Nonlinear analysis tools for proving existence of weak solutions of cross-diffusion problems

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Objective: proving existence of weak solutions of evolution cross-diffusion problems of the Shigesada-Kawasaki-Teramoto (SKT) type.

Steps:

1. Linear scalar heat equation,
2. Nonlinear scalar reaction-diffusion problem,
3. Cross-diffusion problem.

Rules:

- the maximum principle can not be applied, and
- the starting point to construct a solution is the Lax-Milgram's lemma.

The problems are motivated by population dynamics, but the techniques apply to general evolution reaction-convection-diffusion problems.

Along the way, we recall well known results of functional analysis that provide us with powerful tools to tackle these problems.

The contents is partially extracted from:

1. G. Galiano, M. L. Garzón, A. Jüngel, Semi-discretization in time and numerical convergence of solutions of a nonlinear cross-diffusion population model, Numerische Mathematik 93 (2003) 655-673.
2. L. Chen, A. Jüngel, Analysis of a multidimensional parabolic population model with strong cross-diffusion, SIAM J. Mathematical Analysis, 36 (2004) 301-322.
3. G. Galiano, V. Selgas, On a cross-diffusion segregation problem arising from a model of interacting particles, Nonlinear Analysis: Real World Applications 18 (2014) 34-49.

Outline

1. **A linear population model**
   - Formal arguments
   - Time discretization
   - Back to the evolution problem
2. **A nonlinear population model**
   - Formal arguments
   - Time discretization
   - Back to the evolution problem
   - The limit $\tau \to 0$
   - The limit $\varepsilon \to 0$
3. **A cross-diffusion population model**
   - Formal estimates
   - Symmetrization
   - Solving a time discrete approximated symmetric problem
   - Back to the original unknowns
   - Back to the evolution problem
   - The limit $\tau \to 0$
   - The limit $\varepsilon \to 0$

Outline

Problem: Given $T > 0$ and bounded $\Omega \subset \mathbb{R}^N$, find (a non-negative) $u : (0, T) \times \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
&\partial_t u - \Delta u = u && \text{in } Q_T = (0, T) \times \Omega, \\
&\nabla u \cdot n = 0 && \text{on } \Gamma_T = \partial(0, T) \times \Omega, \\
&u(\cdot, 0) = u_0 \geq 0 && \text{in } \Omega.
\end{aligned}
$$

In terms of population dynamics,

- Population diffuses randomly.
- Newborns are proportional to the existent population, and no growth limit. The corresponding kinetics ($\partial_t u = u$) implies exponential growth.

First ingredient: energy estimate $\rightarrow$ notion of weak solution.

Suppose u is smooth solution. We get the energy identity

$$\frac{1}{2}\int_\Omega u(t)^2 + \int_{Q_t} |\nabla u|^2 = \frac{1}{2}\int_\Omega u_0^2 + \int_{Q_t} u^2.$$

Lemma (Gronwall's lemma)

Let $T > 0$, $a \in L^\infty(0,T)$, and $\lambda \in L^1(0,T)$, with $\lambda \geq 0$ in $(0,T)$. Suppose that, for $b \in C([0,T])$ increasing,

$$a(t) \leq b(t) + \int_0^t \lambda(s)a(s)ds \quad \text{a.e. in } (0,T).$$

Using Gronwall's lemma,

$$\int_\Omega u(t)^2 \leq e^{2t}\int_\Omega u_0^2, \quad \text{which implies} \quad \int_{Q_T} u^2 \leq Te^{2T}\int_\Omega u_0^2.$$

Therefore,

$$\|u\|_{L^\infty(L^2)} + \|\nabla u\|_{L^2} \leq C \implies \|u\|_{L^2(H^1)} \leq C.$$

Since

$$\partial_t u = -\operatorname{div}(\nabla u) + u \in L^2(0, T; (H^1(\Omega))'),$$

we don't expect $\partial_t u \in L^p(Q_T)$.

A generic definition of weak solution

$$\int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} \nabla u \cdot \nabla \varphi = \int_{Q_T} u\varphi, \quad \text{for all } \varphi \in V,$$

with V a space of test functions and $\langle \cdot, \cdot \rangle$ the duality product of $V' \times V$.

Formal arguments

Method of proof: Consider a sequence of approximating problems (P_n) such that $(P_n) \to (P)$ as $n \to \infty$.

