106) we also deduce

$$\|[u_{i,\varepsilon}]_-\|_{L^\infty(L^2)} \leq C\sqrt{\varepsilon}, \quad \text{and} \quad \sqrt{\varepsilon}\|F'_\varepsilon(u_{i,\varepsilon})\|_{L^2} \leq C. \tag{1.114}$$

From the bounds (1.110), (1.111), (1.112), (1.113), and (1.114) we deduce the existence of $\mathbf{u} \in L^2(0,T;H^1(\Omega))^2$ and of a subsequence of $\mathbf{u}_\varepsilon$ (not relabeled) such that

$$\begin{aligned}
&\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} && \text{weakly in } L^2(0,T;H^1(\Omega))^2,\\
&\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} && \text{weakly in } L^p(Q_T)^2,\\
&\partial_t\mathbf{u}_\varepsilon \rightharpoonup \partial_t\mathbf{u} && \text{weakly in } L^r(0,T;(W^{1,r'}(\Omega))')^2,\\
&[\mathbf{u}_\varepsilon]_- \rightharpoonup 0 && \text{weakly*-weakly in } L^\infty(0,T;L^2(\Omega))^2
\end{aligned}$$

Compactness and strong convergences

We again use the compactness Aubin-Lions lemma, Lemma 3, to get the existence of a subsequence strongly convergent. Then, like we did in (1.81), (1.82), and (1.83), we get

$$\begin{aligned}
&\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{strongly in } L^q(Q_T)^2, \text{ for any } 1 \leq q < p.\\
&[\mathbf{u}_\varepsilon]_- \to 0 \quad \text{strongly in } L^q(Q_T)^2 \text{ and a.e. in } Q_T, \text{ that is } u_i \geq 0 \text{ a.e. in } Q_T,\\
&a_\varepsilon(u_{i,\varepsilon}) \to u_i \quad \text{strongly in } L^q(Q_T).
\end{aligned}$$

Observe that, in particular, we may choose $2 \leq q < p = (2N+2)/N$.

Convergence

We have to pass to the limit in the expression (1.109). Except for the term involving $F'_\varepsilon(u_{i,\varepsilon})$, the passing to the limit of the rest of terms are justified like in Subsection 2.5. For the former, we have, using (1.114)

$$\varepsilon\int_{Q_T} F'_\varepsilon(u_{i,\varepsilon})\varphi \leq \varepsilon\|F'_\varepsilon(u_{i,\varepsilon})\|_{L^2}\|\varphi\|_{L^2} \leq C\sqrt{\varepsilon} \to 0.$$

Theorem 1.10 *Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz continuous boundary, and let $T > 0$. Suppose that $u_{i0} \in L^2(\Omega)$ are non-negative, for $i = 1,2$. Then, problem (1.84)-(1.87) with, for $i,j = 1,2$ and $i \neq j$,*

$$\begin{aligned}
J_i(u_1,u_2) &= u_i(a_i\nabla u_i + \nabla u_j), \quad \text{with } a_i > 0,\ a_1a_2 > 1,\\
f_i(u_1,u_2) &= u_i\big(\alpha_i - (\beta_{i1}u_1 + \beta_{i2}u_2)\big),
\end{aligned}$$

has a weak solution (u_1,u_2) satisfying $u_i \geq 0$ in Q_T and

$$u_i \in L^2(0,T;H^1(\Omega)) \cap L^p(Q_T) \cap W^{1,r}(0,T;(W^{1,r'}(\Omega))'),$$

where $p = 2(N+1)/N$, $r = 2(N+1)/(2N+1)$, and $r' = 2(N+1)$, in the sense that for all $\varphi \in L^{r'}(0,T;W^{1,r'}(\Omega))$, and $i = 1,2$,

$$\int_0^T < \partial_t u_i, \varphi > + \int_{Q_T} J_i(u_1,u_2)\cdot\nabla\varphi = \int_{Q_T} f_i(u_1,u_2)\,\varphi,$$

with $\langle \cdot, \cdot \rangle$ denoting the duality product between $W^{1,r'}(\Omega)$ and its dual $(W^{1,r'}(\Omega))'$. The initial data is satisfied in the sense

$$\int_0^T < \partial_t u_i, \psi > + \int_{Q_T} (u_i - u_{i0}) \partial_t \psi = 0,$$

for all $\psi \in L^{r'}(0,T;W^{1,r'}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $\psi(T) = 0$ a.e. in Ω.

3.8 Generalizations and final remarks

Linear diffusion. The addition of a linear diffusion term in the flows J_i may be treated as follows:

- Lax-Milgram. The coercivity and continuity of the bilinear form A, defined in (1.95), is not altered when adding the terms $a_{i0} \int_\Omega g'_\varepsilon(v_i) \nabla w_i \cdot \nabla \varphi$, for which we have $a_{i0} \int_\Omega g'_\varepsilon(v_i) |\nabla w_i|^2 \geq 0$.
- Bound in the fixed point. We have $a_{i0} \int_\Omega g'_\varepsilon(w_i) |\nabla w_i|^2 \geq 0$, so the bound (1.99) is not altered.
- Original unknowns. When using $\varphi = F'_\varepsilon(u^k_{i,\varepsilon})$ to get the energy estimate (1.103), the corresponding energy term is well defined, since

$$\varepsilon \tau a_{i0} \int_\Omega |\nabla u^k_{i,\varepsilon}|^2 \leq \tau a_{i0} \int_\Omega F''_\varepsilon(u^k_{i,\varepsilon}) |\nabla u^k_{i,\varepsilon}|^2 \leq \varepsilon^{-1} \tau a_{i0} \int_\Omega |\nabla u^k_{i,\varepsilon}|^2.$$

- The limit $\tau \to 0$. We add to the left hand side of estimate (1.106), the following term

$$a_{i0} \int_{Q_T} \frac{1}{a_\varepsilon(u^{(\tau)}_{i,\varepsilon})} |\nabla u^{(\tau)}_{i,\varepsilon}|^2.$$

  Due to the weak convergence of $\nabla u^{(\tau)}_{i,\varepsilon}$ in $L^2(Q_T)$, we get in the weak formulation (1.105),

$$a_{i0} \int_{Q_T} \nabla u^{(\tau)}_{i,\varepsilon} \cdot \nabla \varphi \to a_{i0} \int_{Q_T} \nabla u_{i,\varepsilon} \cdot \nabla \varphi.$$

- The limit $\varepsilon \to 0$. We add to the left hand side of estimate (1.110), the term

$$4 a_{i0} \int_{Q_T} \frac{1}{|a'_\varepsilon(u_{i,\varepsilon})|^2} |\nabla \sqrt{a_\varepsilon(u_{i,\varepsilon})}|^2.$$

  Again, the weak convergence of $\nabla u_{i,\varepsilon}$ in $L^2(Q_T)$ gives in the weak formulation (1.109),

$$a_{i0} \int_{Q_T} \nabla u_{i,\varepsilon} \cdot \nabla \varphi \to a_{i0} \int_{Q_T} \nabla u_i \cdot \nabla \varphi,$$

  and, in fact, the above energy also passes to the limit (and remains bounded)

$$\int_{Q_T} \frac{1}{|a'_\varepsilon(u_{i,\varepsilon})|^2} |\nabla \sqrt{a_\varepsilon(u_{i,\varepsilon})}|^2 \to \int_{Q_T} |\nabla \sqrt{u_i}|^2.$$

Linear convection. The addition of a linear convective term, e.g. $\mathbf{q} = \nabla\Phi \in L^2(Q_T)$, in the flows J_i may be treated as follows:

- Time discretization. We introduce the time discretization $\mathbf{q}^k(x) = \mathbf{q}(t,x)$ for $t \in (t_{k-1}, t_k]$.
- Lax-Milgram. We add the terms $\int_\Omega a_\varepsilon(v_i)\mathbf{q}^k \cdot \nabla\varphi$ to the linear form F defined in (1.96). The continuity is not altered.
- Bound in the fixed point. We have

$$\int_\Omega a_\varepsilon(w_i)\mathbf{q}^k \cdot \nabla w_i \leq \varepsilon^{-1}\|\mathbf{q}^k\|_{L^2}\|\nabla w_i\|_{L^2} \leq \gamma\|\nabla w_i\|_{L^2}^2 + \frac{4}{\gamma\varepsilon^2}\|\mathbf{q}^k\|_{L^2}^2,$$

  so the bound (1.99) is not altered if we take γ small enough.
- Original unknowns. When using $\varphi = F_\varepsilon'(u_{i,\varepsilon}^k)$, the corresponding term in (1.103) is controled by the gradient:

$$\int_\Omega \mathbf{q}^k \cdot \nabla u_{i,\varepsilon}^k \leq 2a_0 \int_\Omega |\mathbf{q}^k|^2 + \frac{a_0}{2}\int_\Omega |\nabla u_{i,\varepsilon}^k|^2.$$

- The limit $\tau \to 0$. We clearly have $\mathbf{q}^{(\tau)} \to \mathbf{q}$ strongly in $L^2(Q_T)$. Thus,

$$\int_{Q_T} a_\varepsilon(u_{i,\varepsilon}^{(\tau)})\mathbf{q}^{(\tau)} \cdot \nabla\varphi \to \int_{Q_T} a_\varepsilon(u_{i,\varepsilon})\mathbf{q} \cdot \nabla\varphi,$$

  due to the strong convergence $a_\varepsilon(u_{i,\varepsilon}^{(\tau)}) \to a_\varepsilon(u_{i,\varepsilon})$ in $L^2(Q_T)$.
- The limit $\varepsilon \to 0$. The strong convergence $a_\varepsilon(u_{i,\varepsilon}) \to u_i$ in $L^2(Q_T)$ implies

$$\int_{Q_T} a_\varepsilon(u_{i,\varepsilon})\mathbf{q} \cdot \nabla\varphi \to \int_{Q_T} u_i\mathbf{q} \cdot \nabla\varphi.$$

Non-constant coefficients. All the constant coefficients appearing in the problem may be generalized to be $L^\infty(Q_T)$ functions.

The SKT model. The SKT model, i.e., the problem (1.84)-(1.87) with flows given by

$$J_i^{SKT}(\mathbf{u}) = \nabla\big(u_i(a_{i0} + a_i u_1 + u_2)\big) - b_i u_i \nabla\Phi$$

may be treated in a similar way. In particular, the formal estimate obtained by multiplying both equations by $\ln u_i$, integrating by parts, and adding the resulting identities, give for the nonlinear diffusion term

$$\sum_{i=1}^{2}\int_{Q_T}\big(a_i|\nabla u_i|^2 + 2|\nabla\sqrt{u_1 u_2}|^2\big),$$

and thus, the $L^2(0,T;H^1(\Omega))$ estimates of u_1 and u_2 remain valid.

Systems of m equations. The proof of existence of solutions $\mathbf{u} = (u_1, \ldots, u_m)$ for systems of m equations

$$\partial_t u_i - \operatorname{div} J_i(\mathbf{u}) = f_i(\mathbf{u}), \quad i = 1, \ldots, m,$$

with flows of the BT type,

$$J_i(\mathbf{u}) = a_{i0}\nabla u_i + u_i \sum_{j=1}^{m} a_{ij}\nabla u_j - b_i u_i \nabla\Phi, \quad \text{for } i = 1,2,$$

with $a_{i0} \geq 0$, $a_{ij}, b_i \in \mathbb{R}$, and competitive Lotka-Volterra terms

$$f_i(\mathbf{u}) = u_i\big(\alpha_i - \sum_{j=1}^{m} \beta_{ij} u_j\big), \quad \text{for } i = 1,\dots,m,$$

with $\alpha_i, \beta_{ij} \geq 0$, for $i, j = 1,\dots,m$ is straightforward under the condition

$$\mathbf{w}^T a\mathbf{w} \geq a_0 \|\mathbf{w}\|^2, \quad \text{for some } a_0 > 0, \text{ and for all } \mathbf{w} \in \mathbb{R}^m, \tag{1.115}$$

where a is the matrix of diffusion coefficients, $a = (a_{ij})$. Defining the symmetric matrix a^s

$$a^s_{ij} = \begin{cases} a_{ii} & \text{if } i = j, \\ \dfrac{a_{ij} + a_{ji}}{2} & \text{if } i \neq j, \end{cases}$$

we may check that (1.115) is satisfied if a^s is positive definite. Observe that, for $m = 2$, this condition is equivalent to

$$4a_{11}a_{22} > (a_{12} + a_{21})^2. \tag{1.116}$$

However, under the rescaling of the unknowns $U_i = a_{ji}u_i$, for $i = 1,2$, $i \neq j$ we obtained the new matrix

$$\tilde{a} = \begin{pmatrix} a_1 & 1 \\ 1 & a_2 \end{pmatrix},$$

with $a_i = a_{ii}/a_{ji}$, for which the condition simplifies to $a_1 a_2 > 1$, that is, $\det(a) > 0$, which is weaker than (1.116).

Since the rescaling introduced for $m = 2$ has not a clear extension to $m > 2$, the following question arises: may we replace the condition of a^s being positive definite by the condition $\det(a) > 0$ to get the existence of weak solutions?

A similar situation happens for the SKT problem, being both at the moment, open problems.

4 Exercises

1. Prove the estimate
$$(1-r)^{-1} \le \exp(r(1-r)^{-1}) \quad \text{for all } r \in [0,1).$$

2. Prove the following estimates involving the functions a_ε and F_ε defined in Subsection 2.2
   (a) $(s-t)F_\varepsilon'(s) \ge F_\varepsilon(s) - F_\varepsilon(t) + \frac{\varepsilon}{2}(s-t)^2$, for all $s,t \in \mathbb{R}$.
   (b) $F_\varepsilon(s) \ge \frac{\varepsilon}{2}s^2 - 2$ for all $s \ge 0$.
   (c) $F_\varepsilon(s) \ge \frac{s^2}{2\varepsilon}$ for all $s \le 0$.
   (d) $\max\{a_\varepsilon(s), sF_\varepsilon'(s)\} \le 2F_\varepsilon(s) + 1$ for all $s \in \mathbb{R}$.
   (e) $a_\varepsilon(s)F_\varepsilon'(s) \ge s - 1$ for all $s \in \mathbb{R}$.
   (f) $F_\varepsilon(a_\varepsilon(s)) \le F_\varepsilon(s)$ for all $s \in \mathbb{R}$.
   (g) $[1-s]_+ \le 1 - [s]_-$ for all $s \in \mathbb{R}$.

3. Adapt the ideas we employed at the end of Section 1 to show lower and upper bounds of the solution, u, of the linear problem (1.1)-(1.3) to prove that the solution of the time discrete problem (1.13) has similar bounds. Prove then that they also hold in the limit $\tau \to 0$.

4. Using (1.58) and the Poincaré-Wirtinger's inequality (Theorem 1.6), deduce
$$\|u_\varepsilon^{(\tau)}\|_{L^2}^2 \le \frac{1}{\Omega}\|u_\varepsilon^{(\tau)}\|_{L^1}^2 + C\|\nabla u_\varepsilon^{(\tau)}\|_{L^2}^2 \le C.$$

5. Use the Gagliardo-Nirenberg inequality (Theorem 1.7) to prove estimate (1.77)

6. Use a similar argument to that employed at the end of Section 1 to get upper and lower bounds for the solution, u, of the nonlinear problem (1.23)-(1.25), in terms of the initial data.