Suppose the energy estimate is satisfied by u_n, slution of

$$\int_0^T \langle \partial_t u_n, \varphi \rangle + \int_{Q_T} \nabla u_n \cdot \nabla \varphi = \int_{Q_T} u_n \varphi, \quad \text{for all } \varphi \in V, \qquad (P_n)$$

that is

$$\|u_n\|_{L^\infty(L^2)} + \|u_n\|_{L^2(H^1)} \leq C.$$

Then, there exists a subsequence of u_n (that we do not relabel) and $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that

$$u_n \rightharpoonup u \quad \text{weakly*-weakly in } L^\infty(0,T;L^2(\Omega)),$$
$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(Q_T).$$

Then, for all $\varphi \in V \subset L^2(0,T;H^1(\Omega))$,

$$\int_{Q_T} \nabla u_n \cdot \nabla \varphi \to \int_{Q_T} \nabla u \cdot \nabla \varphi,$$
$$\int_{Q_T} u_n \varphi \to \int_{Q_T} u \varphi.$$

Second ingredient: Estimate for the time derivative.

Definition

Let V be a normed space, and $\psi : V \to \mathbb{R}$ be a linear functional. Then *the norm of* ψ *on the dual space* V' *of* V is defined by

$$\|\psi\|_{V'} = \sup_{x \in V} \frac{\langle \psi, x \rangle_{V' \times V}}{\|x\|_V}.$$

Fix $V = L^2(0, T; H^1(\Omega))$,

$$\int_0^T \langle \partial_t u_n, \varphi \rangle \leq \int_{Q_T} |\nabla u_n||\nabla \varphi| + \int_{Q_T} |u_n||\varphi|$$
$$\leq \|\nabla u_n\|_{L^2} \|\nabla \varphi\|_{L^2} + \|u_n\|_{L^2} \|\varphi\|_{L^2} \leq C \|\varphi\|_{L^2(H^1)}.$$

Thus $\|\partial_t u_n\|_{L^2((H^1)')} \leq C$, implying

$$\partial_t u_n \rightharpoonup z \quad \text{weakly in } L^2(0, T; (H^1(\Omega))').$$

Identification $z=\partial_t u$:

For $\psi\in C_c^\infty(0,T;H^1(\Omega))$, dense in $L^2(0,T;H^1(\Omega))$,

$$\int_0^T \langle \partial_t u_n,\psi\rangle \to \int_0^T \langle z,\psi\rangle,$$

Using the weak convergence in e.g. $L^2(Q_T)$

$$\int_0^T \langle \partial_t u_n,\psi\rangle = -\int_0^T \langle u_n,\partial_t\psi\rangle = -\int_0^T\int_\Omega u_n\partial_t\psi \to -\int_0^T\int_\Omega u\partial_t\psi = \int_0^T \langle \partial_t u,\psi\rangle,$$

and, density plus uniqueness of the limit

$$\int_0^T \langle z,\varphi\rangle = \int_0^T \langle \partial_t u,\varphi\rangle,$$

for all $\varphi\in L^2(0,T;H^1(\Omega))$. That is, $z=\partial_t u$.

Therefore, u_n solution of (P_n) converges to u solution of (P).

Sense of initial data.

For $\psi \in C^\infty(Q_T)$

$$\int_0^T \langle \partial_t(u-u_0), \psi\rangle = -\int_{Q_T}(u-u_0)\partial_t\psi + \int_\Omega (u(T)-u_0)\psi(T) - \int_\Omega (u(0)-u_0)\psi(0).$$

Choosing $\psi \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, with $\psi(T)=0$ the initial condition is satisfied in the sense

$$\int_0^T \langle \partial_t u, \psi\rangle + \int_{Q_T}(u-u_0)\partial_t\psi = 0.$$

Theorem (Sobolev's embedding theorem)

Let $\Omega \subset \mathbb{R}^N$ be bounded and of class C^1, and $1 \leq p \leq \infty$. The following injections are continuous:

- $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, *with* $p^* = Np/(N-p)$, *if* $p < N$,
- $W^{1,p}(\Omega) \subset L^q(\Omega)$, *for all* $1 \leq q < \infty$, *if* $p = N$,
- $W^{1,p}(\Omega) \subset C(\bar{\Omega})$, *if* $p > N$.

Thus, the injection $H^1(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))$ is continuous, so it makes sense to set $\psi(T) = 0$ in Ω.

If the solution is more regular, say $\partial_t u \in L^2(Q_T)$, then

$$0 = \int_{Q_T} \partial_t u \psi + \int_{Q_T} (u - u_0)\partial_t \psi = -\int_\Omega u(0)\psi(0) - \int_{Q_T} u_0 \partial_t \psi = \int_\Omega (u_0 - u(0))\psi(0)$$

for all $\psi(0) \in L^2(\Omega)$, implying $u(0) = u_0$ a.e. in Ω.