7. Check that for $r = (2N+2)/(2N+1)$, and $p > 2$ we have $r' \ge \max\{p', (p/2)'\}$. Show that the integrals
$$\int_\Omega uv, \quad \int_\Omega u^2 v, \quad \text{for } u \in L^p(\Omega),\ v \in L^{r'}(\Omega),$$
are well defined.

Chapter 2

Turing instability for the Lotka-Volterra model with cross-diffusion

In this work we study the phenomena of pattern formation for the competitive Lotka-Volterra model under two types of diffusive terms: linear self-diffusion, and nonlinear cross-diffusion of the Shigesada-Kawasaki-Teramoto type.

We first analyze the conditions for linear stability of the linear diffusion system for a generic reaction term. Then, the application of these results to the Lotka-Volterra case shows that the system is linearly stable.

Then, we pass to analyze the model with cross-diffusion. After deducing the conditions on linear stability, we show how cross-diffusion destabilizes the uniform equilibrium and is responsible for the initiation of spatial patterns.

Near marginal stability, through a weakly nonlinear analysis, we are able to predict the shape and the amplitude of the pattern. For the amplitude, in the supercritical and case, we derive the cubic Stuart-Landau equation.

This work has been extracted from:

• G. Gambino, M.C. Lombardo, M. Sammartino, Turing instability and traveling fronts for a nonlinear reaction-diffusion system with cross-diffusion, Mathematics and Computers in Simulation 82(6) (2012) 1112-1132.

Generically, bifurcation consists on a qualitative change in the behavior of the equilibrium solutions of a system. It occurs due to a change in the value of a bifurcation parameter, resulting in the emergence of a new steady state or a change in the stability of the steady states.

The stability of the uniform steady states is associated to the distribution of the eigenvalues of the corresponding linearized system. If the steady state is initially stable, the eigenvalues have both negative real part. At the bifurcation, at least one eigenvalue crosses the imaginary axis. If one considers a two-dimensional system of the form (2.1)-(2.4), then there exist two roots of the characteristic polynomial with real (negative) coefficients, and two different scenarios of bifurcation can occur: Turing bifurcation, in which one eigenvalue crosses the origin, or Hopf bifurcation, where a pair of imaginary eigenvalues crosses the real axis and it results in a limit cycle with oscillations.

Here, we will analyze the occurence of Turing bifurcation.

1 Linear self-diffusion problem

We start considering the linear diffusion nonlinear reaction system

$$\partial_t u_1 - d_1 \Delta u_1 = \gamma f_1(u_1, u_2) \qquad \text{in } Q_T = (0,T)\times\Omega, \tag{2.1}$$

$$\partial_t u_2 - d_2 \Delta u_2 = \gamma f_2(u_1, u_2) \qquad \text{in } Q_T, \tag{2.2}$$

$$\nabla u_1 \cdot n = \nabla u_2 \cdot n = 0 \qquad \text{on } \Gamma_T = \partial(0,T)\times\Omega, \tag{2.3}$$

$$u_1(\cdot,0) = u_{10}, \quad u_2(\cdot,0) = u_{20} \qquad \text{in } \Omega. \tag{2.4}$$

Here, (u_1, u_2) are concentrations or densities of some specie, e.g. of populations. The parameter γ regulates the relative strength of the kinetic terms, or, alternatively, it gives the size of the spatial domain and the time scale, i.e. $\sqrt{\gamma}$ is proportional to the linear dimension of the domain or, in the case of a 2D spatial domain, it is proportional to its area. The reaction terms, (f_1, f_2) are supposed to be nonlinear, e.g. competitive Lotka-Volterra terms.

We write (2.1)-(2.2) in vector form as

$$\partial_t \mathbf{u} - d\Delta\mathbf{u} = \gamma\mathbf{f}(\mathbf{u}), \tag{2.5}$$

with $\mathbf{u} = (u_1, u_2)$, $\mathbf{f} = (f_1, f_2)$, and d the diagonal matrix with diagonal (d_1, d_2). Non-trivial homogeneous stationary state $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)$ are constant positive solutions of

$$\mathbf{f}(\tilde{\mathbf{u}}) = \mathbf{0}.$$

1.1 Linearization

Linearization around $\tilde{\mathbf{u}}$ gives the following system for $\mathbf{w} = \mathbf{u} - \tilde{\mathbf{u}}$,

$$\partial_t \mathbf{w} - d\Delta\mathbf{w} = \gamma D\mathbf{f}(\tilde{\mathbf{u}})\mathbf{w}, \tag{2.6}$$

where $D\mathbf{f} = \big(\partial_i f_j(\mathbf{u})\big)$ is the Jacobian matrix of $\mathbf{f}$:

$$D\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \partial_1 f_1(\mathbf{u}) & \partial_2 f_1(\mathbf{u}) \\ \partial_1 f_2(\mathbf{u}) & \partial_2 f_2(\mathbf{u}) \end{pmatrix}.$$

System (2.6) is linear with constant coefficients. Then, given the boundary conditions (2.3), we look for a particular solution of the form

$$\mathbf{w} = \exp(\lambda_k t + ikx)\mathbf{u}_k, \tag{2.7}$$

where $\mathbf{u}_k$ is a constant vector, λ_k represents the linear growth rate and k is the wavenumber of the perturbation. Upon substitution of (2.7) into (2.6), one gets the following eigenvalue problem

$$A_k\mathbf{w} = \lambda_k\mathbf{w}, \quad \text{with } A_k = \gamma D\mathbf{f}(\tilde{\mathbf{u}}) - k^2 d.$$

For each wavenumber, k, there exists an eigenvalue problem which in general admits two linearly independent eigenvectors $\mathbf{u}_{jk}$, $j = 1,2$. If λ_{jk} are the corresponding eigenvalues, then the particular solution associated to the wavenumber k is of the form

$$\big(c_{1k}\mathbf{u}_{1k}e^{\lambda_{1k}t} + c_{2k}\mathbf{u}_{2k}e^{\lambda_{2k}t}\big)e^{ikx}, \tag{2.8}$$

where the coefficients c_{jk} are complex constants which depend on the initial data.

Then, the general solution can be expressed by the sum of particular solutions of the form (2.8)

$$\mathbf{w}(t,x) = \sum_k \big(c_{1k}\mathbf{u}_{1k}e^{\lambda_{1k}t} + c_{2k}\mathbf{u}_{2k}e^{\lambda_{2k}t}\big)e^{ikx}. \tag{2.9}$$

Observe that the characteristic polynomial associated to the eigenvalue problem (2.6) is given by

$$\lambda_k^2 - \mathrm{tr}(A_k)\lambda_k + \det(A_k) = 0,$$

where

$$\begin{aligned} \mathrm{tr}(A_k) &= \gamma(\partial_1 f_1(\tilde{\mathbf{u}}) + \partial_2 f_2(\tilde{\mathbf{u}})) - k^2(d_1 + d_2), \\ \det(A_k) &= d_1 d_2 k^4 - \gamma(d_2\partial_1 f_1(\tilde{\mathbf{u}}) + d_1\partial_2 f_2(\tilde{\mathbf{u}}))k^2 + \gamma^2\det(D\mathbf{f}(\tilde{\mathbf{u}})), \end{aligned} \tag{2.10}$$

having the roots

$$\lambda_k = \frac{1}{2}\Big(\mathrm{tr}(A_k) \pm \sqrt{\mathrm{tr}(A_k) - 4\det(A_k)}\Big).$$

1.2 Conditions for linear instability

Since we are interested in diffusion-driven instability, we suppose that when spatial variations are neglected ($k = 0$), the steady state is linearly stable, that is $\mathrm{Re}(\lambda_{j0}) < 0$ for $j = 1,2$. This implies

$$\mathrm{tr}(A_0) = \mathrm{tr}(D\mathbf{f}(\tilde{\mathbf{u}})) < 0, \quad \det(A_0) = \det(D\mathbf{f}(\tilde{\mathbf{u}})) > 0. \tag{2.11}$$

Returning to spatial-dependent problem, we look for the situation in which the variation of some system parameters (diffusion coefficients) changes the sign of some $\mathrm{Re}(\lambda_k)$, for $k \neq 0$, implying that the uniform steady state becomes linearly unstable.