Time discretization

We start here the rigorous proof. Take

$$K \in \mathbb{N}, \quad \tau = T/K, \quad t_k = k\tau, \quad (0,T] = \cup_{k=0}^{K} (t_{k-1}, t_k].$$

Problem: Given $u^{k-1} \in L^2(\Omega)$, find $u^k : \Omega \to \mathbb{R}$ such that

$$\frac{1}{\tau}\int_\Omega (u^k - u^{k-1})\varphi + \int_\Omega \nabla u^k \cdot \nabla \varphi = \int_\Omega u^k \varphi \quad \text{for all } \varphi \in H^1(\Omega).$$

Lemma (Lax-Milgram)

Let H be a Hilbert space and assume that $A : H \times H \to \mathbb{R}$ is a continuous coercive bilinear form. Then, given any $F \in H'$, there exists a unique element $u \in H$ such that $A(u,\varphi) = \langle F, \varphi\rangle$ for all $\varphi \in H$.

Set $H = H^1(\Omega)$, and

$$A(u,\varphi) = \int_\Omega \nabla u \cdot \nabla \varphi + \frac{1}{\tau}\int_\Omega u\varphi, \qquad F = \frac{1}{\tau}u^{k-1} \in L^2(\Omega) \subset (H^1(\Omega))'$$

A is clearly continuous and coercive in $H^1(\Omega)$. Then, (Lax-Milgram) there exists a weak solution, $u^k \in H^1(\Omega)$ of the time discrete problem.

Uniform estimates.
Use $\varphi = u^k$ as test function

$$(1-\tau)\int_\Omega |u^k|^2 + \tau \int_\Omega |\nabla u^k|^2 = \int_\Omega u^{k-1}u^k.$$

Youngs' inequality gives

$$(\frac{1}{2}-\tau)\int_\Omega |u^k|^2 + \tau \int_\Omega |\nabla u^k|^2 \leq \frac{1}{2}\int_\Omega |u^{k-1}|^2.$$

Taking $\tau < 1/2$, and using $(1-r)^{-1} \leq \exp(r(1-r)^{-1})$ for all $r \in [0,1)$,

$$\int_\Omega |u^k|^2 \leq e^{4T}\int_\Omega |u_0|^2 \leq C.$$

Summing for $k = 1, \ldots, K$, we obtain

$$\frac{1}{2}\int_\Omega |u^K|^2 + \tau \sum_{k=1}^K \int_\Omega |\nabla u^k|^2 \le \frac{1}{2}\int_\Omega |u_0|^2 + \tau \sum_{k=1}^K \int_\Omega |u^k|^2,$$

and thus, using $K\tau = T$,

$$\tau \sum_{k=1}^K \int_\Omega |\nabla u^k|^2 \le \frac{1}{2}\int_\Omega |u_0|^2 + TC \le C.$$

Therefore

$$\max_{k=1,\ldots,K} \int_\Omega |u^k|^2 + \tau \sum_{k=1}^K \int_\Omega |\nabla u^k|^2 \le C.$$

Back to the evolution problem

Introduce piecewise constant and linear interpolators in time,

$$u^{(\tau)}(t,x) = u^k(x), \quad \tilde{u}^{(\tau)}(t,x) = u^k(x) + \frac{t_k - t}{\tau}(u^{k-1}(x) - u^k(x)),$$

for $(t,x) \in (t_{k-1}, t_k] \times \Omega$, for $k = 1, \ldots, K$. Then

$$\max_{t\in(0,T)} \int_\Omega |u^{(\tau)}|^2 + \int_{Q_T} |\nabla u^{(\tau)}|^2 \leq C,$$

and since $t_k - t < \tau$, we also deduce

$$\max_{t\in(0,T)} \int_\Omega |\tilde{u}^{(\tau)}|^2 + \int_{Q_T} |\nabla \tilde{u}^{(\tau)}|^2 \leq C.$$

Replacing $u^{(\tau)}$ and $\tilde{u}^{(\tau)}$ in the weak formulation we get

$$\int_{Q_T} \partial_t \tilde{u}^{(\tau)} \varphi + \int_{Q_T} \nabla u^{(\tau)} \cdot \nabla \varphi = \int_{Q_T} u^{(\tau)} \varphi \quad \text{for all } \varphi \in L^2(0,T;H^1(\Omega)).$$

From this identity and the above estimates, we obtain, like in the formal computation,

$$\|\partial_t \tilde{u}^{(\tau)}\|_{L^2((H^1)')} \leq C.$$

Therefore, we deduce the existence of $u, z \in L^2(0, T; H^1(\Omega))$ and subsequences of $u^{(\tau)}$ and $\tilde{u}^{(\tau)}$ such that

$$
\begin{aligned}
u^{(\tau)} &\rightharpoonup u && \text{weakly in } L^2(0, T; H^1(\Omega)),\\
u^{(\tau)} &\rightharpoonup u && \text{weakly*-weakly in } L^\infty(0, T; L^2(\Omega))),\\
\tilde{u}^{(\tau)} &\rightharpoonup z && \text{weakly in } L^2(0, T; H^1(\Omega)),\\
\tilde{u}^{(\tau)} &\rightharpoonup z && \text{weakly*-weakly in } L^\infty(0, T; L^2(\Omega))),\\
\partial_t \tilde{u}^{(\tau)} &\rightharpoonup \partial_t z && \text{weakly in } L^2(0, T; ((H^1(\Omega))').
\end{aligned}
$$

Identification $z = u$.

For $t \in (t_{k-1}, t_k]$,

$$|\tilde{u}^{(\tau)}(t,x) - u^{(\tau)}(t,x)| = |(t_k - t)\frac{u^{k-1}(x) - u^k(x)}{\tau}| \leq \tau|\partial_t \tilde{u}^{(\tau)}(t,x)|,$$

we deduce

$$\|\tilde{u}^{(\tau)} - u^{(\tau)}\|_{L^2((H^1)')} \leq \tau\|\partial_t \tilde{u}^{(\tau)}\|_{L^2((H^1)')} \to 0 \quad \text{as } \tau \to 0,$$

and hence $z = u$.

Therefore, we may pass to the limit $\tau \to 0$, obtaining

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz continuous boundary, and let $T > 0$. Suppose that $u_0 \in L^2(\Omega)$. Then, there exists $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$, such that, for all $\varphi \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T < \partial_t u, \varphi > + \int_{Q_T} \nabla u \cdot \nabla \varphi = \int_{Q_T} u\,\varphi,$$

with $< \cdot, \cdot >$ denoting the duality product between $H^1(\Omega)$ and its dual $(H^1(\Omega))'$. In addition, the initial data is satisfied in the sense

$$\int_0^T < \partial_t u, \psi > + \int_{Q_T} (u - u_0)\partial_t \psi = 0,$$

for all $\psi \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $\psi(T) = 0$ a.e. in Ω.

Outline

Problem: Find $u : (0, T) \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} &\partial_t u - \operatorname{div}(u\nabla u) = f(u) = u(\alpha - \beta u) && \text{in } Q_T, \\ &u\nabla u \cdot n = 0 && \text{on } \Gamma_T, \\ &u(\cdot, 0) = u_0 \geq 0 && \text{in } \Omega. \end{aligned}$$

In terms of population dynamics

- Population diffuses to avoid overcrowding (maxima of u).
- Newborns are proportional to the existent population. Growth limit in terms of the so-called *carrying capacity* of the habitat.
  The corresponding kinetics ($\partial_t u = f(u)$) has a stable equilibrium at $u = \alpha/\beta$.

Generic form of weak solution:

$$\int_{Q_T} \langle \partial_t u, \varphi \rangle + \int_{Q_T} u\nabla u \cdot \nabla \varphi = \int_{Q_T} f(u)\varphi, \quad \text{for all } \varphi \in V.$$

Formal arguments

Formal estimates:

- Using $\varphi = \ln(u)$ we get, for $F(s) = s(\ln(s) - 1) + 1 \geq 0$,

$$\int_\Omega F(u(T)) + \int_{Q_T} |\nabla u|^2 = \int_\Omega F(u_0) + \int_{Q_T} f(u)\ln(u).$$

The term $E(t) = \int_\Omega F(u(t))$ is called the *entropy* of the system. This identity only makes sense if $u > 0$.

- Using $\varphi = 1$ in we get (if $u \geq 0$)

$$\int_\Omega u(T) \leq \int_\Omega u_0 + \alpha \int_{Q_T} u,$$

and then Gronwall's lemma implies

$$\int_\Omega u(T) \leq e^{\alpha T} \int_\Omega u_0 \leq C.$$

Suppose that the RHS in the weak formulation may be controled by the LHS:

$$\max_{t\in(0,T)} \int_\Omega F(u(t)) + \|u\|_{L^\infty(L^1)} + \|\nabla u\|_{L^2} \leq C,$$

so $\|u\|_{L^2(H^1)} \leq C$ (we shall see later why).

Introduce again a sequence of aproximated problems (P_n), e.g.

$$\int_{Q_T} \partial_t u_n \, \varphi + \int_{Q_T} \psi_n(u_n)\nabla u_n \cdot \nabla\varphi = \int_{Q_T} f(u_n)\varphi, \quad \text{for all } \varphi \in V, \qquad (P_n)$$

with $\psi_n \to id$, and suppose u_n satisfies the above estimate. That is, $\|u_n\|_{L^2(H^1)} \leq C$. Then,

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(Q_T).$$

The gradient estimate is the first ingredient to prove the strong compactness of u_n in some L^p.

We need strong convergence in L^p, and a.e. convergence in Q_T to pass to the limit in the nonlinear terms.