First, we observe that from (2.11) and (2.10) we obtain $\mathrm{tr}(A_k) < 0$. Thus the only way to have $\mathrm{Re}(\lambda_k) > 0$ for some $k \neq 0$ is that $\det(A_k)$ becomes negative. Since $\det(A_k)$, as a function of k^2, is a convex parabola, we analyze the point of minimum

$$k_c^2 = \gamma \frac{d_2 \partial_1 f_1(\tilde{\mathbf{u}}) + d_1 \partial_2 f_2(\tilde{\mathbf{u}})}{2 d_1 d_2}, \tag{2.12}$$

and the corresponding minimum value

$$h(k_c^2) = \gamma^2 \left[\det(D\mathbf{f}(\tilde{\mathbf{u}})) - \frac{\big(d_2 \partial_1 f_1(\tilde{\mathbf{u}}) + d_1 \partial_2 f_2(\tilde{\mathbf{u}})\big)^2}{4 d_1 d_2} \right].$$

For fixed values of the kinetic parameters, we obtain that $h(k_c^2) = 0$ (bifurcation) if d_c is a positive root of

$$\big(\partial_1 f_1(\tilde{\mathbf{u}})\big)^2 d_c^2 + 2\big(2\partial_2 f_1(\tilde{\mathbf{u}})\partial_1 f_2(\tilde{\mathbf{u}}) - \partial_1 f_1(\tilde{\mathbf{u}})\partial_2 f_2(\tilde{\mathbf{u}})\big) d_c + \big(\partial_2 f_2(\tilde{\mathbf{u}})\big)^2 = 0, \tag{2.13}$$

where d_c is the *critical diffusion ratio*. If such d_c does exist, then the corresponding *critical wavenumber* is obtained from (2.12) replacing d_1, d_2 by the critical diffusion coefficients d_1^c, d_2^c, satisfying $d_2^c / d_1^c = d_c$.

Observe that for a diffusion ratio $d^* > d_c$, there exists a range of unstable wavenumbers contained in the interval $[k_1^2, k_2^2]$, where k_1, k_2 are such that $\det(A_{k_1}) = \det(A_{k_2}) = 0$. Moreover, since the eigenvalue problem is defined in a finite domain, the wavenumbers are discrete and within a well defined range, thus there will be only a finite number of possible unstable wavenumbers contained in $[k_1^2, k_2^2]$.

Within this range of possible unstable wavenumbers, $\mathrm{Re}(\lambda_k)$ is positive and assumes its maximum value for k_c^2. Therefore, there exists a fastest growing mode in the solution (2.9), and the dominant contributions as t increases are given by the modes for which $\mathrm{Re}\,\lambda_k > 0$, i.e. for large t the perturbation solution can be expressed in the following form

$$\mathbf{w}(t,x) \approx \sum_{k=k_1}^{k_2} \mathbf{u}_k e^{\lambda_k t} e^{ikx}.$$

Summarizing, the Turing instability analysis allows to derive the conditions on the system parameters under which a reaction-diffusion system can exhibit Turing patterns, that is a homogeneous steady state becomes unstable in response to infinitesimal disturbances. Furthermore, the dominant unstable mode wavelength, i.e. the length scale $1/k_c$, and the range of unstable wavenumbers can be determined.

1.3 Linear stability of the competitive Lotka-Volterra system

Let us consider the reaction terms

$$f_i(\mathbf{u}) = \alpha_i u_i - (\beta_{i1} u_1 + \beta_{i2} u_2) u_i,$$

with $\alpha_i, \beta_{ij} \geq 0$, for $i, j = 1, 2$. We then have a unique non-trivial co-existence equilibrium

$$\tilde{\mathbf{u}} = \Big(\frac{\beta_{22}\alpha_1 - \beta_{12}\alpha_2}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}, \frac{\beta_{11}\alpha_2 - \beta_{21}\alpha_1}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} \Big), \tag{2.14}$$

with $\tilde{u}_i > 0$, for which

$$D\mathbf{f}(\mathbf{u}) = \begin{pmatrix} -\beta_{11}\tilde{u}_1 & -\beta_{12}\tilde{u}_1 \\ -\beta_{21}\tilde{u}_2 & -\beta_{22}\tilde{u}_2 \end{pmatrix}. \tag{2.15}$$

The equilibrium $\tilde{\mathbf{u}}$ is stable for the dynamical system if the eigenvalues, μ_i, of $D\mathbf{f}(\mathbf{u})$ are negative. The characteristic equation is

$$\mu^2 - \mathrm{tr}(D\mathbf{f}(\mathbf{u})) + \det(D\mathbf{f}(\mathbf{u})) = 0.$$

Thus, for stability of the dynamical system, the following conditions are required

$$\begin{aligned} &\mathrm{tr}(D\mathbf{f}(\mathbf{u})) < 0, \quad \text{and} \quad \det(D\mathbf{f}(\mathbf{u})) > 0 && \text{(for negative real part)}, \\ &\mathrm{tr}(D\mathbf{f}(\mathbf{u}))^2 - 4\det(D\mathbf{f}(\mathbf{u})) \geq 0 && \text{(for null imaginary part)}. \end{aligned}$$

It is easy to see that the second condition is equivalent to

$$(\beta_{11}\tilde{u}_1 - \beta_{22}\tilde{u}_2)^2 + 4\beta_{12}\beta_{21}\tilde{u}_1\tilde{u}_2 \geq 0.$$

Thus, both conditions are satisfied if we assume $\beta_{ij} \geq 0$, for $i, j = 1, 2$, and

$$\mathrm{tr}(B) > 0, \quad \text{and} \quad \det(B) > 0, \quad \text{with} \quad B = (\beta_{ij}). \tag{2.16}$$

Returning to the spatial-dependent problem and writing (2.13) as $ad_c^2 + bd_c + c = 0$, the solutions are given by $d_c = \frac{1}{2}(-b \pm \sqrt{b^2 - 4ac})$. For any of the solutions to be real and positive, it is necessary that

$$b^2 - 4ac > 0 \quad \text{and} \quad b < 0.$$

After some computations, we see that

$$\begin{aligned} &b^2 - 4ac > 0 \iff \beta_{12}\beta_{21} > \beta_{11}\beta_{22}, \\ &b < 0 \iff \beta_{11}\beta_{22} > 2\beta_{12}\beta_{21}, \end{aligned}$$

which are incompatible. In fact, the first condition is $\det(B) < 0$, which contradicts the stability assumption (2.16) for the dynamical system. Thus, in the case of linear self-diffusion, $\det(A_k) > 0$ for all k, and therefore, $\mathrm{Re}(\lambda_k)$ is never positive. In conclusion, $\tilde{\mathbf{u}}$ is linearly stable for any choice of the diffusion coefficients.