The second ingredient is an estimate for the time derivative.

Lemma (Simon, Aubin-Lions)

Let X, B, and Y be Banach spaces with $X \subset B \subset Y$ such that

- *X is compactly embedded in B.*
- *B is continuously embedded in Y.*

Suppose that the sequence u_n satisfies:

- *u_n is bounded in $L^q(0,T;X) \cap L^1_{loc}(0,T;X)$, for $1 < q \leq \infty$.*
- *$\partial_t u_n$ is bounded in $L^1_{loc}(0,T;Y)$.*

Then, for all $p < q$, there exists a subsequence of u_n (not relabeled) and an element $u \in L^p(0,T;B)$ such that

$$u_n \to u \quad \text{strongly in } L^p(0,T;B) \text{ and a.e. in } Q_T.$$

A usual situation is: $X = H^1(\Omega)$, and $B = L^2(\Omega)$. Indeed,

Theorem (Rellich-Kondrachov)

Let $\Omega \subset \mathbb{R}^N$ be bounded and of class C^1, and $1 \leq p \leq \infty$. The following injections are ***compact****:*

- $W^{1,p}(\Omega) \subset L^q(\Omega)$, *for all* $1 \leq q < p^*$, *with* $p^* = Np/(N-p)$, *if* $p < N$,
- $W^{1,p}(\Omega) \subset L^q(\Omega)$, *for all* $p \leq q < \infty$, *if* $p = N$,
- $W^{1,p}(\Omega) \subset C(\bar{\Omega})$, *if* $p > N$.

Then we get that

$$\partial_t u_n \text{ bounded in } L^1(0,T;Y) \Longrightarrow u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.$$

Summarizing, if the RHS may be absorbed by LHS, and the time derivative estimate is available, we have

$$\begin{aligned} &\nabla u_n \rightharpoonup \nabla u && \text{weakly in } L^2(Q_T), \\ &\partial_t u_n \rightharpoonup \partial_t u && \text{weakly in } L^1(0,T;Y), \\ &u_n \to u && \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \end{aligned}$$

With these kind of estimates (and others), we have to justify the limits

$$\int_{Q_T} \langle \partial_t u_n, \varphi \rangle \to \int_{Q_T} \langle \partial_t u, \varphi \rangle,$$

$$\int_{Q_T} \psi_n(u_n) \nabla u_n \cdot \nabla \varphi \to \int_{Q_T} u \nabla u \cdot \nabla \varphi,$$

$$\int_{Q_T} (\alpha u_n - \beta u_n^2)\varphi = \int_{Q_T} f(u_n)\varphi \to \int_{Q_T} f(u)\varphi.$$

Time discretization

First (non-successful) attempt.

Nonlinear problem: Given $u^{k-1} \in V$, find $u^k : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} &\frac{1}{\tau}(u^k - u^{k-1}) - \operatorname{div}(u^k \nabla u^k) = f(u^k) && \text{in } \Omega, \\ &u^k \nabla u^k \cdot n = 0 && \text{on } \partial\Omega. \end{aligned}$$

Linear problem (for Lax-Milgram): Given $u^{k-1},\ v \in V$, find $u^k : \Omega \to \mathbb{R}$

$$\begin{aligned} &\frac{1}{\tau}(u^k - u^{k-1}) - \operatorname{div}(v \nabla u^k) = f(v) && \text{in } \Omega, \\ &v \nabla u^k \cdot n = 0 && \text{on } \partial\Omega. \end{aligned}$$

Like in the linear case, we would like to take $H = H^1(\Omega)$, and define

$$A(u, \varphi) = \int_\Omega v \nabla u \cdot \nabla \varphi + \frac{1}{\tau} \int_\Omega u \varphi.$$

However, $A(u, \varphi)$ is not coercive (v might vanish). And we need, for using $\varphi = \ln u$, to avoid $u = 0$. We adopt the following approximation.

Approximation of the linear problem

Regularized linear problem.
Let $\varepsilon > 0$. Given $u_\varepsilon^{k-1},\ v \in V$, find $u_\varepsilon^k : \Omega \to \mathbb{R}$

$$\frac{1}{\tau}(u_\varepsilon^k - u_\varepsilon^{k-1}) - \operatorname{div}(a_\varepsilon(v)\nabla u_\varepsilon^k)) = f_\varepsilon(v) \qquad \text{in } \Omega,$$
$$a_\varepsilon(v)\nabla u_\varepsilon^k \cdot n = 0 \qquad \text{on } \partial\Omega,$$

with $f_\varepsilon(s) = \alpha s - \beta a_\varepsilon(s)^2$. Here, $a_\varepsilon(s) \to s$, to which we impose

$$\varepsilon^{-1} \geq a_\varepsilon(s) \geq \varepsilon \quad \text{for all } s \in \mathbb{R}.$$