2 Cross-diffusion problem

Since the Lotka-Volterra system with linear self-diffusion is stable under perturbations of the co-existence equilibrium, one may ask if this is still the case for more complex situations. In this section, we study the case in which cross-diffusion terms of the SKT[1] type are considered,

$$\partial_t u_1 - \operatorname{div} J_1(\mathbf{u}) = \gamma f_1(u_1,u_2) \quad \text{in } Q_T, \tag{2.17}$$
$$\partial_t u_2 - \operatorname{div} J_2(\mathbf{u}) = \gamma f_2(u_1,u_2) \quad \text{in } Q_T, \tag{2.18}$$
$$J_1(\mathbf{u})\cdot n = J_2(\mathbf{u})\cdot n = 0 \quad \text{on } \Gamma_T, \tag{2.19}$$
$$u_1(\cdot,0) = u_{10}, \quad u_2(\cdot,0) = u_{20} \quad \text{in } \Omega. \tag{2.20}$$

with flows and reaction terms given by

$$J_i(\mathbf{u}) = \nabla\big(u_i(d_1 + a_{i1}\nabla u_1 + a_{i2}\nabla u_2)\big)$$
$$f_i(\mathbf{u}) = \alpha_i u_i - (\beta_{i1}u_1 + \beta_{i2}u_2)u_i,$$

for $\alpha_i, \beta_{ij} \geq 0$, $i,j = 1,2$. The diffusion coefficients d_i, and $A = (a_{ij})$ are assumed to be non-negative, with $a_{ii} > 0$, whereas the inter- and intra-competitive coefficients $B = (\beta_{ij})$ are assumed to satisfy the kinetic stability conditions (2.16).

We consider the co-existence homogeneous stationary state given by (2.14), for which, see (2.15),

$$K := D\mathbf{f}(\tilde{\mathbf{u}}) = \begin{pmatrix} -\beta_{11}\tilde{u}_1 & -\beta_{12}\tilde{u}_1 \\ -\beta_{21}\tilde{u}_2 & -\beta_{22}\tilde{u}_2 \end{pmatrix}.$$

Linearization around $\tilde{\mathbf{u}}$ gives the following system for $\mathbf{w} = \mathbf{u} - \tilde{\mathbf{u}}$

$$\partial_t \mathbf{w} - D\Delta\mathbf{w} = \gamma K\mathbf{w},$$

with

$$D = \begin{pmatrix} d_1 + 2a_{11}\tilde{u}_1 + a_{12}\tilde{u}_2 & a_{12}\tilde{u}_1 \\ a_{21}\tilde{u}_2 & d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2 \end{pmatrix}. \tag{2.21}$$

The corresponding eigenvalue problem leads to the characteristic polynomial

$$\lambda_k^2 - \operatorname{tr}(A_k)\lambda_k + h(k^2) = 0,$$

with $A_k = \gamma K - k^2 D$, and

$$h(k^2) = \det(A_k) = \det(D)k^4 + \gamma q k^2 + \gamma^2 \det(K),$$

being

$$q = \beta_{11}\tilde{u}_1(2a_{22}\tilde{u}_2 + d_2) + \beta_{22}\tilde{u}_2(2a_{11}\tilde{u}_1 + d_1) + a_{12}\tilde{u}_2(\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1) + a_{21}\tilde{u}_1(\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2). \tag{2.22}$$

[1] Shigesada-Kawasaki-Teramoto

2.1 Conditions for linear instability

Spatial patterns arise in correspondence of those modes k for which $\mathrm{Re}(\lambda_k) > 0$. Since we assume that $\tilde{\mathbf{u}}$ is stable for the kinetics, one has that $\mathrm{tr}(K) < 0$, and thus $\mathrm{tr}(A_k) < 0$ implying that the only way to have $\mathrm{Re}(\lambda_k) > 0$ for some $k \neq 0$ is that $h(k^2)$ becomes negative. Thus, the condition for the marginal stability at some $k = k_c$ is

$$\min(h(k_c^2)) = 0. \tag{2.23}$$

The minimum of h is attained for

$$k_c^2 = -\frac{\gamma q}{2\det(D)}, \tag{2.24}$$

which requires $q < 0$. The first two terms of q are non-negative: it follows that the only potential destabilizing mechanism is the presence of the cross-diffusion terms.

Election of the bifurcation parameter

The conditions on the positiveness and stability of the equilibrium point $\tilde{\mathbf{u}}$ imply that only one of the two following inequalities can be satisfied

$$\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1 < 0 \quad \text{or} \quad \beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2 < 0.$$

Indeed, using the explicit definitiion of $\tilde{\mathbf{u}}$ given in (2.14), we get, for $s = \beta_{11}\beta_{22} + \beta_{12}\beta_{21}$,

$$\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1 = \frac{1}{\det(B)}(\alpha_2 s - 2\alpha_1\beta_{21}\beta_{22}),$$
$$\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2 = \frac{1}{\det(B)}(\alpha_1 s - 2\alpha_2\beta_{11}\beta_{12}).$$

Assuming that both are negative, we deduce

$$\frac{\alpha_2}{\alpha_1} < \frac{2\beta_{21}\beta_{22}}{s} \quad \text{and} \quad \frac{\alpha_2}{\alpha_1} > \frac{s}{2\beta_{11}\beta_{12}},$$

implying $s^2 < 4\beta_{11}\beta_{12}\beta_{21}\beta_{22}$, which is not possible.

Therefore when a_{12} has a destabilizing effect then a_{21} acts as a stabilizer and vice versa. In what follows we shall choose the case $\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1 < 0$ without loss of generality and

$$b := a_{12} \quad \text{as the bifurcation parameter.}$$

Critical value of the bifurcation parameter

Since the graph of $h(k^2)$ depends on b, from (2.23) one gets the bifurcation value of b and the corresponding value of k_c^2, if they do exist. Consider the non-negative quantities

$$m_1 = \tilde{u}_2(\beta_{21}\tilde{u}_1 - \beta_{22}\tilde{u}_2),$$
$$m_2 = \beta_{11}\tilde{u}_1(2a_{22}\tilde{u}_2 + d_2) + \beta_{22}\tilde{u}_2(2a_{11}\tilde{u}_1 + d_1) + a_{21}\tilde{u}_1(\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2),$$

so that $q = -m_1 b + m_2$. Using (2.24), we see that the minimum value of $h(k^2)$ is

$$\min(h(k_c^2)) = \gamma^2 \Big(\det(K) - \frac{(-m_1 b + m_2)^2}{4\det(D)} \Big). \tag{2.25}$$

Let $\xi \in \mathbb{R}$, to be determined, and set $b = m_2/m_1 + \xi$. Introducing this expression in (2.25), we get the marginal stability condition

$$\frac{m_1^2}{4\det(K)} \xi^2 - \det(D) = 0. \tag{2.26}$$

Replacing $a_{12} \equiv b = m_2/m_1 + \xi$ in (2.21), we find

$$\det(D) = \tilde{u}_2(d_2 + 2a_{22}\tilde{u}_2)\xi + \Big(\frac{m_2}{m_1} \tilde{u}_2(d_2 + 2a_{22}\tilde{u}_2) + (d_1 + 2a_{11}\tilde{u}_1)(d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2) \Big).$$

Therefore, the second order polynomial (2.26) admits a positive root, which we denote by ξ^+, giving the critical value for the bifurcation parameter b,

$$b^c = \frac{m_2}{m_1} + \xi^+. \tag{2.27}$$

Observe that, under the election of the positive root ξ^+, the condition $q < 0$ is guaranteed.

Thus, for $b > b^c$ the system has a finite k pattern-forming stationary instability. The unstable wavenumbers stay in between the roots of $h(k^2)$, denoted by k_1^2 and k_2^2. It is straightforward to check that these roots are proportional to γ. Hence, to have the possibility of the pattern formation, γ must be big enough so that at least one of the modes allowed by the boundary conditions falls within the interval $[k_1^2, k_2^2]$.

2.2 Amplitude equations and weakly nonlinear analysis

The linear stability theory represents a useful first step in understanding pattern formation, but it gives only a rough indication of the patterns we should expect. Through the linear analysis we determine both the conditions on the system parameters for the onset of instability to infinitesimal disturbances and the length scale of the pattern formation, $1/k_c$.

Moreover the linear analysis displays the important physical processes and shows how the diffusion is the key mechanism for pattern formation.

Nevertheless, the exponentially growing solutions obtained via the linear theory are physically meaningless. To predict the amplitude and the form of the pattern close to the threshold the nonlinear terms must be included into the analysis. We shall perform a weakly nonlinear analysis based on the method of multiple scales.

Nonlinear expansion

In the Turing bifurcation, close to the bifurcation the eigenvalues are negative. Thus, the linear instability of the steady state must be preceded by the presence of a null eigenvalue for the linearized operator. In particular, this implies that the pattern evolves on a slow temporal scale, like $e^{\lambda t}$, with $\lambda \approx 0$. Thus, new scaled coordinates are considered, and treated as separate variables.

Firstly, we fix a small control parameter ε, representing the dimensionless distance from the critical threshold. Here we choose $\varepsilon^2 = (b - b_c)/b_c$. Secondly, the solution of the original system (2.5) is written as a weakly nonlinear expansion in terms of ε^2.

Considering a random perturbation, $\mathbf{w}$, around the steady state, we can recast the original system (2.17)-(2.18) in the form

$$\partial_t \mathbf{w} = \mathcal{L}^b \mathbf{w} + \mathcal{N}^b \mathbf{w}, \tag{2.28}$$

where $\mathcal{L}^b = \gamma K + D^b \Delta$, and $\mathcal{N}^b$ is the nonlinear operator containing the second order terms, which we decompose as

$$\mathcal{N}^b = \frac{1}{2} Q_K(\mathbf{w},\mathbf{w}) + \frac{1}{2} \Delta Q_D^b(\mathbf{w},\mathbf{w}),$$

with the bilinear forms

$$Q_K(\mathbf{x},\mathbf{y}) = \gamma \begin{pmatrix} -2\beta_{11} x^1 y^1 - \beta_{12}(x^1 y^2 + x^2 y^1) \\ -2\beta_{22} x^2 y^2 - \beta_{21}(x^1 y^2 + x^2 y^1) \end{pmatrix},$$

$$Q_D^b(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 2a_{11} x^1 y^1 + b(x^1 y^2 + x^2 y^1) \\ 2a_{22} x^2 y^2 + a_{21}(x^1 y^2 + x^2 y^1) \end{pmatrix}.$$

The idea is to expand the perturbation $\mathbf{w}$ in terms of the control parameter ε, so that the leading term of the expansion is the product of a slowly varying amplitude, A, and a basic pattern with wavenumber k_c, to derive an equation describing the evolution of the amplitude. We expand b, $\mathbf{w}$, and the time variable as

$$\begin{aligned} b &= b^c + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 b_3 + O(\varepsilon^4), \\ \mathbf{w} &= \varepsilon \mathbf{w}_1 + \varepsilon^2 \mathbf{w}_2 + \varepsilon^3 \mathbf{w}_3 + O(\varepsilon^4), \\ \partial_t &= \varepsilon \partial_{T_1} + \varepsilon^2 \partial_{T_2} + \varepsilon^3 \partial_{T_3} + O(\varepsilon^4). \end{aligned}$$

Then, we decompose the diffusion matrix, D^b, as

$$D^b = \begin{pmatrix} d_1 + 2a_{11}\tilde{u}_1 + b\tilde{u}_2 & b\tilde{u}_1 \\ a_{21}\tilde{u}_2 & d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2 \end{pmatrix} = D^{b^c} + \sum_{j=1}^{3} \varepsilon^j \begin{pmatrix} b_j \tilde{u}_2 & b_j \tilde{u}_1 \\ 0 & 0 \end{pmatrix} + O(\varepsilon^4),$$

so that $\mathcal{L}^b = \gamma K + D^b \Delta$ takes the form

$$\mathcal{L}^b = \mathcal{L}^{b^c} + \sum_{j=1}^{3} \varepsilon^j \begin{pmatrix} b_j \tilde{u}_2 & b_j \tilde{u}_1 \\ 0 & 0 \end{pmatrix} \Delta + O(\varepsilon^4), \quad \text{with } \mathcal{L}^{b^c} = \gamma K + D^{b^c} \Delta.$$

For the quadratic terms, we have

$$\begin{aligned} Q_K(\mathbf{w},\mathbf{w}) &= \varepsilon^2 Q_K(\mathbf{w}_1,\mathbf{w}_1) + 2\varepsilon^3 Q_K(\mathbf{w}_1,\mathbf{w}_2) + O(\varepsilon^4), \\ Q_D^b(\mathbf{w},\mathbf{w}) &= \varepsilon^2 Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_1) + 2\varepsilon^3 \left(Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) + (b_1 w_1^1 w_1^2, 0)^t \right) + O(\varepsilon^4). \end{aligned}$$

Finally, considering the time derivative expansion, we get

$$\partial_t \mathbf{w} = \varepsilon^2 \partial_{T_1} \mathbf{w}_1 + \varepsilon^3 (\partial_{T_1} \mathbf{w}_2 + \partial_{T_2} \mathbf{w}_1) + O(\varepsilon^4).$$

Introducing these expansions in (2.28), and equating in terms of the order of ε, leads to

$$O(\varepsilon): \quad \mathcal{L}^{b^c}\mathbf{w}_1 = 0, \tag{2.29}$$

$$O(\varepsilon^2): \quad \mathcal{L}^{b^c}\mathbf{w}_2 = \partial_{T_1}\mathbf{w}_1 - \frac{1}{2}\left(Q_K(\mathbf{w}_1,\mathbf{w}_1) + \Delta Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_1)\right) - b_1\begin{pmatrix}\tilde{u}_2 & \tilde{u}_1\\ 0 & 0\end{pmatrix}\Delta\mathbf{w}_1 =: \mathbf{F}, \tag{2.30}$$

$$O(\varepsilon^3): \quad \mathcal{L}^{b^c}\mathbf{w}_3 = \partial_{T_1}\mathbf{w}_2 + \partial_{T_2}\mathbf{w}_1 - Q_K(\mathbf{w}_1,\mathbf{w}_2) - \Delta Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) - b_1\Delta\begin{pmatrix}w_1^1 w_1^2\\ 0\end{pmatrix} - \begin{pmatrix}\tilde{u}_2 & \tilde{u}_1\\ 0 & 0\end{pmatrix}(b_1\Delta\mathbf{w}_2 + b_2\Delta\mathbf{w}_1) =: \mathbf{G}. \tag{2.31}$$

Studying the orders of the expansion

Order ε: The solution of the linear problem (2.29) in $(0, 2\pi/k_c)$ satisfying the Neumann boundary conditions is given by

$$\mathbf{w}_1 = A(T_1,T_2)\rho\cos(k_c x), \quad \text{with } \rho \in \ker(\gamma K - k_c^2 D^{b^c}),$$

where A is the amplitude of the pattern and it is still arbitrary at this level. The vector ρ is defined up to a multiplicative constant, and we shall make the normalization in the following way

$$\rho = (1, M)^t, \quad \text{with } M = \frac{-\gamma K_{21} + D_{21}^{b^c} k_c^2}{\gamma K_{22} - D_{22}^{b^c} k_c^2}, \tag{2.32}$$

where $K_{ij}, D_{ij}^{b^c}$ are the i,j-entries of the matrices K and D^{b^c}.