In weak form:

$$\frac{1}{\tau}\int_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})\varphi + \int_\Omega a_\varepsilon(v)\nabla u_\varepsilon^k \cdot \nabla\varphi = \int_\Omega f_\varepsilon(v)\varphi, \quad \text{for all } \varphi \in H^1(\Omega).$$

Now we can take $H = H^1(\Omega)$, $V = L^2(\Omega)$ in Lax-Milgram, and define

$$A_\varepsilon(u,\varphi) = \int_\Omega a_\varepsilon(v)\nabla u \cdot \nabla\varphi + \frac{1}{\tau}\int_\Omega u\varphi, \quad F = f_\varepsilon(v) + \frac{1}{\tau}u_\varepsilon^{k-1} \in L^2(\Omega).$$

A_ε is clearly continuous and coercive in $H^1(\Omega)$, and therefore there exists a weak solution $u_\varepsilon^k \in H^1(\Omega)$.

At this point, we reformulate the nonlinear time-discrete problem.
Given $u_\varepsilon^{k-1} \in L^2(\Omega)$, find $u^k : \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\frac{1}{\tau}(u_\varepsilon^k - u_\varepsilon^{k-1}) - \mathrm{div}(a_\varepsilon(u_\varepsilon^k)\nabla u_\varepsilon^k)) = f_\varepsilon(u_\varepsilon^k) && \text{in } \Omega,\\
&a_\varepsilon(u_\varepsilon^k)\nabla u_\varepsilon^k \cdot n = 0 && \text{on } \partial\Omega.
\end{aligned}$$

Now, observe the following inconvenient:

Assuming that we may use $\varphi = F'(u_\varepsilon^k) = \ln(u_\varepsilon^k)$, we obtain

$$\int_\Omega a_\varepsilon(u_\varepsilon^k) F''(u_\varepsilon^k)|\nabla u_\varepsilon^k|^2 = \int_\Omega \frac{a_\varepsilon(u_\varepsilon^k)}{u_\varepsilon^k}|\nabla u_\varepsilon^k|^2,$$

instead of the original formal identity

$$\int_\Omega uF''(u)|\nabla u|^2 = \int_\Omega |\nabla u|^2.$$

Thus, we also need to approximate F by a suitable F_ε which allows us to obtain an L^2 estimate of ∇u_ε.

The definition of a_ε and F_ε

For $\varepsilon > 0$, we want to produce approximations:

- a_ε such that $a_\varepsilon(s) \to s$ as $\varepsilon \to 0$, with $\varepsilon^{-1} \geq a_\varepsilon(s) \geq \varepsilon$ for all $s \in \mathbb{R}$.
- F_ε non-negative and smooth such that $F_\varepsilon(s) \to F(s) = s(\ln(s) - 1) + 1$, as $\varepsilon \to 0$.
- $a_\varepsilon(s) F_\varepsilon''(s) = 1$ for all $s \in \mathbb{R}$.

Let $a_\varepsilon : \mathbb{R} \to [\varepsilon, \varepsilon^{-1}]$ be given by the truncature function

$$a_\varepsilon(s) := \begin{cases} \varepsilon & \text{if } s \leq \varepsilon, \\ s & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}, \\ \varepsilon^{-1} & \text{if } \varepsilon^{-1} \leq s. \end{cases}$$

Using the third condition, we set $F_\varepsilon''(s) = 1/a_\varepsilon(s)$. Integrating and adjusting the integration constants for continuity, we get $F_\varepsilon \in C^{2,1}(\mathbb{R}, \mathbb{R}_+)$ given by

$$F_\varepsilon(s) := \begin{cases} \dfrac{s^2 - \varepsilon^2}{2\varepsilon} + s(\ln \varepsilon - 1) + 1 & \text{if } s \le \varepsilon, \\ s(\ln s - 1) + 1 & \text{if } \varepsilon \le s \le \varepsilon^{-1}, \\ \dfrac{\varepsilon(s^2 - \varepsilon^{-2})}{2} + s(\ln \varepsilon^{-1} - 1) + 1 & \text{if } \varepsilon^{-1} \le s, \end{cases}$$

with

$$F_\varepsilon'(s) := \begin{cases} \dfrac{s}{\varepsilon} + \ln \varepsilon - 1 & \text{if } s \le \varepsilon, \\ \ln s & \text{if } \varepsilon \le s \le \varepsilon^{-1}, \\ \varepsilon s + \ln \varepsilon^{-1} - 1 & \text{if } \varepsilon^{-1} \le s. \end{cases}$$

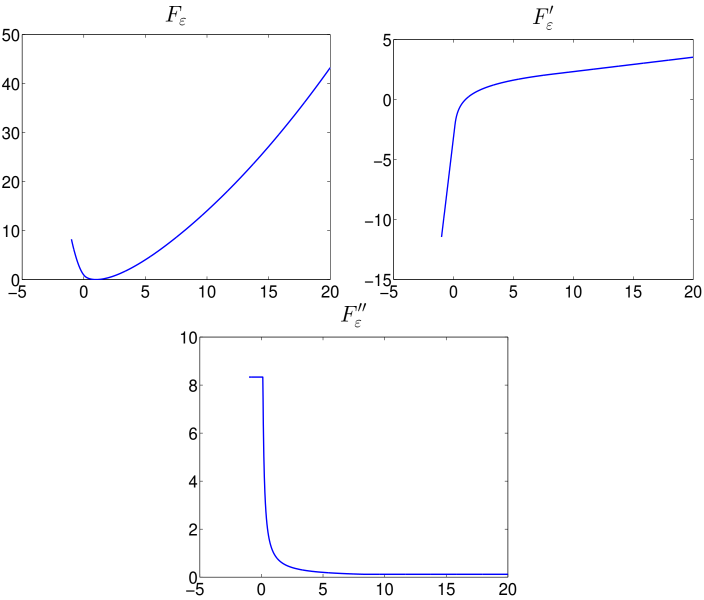

Figure : The convex function F_ε and its derivatives.

Fixed point method to couple the nonlinearities

Theorem (Leray-Schauder fixed point theorem)

Let V be a Banach space and let $S : V \times [0,1] \to V$ be a continuous and compact map such that

- $S(v,0) = 0$ *for all* $v \in V$.
- *For each pair* $(v,\sigma) \in V \times [0,1]$ *satisfying* $v = S(v,\sigma)$, *there exists a positive constant* C, *such that* $\|v\|_V \leq C$.

Then there exist a fixed point, $w \in V$, *of the map* $S(v,1)$, *i.e.* $w = S(w,1)$.

To solve the nonlinear time-discrete problem we define $S : L^2(\Omega) \times [0,1] \to L^2(\Omega)$ such that, for $u_\varepsilon^{k-1} \in L^2(\Omega)$ given,

$$(v,\sigma) \mapsto u_\varepsilon^{k,\sigma}$$

solution of

$$\begin{aligned} &\frac{1}{\tau} u_\varepsilon^{k,\sigma} - \operatorname{div}(a_\varepsilon(v)\nabla u_\varepsilon^{k,\sigma})) = \sigma\Big(f_\varepsilon(v) + \frac{1}{\tau} u_\varepsilon^{k-1}\Big) && \text{in } \Omega, \\ &a_\varepsilon(v)\nabla u_\varepsilon^{k,\sigma} \cdot n = 0 && \text{on } \partial\Omega. \end{aligned}$$

Lax-Milgram shows that there exists a unique solution $u_\varepsilon^{k,\sigma} \in H^1(\Omega)$ of the above problem. Thus, S is well defined.

To apply Leray-Schauder's theorem, we have to check the following:

1. Continuity: Let $v_n \in L^2(\Omega)$, $\sigma_n \in [0,1]$, with

$$v_n \to v \quad \text{strongly in } L^2(\Omega), \quad \sigma_n \to \sigma.$$

   Denote by $u^k_{\varepsilon,n}$ to the solution of the linear problem corresponding to (v_n, σ_n), that is $S(v_n, \sigma_n)$. We must check

$$u^k_{\varepsilon,n} \to u_\varepsilon^{k,\sigma} \quad \text{strongly in } L^2(\Omega), \quad \text{as } n \to \infty.$$

2. Compactness: Since we start with $v \in L^2(\Omega)$ and finish in $S(v,\sigma) = u_\varepsilon^{k,\sigma} \in H^1(\Omega)$, we deduce that S is compact.
3. $S(v,0) = 0$, which is inmmediate. Use $\varphi = u_\varepsilon^{k,\sigma}$ as test function.
4. If $v = S(v,\sigma)(= u_\varepsilon^{k,\sigma})$ for $(v,\sigma) \in L^2(\Omega) \times [0,1]$ then $\|u_\varepsilon^{k,\sigma}\|_{L^2} \leq C$.