Order ε^2: Observing that

$$Q_K(\mathbf{w}_1,\mathbf{w}_1) = A(T_1,T_2)^2 Q_K(\rho,\rho)\cos^2(k_c x)$$
$$Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_1) = A(T_1,T_2)^2 Q_D^{b^c}(\rho,\rho)\cos^2(k_c x),$$

and using standard trigonometric identities, we find that (mind the Laplacian operator)

$$\begin{aligned}\frac{1}{2}\left(Q_K(\mathbf{w}_1,\mathbf{w}_1) + \Delta Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_1)\right) &= \frac{1}{4}A(T_1,T_2)^2\Big(Q_K(\rho,\rho) + \left(Q_K(\rho,\rho) - 4k_c^2 Q_D^{b^c}(\rho,\rho)\right)\cos(2k_c x)\Big)\\ &= \frac{1}{4}A(T_1,T_2)^2\sum_{j=0,2}\mathcal{M}_j(\rho,\rho)\cos(jk_c x),\end{aligned}$$

with

$$\mathcal{M}_j = Q_K - j^2 k_c^2 Q_D^{b^c}. \tag{2.33}$$

Therefore, $\mathbf{F}$ given in (2.30) may be expressed as

$$\mathbf{F} = -\frac{1}{4}A(T_1,T_2)^2\sum_{j=0,2}\mathcal{M}_j(\rho,\rho)\cos(jk_c x) + \Big(\partial_{T_1}A_1\rho + b_1 k_c^2 A_1(\tilde{u}_2 + \tilde{u}_1 M, 0)^t\Big)\cos(k_c x).$$

By the Fredholm alternative, (2.30) admits a solution if and only if $\langle \mathbf{F}, \psi \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(0, 2\pi/k_c)$, and the normalized vector $\psi \in \ker((\mathcal{L}^{b^c})^*)$ is given by

$$\psi = (1, M^*)^t \cos(k_c x), \quad \text{with } M^* = \frac{-\gamma K_{12} + D_{12}^{b^c} k_c^2}{\gamma K_{22} - D_{22}^{b^c} k_c^2}. \tag{2.34}$$

The compatibility condition reads

$$0 = \langle \mathbf{F}, \psi \rangle = -\frac{1}{4} A^2 \sum_{j=0,2} \mathcal{M}_j(\rho, \rho)(1, M^*)^t \int_0^{\frac{2\pi}{k_c}} \cos(jk_c x)\cos(k_c x) dx$$
$$+ \left(\partial_{T_1} A \rho + b_1 k_c^2 A (\tilde{u}_2 + \tilde{u}_1 M, 0)^t \right)(1, M^*)^t \int_0^{\frac{2\pi}{k_c}} \cos^2(k_c x) dx.$$

The first integrand at the right hand side vanishes, so we obtain

$$\partial_{T_1} A(T_1, T_2) = \chi A(T_1, T_2), \quad \text{with } \chi = -\frac{b_1 k_c (\tilde{u}_2 + \tilde{u}_1 M)}{1 + MM^*},$$

which does not give any indication on the asymptotic behavior of the pattern amplitude. Therefore, to suppress the secular terms appearing in $\mathbf{F}$, one imposes $T_1 = 0$ and $b_1 = 0$, and then the compatibility condition is automatically satisfied. With this choice, we have that $\mathbf{F}$ reduces to

$$\mathbf{F} = -\frac{1}{4} A(T_2)^2 \sum_{j=0,2} \mathcal{M}_j(\rho, \rho) \cos(jk_c x), \tag{2.35}$$

and the solution to (2.30) is then explicitly computed in terms of the parameters of the full system. Writing

$$\mathbf{w}_2 = A(T_2)^2 \sum_{j=0,2} \mathbf{w}_{2j} \cos(jk_c x),$$

we get

$$\mathcal{L}^{b^c} \mathbf{w}_2 = (\gamma K + D^{b^c} \Delta) \mathbf{w}_2 = A(T_2)^2 \sum_{j=0,2} \left(\gamma K - (jk_c)^2 D^{b^c} \right) \mathbf{w}_{2j} \cos(jk_c x),$$

and then for $\mathbf{w}_2$ to satisfy $\mathcal{L}^{b^c} \mathbf{w}_2 = \mathbf{F}$, with $\mathbf{F}$ given by (2.35), the vectors $\mathbf{w}_{2j}$ must satisfy the following linear systems

$$L_j \mathbf{w}_{2j} = -\frac{1}{4} \mathcal{M}_j(\rho, \rho), \quad \text{for } j = 0, 2, \tag{2.36}$$

with $L_j = \gamma K - j^2 k_c^2 D^{b^c}$.

Order ε^3: Since $T_1 = b_1 = 0$, we have that $\mathbf{G}$ given in problem (2.31) reduces to

$$\mathbf{G} = \partial_{T_2} \mathbf{w}_1 - Q_K(\mathbf{w}_1, \mathbf{w}_2) - \Delta Q_D^{b^c}(\mathbf{w}_1, \mathbf{w}_2) - \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} b_2 \Delta \mathbf{w}_1, \tag{2.37}$$

where we recall

$$\mathbf{w}_1 = A(T_2) \rho \cos(k_c x),$$
$$\mathbf{w}_2 = A(T_2)^2 (\mathbf{w}_{20} + \mathbf{w}_{22} \cos(2k_c x)).$$

We easily see that the first and the last terms of (2.37) may be expressed as

$$\partial_{T_2}\mathbf{w}_1 = \rho\cos(k_c x)\partial_{T_2}A(T_2), \tag{2.38}$$

$$-\begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} b_2\Delta\mathbf{w}_1 = A(T_2)k_c^2\cos(k_c x)b_2(\tilde{u}_2+\tilde{u}_1 M,0)^t. \tag{2.39}$$

Using that Q_K and $Q_D^{b^c}$ are bilinear, and the trigonometric identity $2\cos(x)\cos(y) = \cos(x+y)+\cos(x-y)$, we get

$$\begin{aligned} Q_K(\mathbf{w}_1,\mathbf{w}_2) &= A(T_2)^2 Q_K(\mathbf{w}_1,\mathbf{w}_{20}) + A(T_2)^2\cos(2k_c x)Q_K(\mathbf{w}_1,\mathbf{w}_{22}) \\ &= A(T_2)^3\cos(k_c x)Q_K(\rho,\mathbf{w}_{20}) + A(T_2)^3\cos(2k_c x)\cos(k_c x)Q_K(\rho,\mathbf{w}_{22}) \\ &= A(T_2)^3\Big(\cos(k_c x)\big(Q_K(\rho,\mathbf{w}_{20})+\frac{1}{2}Q_K(\rho,\mathbf{w}_{22})\big)+\frac{1}{2}\cos(3k_c x)Q_K(\rho,\mathbf{w}_{22})\Big), \end{aligned}$$

and similarly

$$\Delta Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) = A(T_2)^3\Big(-k_c^2\cos(k_c x)\big(Q_D^{b^c}(\rho,\mathbf{w}_{20})+\frac{1}{2}Q_D^{b^c}(\rho,\mathbf{w}_{22})\big)-\frac{9}{2}k_c^2\cos(3k_c x)Q_D^{b^c}(\rho,\mathbf{w}_{22})\Big).$$

Recalling the definition of $\mathcal{M}$, see (2.33), we get

$$\begin{aligned} Q_K(\mathbf{w}_1,\mathbf{w}_2)+\Delta Q_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) =& A(T_2)^3\Big(\cos(k_c x)\big(\mathcal{M}_1(\rho,\mathbf{w}_{20})+\frac{1}{2}\mathcal{M}_1(\rho,\mathbf{w}_{22})\big) \\ &+\frac{1}{2}\cos(3k_c x)\mathcal{M}_3(\rho,\mathbf{w}_{22})\Big). \end{aligned} \tag{2.40}$$

Thus, from (2.37), and gathering (2.38), (2.39) and (2.40), we obtain

$$\mathbf{G} = \Big(\rho\partial_{T_2}A + \mathbf{G}_1^{(1)}A + \mathbf{G}_1^{(3)}A^3\Big)\cos(k_c x) + \mathbf{G}_3 A^3\cos(3k_c x),$$

with

$$\begin{aligned} \mathbf{G}_1^{(1)} &= (\tilde{u}_2+\tilde{u}_1 M)k_c^2 b_2(1,0)^t, \\ \mathbf{G}_1^{(3)} &= -\big(\mathcal{M}_1(\rho,\mathbf{w}_{20})+\frac{1}{2}\mathcal{M}_1(\rho,\mathbf{w}_{22})\big), \\ \mathbf{G}_3 &= -\frac{1}{2}\mathcal{M}_3(\rho,\mathbf{w}_{22}). \end{aligned}$$

The solvability condition for problem (2.31) is $\langle\mathbf{G},\psi\rangle = 0$, with ψ given by (2.34). This condition leads to

$$\langle\rho\cos(k_c x),\psi\rangle\partial_{T_2}A + \langle\mathbf{G}_1^{(1)}\cos(k_c x),\psi\rangle A + \langle\mathbf{G}_1^{(3)}\cos(k_c x),\psi\rangle A^3 = 0.$$

Thus, recalling the definition of ψ in (2.34), and for $\eta = (1,M^*)^t$,

$$\sigma = \frac{\mathbf{G}_1^{(1)}\cdot\eta}{\rho\cdot\eta}, \qquad L = \frac{\mathbf{G}_3^{(1)}\cdot\eta}{\rho\cdot\eta}, \tag{2.41}$$

we write the resulting Stuart-Landau equation as

$$\partial_{T_2}A = \sigma A - LA^3. \tag{2.42}$$

Since the growth rate coefficient σ is always positive, the dynamics of the Stuart-Landau equation (2.42) can be divided into two qualitatively different cases depending on the sign of the Landau constant L: the supercritical case, when L is positive, and the subcritical case, when L is negative.

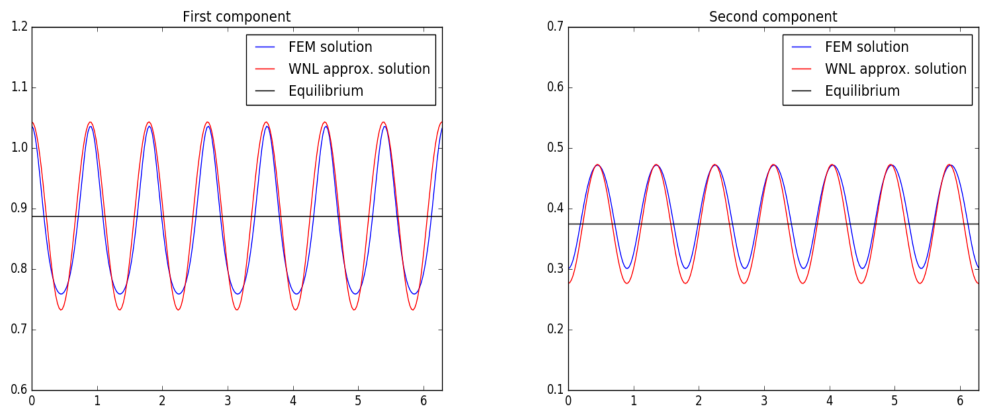
2.3 The supercritical case

If the coefficients σ and L, appearing into (2.42), are both positive, then there exists the stable equilibrium solution $A_\infty = \sqrt{\sigma/L}$, which represents the asymptotic value of the amplitude A. Therefore, we are now able to predict the amplitude and the form of the pattern. According to the weakly nonlinear theory the asymptotic (in time) behavior of the solution $\mathbf{w} = \varepsilon\mathbf{w}_1 + \varepsilon^2\mathbf{w}_2 + O(\varepsilon^3)$, is given by

$$\widetilde{\mathbf{w}} = \varepsilon\rho\sqrt{\frac{\sigma}{L}}\cos(k_c x) + \varepsilon^2\frac{\sigma}{L}\big(\mathbf{w}_{20} + \mathbf{w}_{22}\cos(2k_c x)\big) + O(\varepsilon^3).$$

In the above expression ρ is given in (2.32), while the $\mathbf{w}_{2j}$ are the solutions of the systems (2.36). Clearly, in general, the above solution is not compatible with the Neumann boundary conditions, that require k_c to be integer or semi-integer. We therefore define $\bar{k}_c$ as the first integer or semi-integer to become unstable when b passes the critical value b_c, and take as the weakly nonlinear approximation the following expression

$$\widetilde{\mathbf{w}} = \varepsilon\rho\sqrt{\frac{\sigma}{L}}\cos(\bar{k}_c x) + \varepsilon^2\frac{\sigma}{L}\big(\mathbf{w}_{20} + \mathbf{w}_{22}\cos(2\bar{k}_c x)\big) + O(\varepsilon^3). \tag{2.43}$$



What to compute

We want to compute

$$\widetilde{\mathbf{w}} = \varepsilon\rho\sqrt{\frac{\sigma}{L}}\cos(k_c x) + \varepsilon^2\frac{\sigma}{L}\big(\mathbf{w}_{20} + \mathbf{w}_{22}\cos(2k_c x)\big) + O(\varepsilon^3).$$

1. We start determining the equilibrium solution $\tilde{\mathbf{u}}$ given by (2.14).
2. b^c: Then, we compute the critical bifurcation value, b^c, given by 2.27. This is done by computing the positive root of the polynomial (2.26), whose coefficients only depend on the other diffusion coefficients, $B = (\beta_{ij})$, and $\tilde{\mathbf{u}}$.
3. We then evaluate $q \equiv q(b)$ given by (2.22), and $D \equiv D^b$, given by (2.21) at $a_{12} \equiv b = b^c$. From this and (2.24), we obtain k_c. Then, we define $\bar{k}_c$ as the first integer or semi-integer to become unstable when b passes the critical value b_c.

4. We determine ρ from (2.32), which only depends on γ, K, D^{b^c}, and k_c.

5. We determine $\mathbf{w}_{20}$ and $\mathbf{w}_{22}$ by solving the linear algebraic systems (2.36), that is

$$\gamma K \mathbf{w}_{20} = -\frac{1}{4} Q_K(\rho,\rho),$$
$$(\gamma K - 4k_c^2)\mathbf{w}_{22} = -\frac{1}{4} Q_K(\rho,\rho) + k_c^2 Q_D^{b^c}(\rho,\rho).$$

6. We determine σ from (2.41),

$$\sigma = \frac{\mathbf{G}_1^{(1)} \cdot \eta}{\rho \cdot \eta}.$$

Here, $\eta = (1, M^*)$, with M^* given in (2.34), and $\mathbf{G}_1^{(1)} = (\tilde{u}_2 + \tilde{u}_1 M) k_c^2 b_2 (1,0)^t$. Observe that, using the expansion of b and the definition of ε, we have

$$b = b^c + \frac{b - b^c}{b^c} b_2 + O(\varepsilon^3),$$

implying $b_2 = b^c + O(\varepsilon)$.

7. We determine L from (2.41). Here,

$$\mathbf{G}_1^{(3)} = -\left(\mathcal{M}_1(\rho, \mathbf{w}_{20}) + \frac{1}{2}\mathcal{M}_1(\rho, \mathbf{w}_{22})\right),$$

with

$$\mathcal{M}_1(\rho,\zeta) = Q_K(\rho,\zeta) - k_c^2 Q_D^{b^c}(\rho,\zeta).$$

Index