We start proving the **continuity**. Using $\varphi = u^k_{\varepsilon,n} \in H^1(\Omega)$

$$\frac{1}{\tau}\int_\Omega |u^k_{\varepsilon,n}|^2 + \int_\Omega a_\varepsilon(v_n)|\nabla u^k_{\varepsilon,n}|^2 = \sigma_n \int_\Omega f_\varepsilon(v_n)u^k_{\varepsilon,n} + \frac{\sigma_n}{\tau}\int_\Omega u^{k-1}_\varepsilon u^k_{\varepsilon,n}.$$

Since $a_\varepsilon(s) \geq \varepsilon$ for all $s \in \mathbb{R}$, we have

$$\int_\Omega |u^k_{\varepsilon,n}|^2 + \tau\varepsilon\int_\Omega |\nabla u^k_{\varepsilon,n}|^2 \leq \tau\alpha\sigma_n \int_\Omega v_n u^k_{\varepsilon,n} - \tau\beta\sigma_n \int_\Omega a_\varepsilon(v_n)^2 u^k_{\varepsilon,n} + \sigma_n \int_\Omega u^{k-1}_\varepsilon u^k_{\varepsilon,n}$$

Using Young's inequality in the form $ab \leq \gamma a^2 + \dfrac{b^2}{\gamma}$, and $\sigma_n \leq 1$, we get,

$$\frac{1}{4}\int_\Omega |u^k_{\varepsilon,n}|^2 + \tau\varepsilon\int_\Omega |\nabla u^k_{\varepsilon,n}|^2 \leq 4\tau^2\alpha^2\int_\Omega v_n^2 + 4\tau^2\beta^2\int_\Omega |a_\varepsilon(v_n)|^4 + 4\int_\Omega |u^{k-1}_\varepsilon|^2.$$

Thus, since $v_n,\ u_\varepsilon^{k-1} \in L^2(\Omega)$, and $a_\varepsilon(s) \leq \varepsilon^{-1}$, we obtain

$$\int_\Omega |u_{\varepsilon,n}^k|^2 + \tau\varepsilon \int_\Omega |\nabla u_{\varepsilon,n}^k|^2 \leq C(1+\tau^2\varepsilon^{-4}),$$

implying that $\|u_{\varepsilon,n}^k\|_{H^1(\Omega)}$ is bounded.

The compact embedding $L^2(\Omega) \subset H^1(\Omega)$, implies

$$\begin{aligned} u_{\varepsilon,n}^k &\rightharpoonup z \quad \text{weakly in } H^1(\Omega), \\ u_{\varepsilon,n}^k &\to z \quad \text{strongly in } L^2(\Omega), \text{ and a.e. in } \Omega. \end{aligned}$$

Finally, the continuity will be proven if we identify z as $S(v,\sigma)(= u_\varepsilon^{k,\sigma})$.

Let's take the limit $n \to \infty$ in the weak formulation

$$\frac{1}{\tau}\int_\Omega u^k_{\varepsilon,n}\varphi + \int_\Omega a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n}\cdot\nabla\varphi = \sigma_n\int_\Omega f_\varepsilon(v_n)\varphi + \frac{\sigma_n}{\tau}\int_\Omega u^{k-1}_\varepsilon\varphi.$$

By assumption, $v_n \to v$ strongly in $L^2(\Omega)$.

Since a_ε is Lipschitz continuous, we have

$$\|a_\varepsilon(v_n) - a_\varepsilon(v)\|_{L^2} \leq \|v_n - v\|_{L^2},$$

and thus $a_\varepsilon(v_n) \to a_\varepsilon(v)$ strongly in $L^2(\Omega)$ and a.e. in Ω, as $n \to \infty$.

Similarly,

$$f_\varepsilon(v_n) \to f_\varepsilon(v) \quad \text{strongly in } L^2(\Omega).$$

Theorem (Dominated convergence theorem)

Let f_n be a sequence of functions of $L^1(\Omega)$ satisfying

- $f_n(x) \to f(x)$ *a.e. in* Ω,
- *there is a function* $g \in L^p(\Omega)$, *with* $1 \leq p < \infty$, *such that, for all* n, $|f_n(x)| \leq g(x)$ *a.e. in* Ω.

Then $f \in L^p(\Omega)$ *and* $f_n \to f$ *strongly in* $L^p(\Omega)$.

Being $a_\varepsilon(v_n) \leq \varepsilon^{-1}$ for all n, we may use the DCT to deduce

$$a_\varepsilon(v_n) \to a_\varepsilon(v) \quad \text{strongly in } L^p(\Omega), \text{ for all } p < \infty.$$

Thus,

$$a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n} \rightharpoonup a_\varepsilon(v)\nabla z \quad \text{weakly in } L^q(\Omega), \text{ for } q = \frac{2p}{p-2} < 2, \text{ and } 2 < p < \infty.$$

Since $\nabla\varphi \in L^2(\Omega)$, the above convergence is not enough to pass to the limit in

$$\int_\Omega a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n} \cdot \nabla\varphi.$$

However, having the bound

$$\|a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n}\|_{L^2} \leq \|a_\varepsilon(v_n)\|_{L^\infty}\|\nabla u^k_{\varepsilon,n}\|_{L^2} \leq C,$$

we deduce that, in fact, up to a subsequence,

$$a_\varepsilon(v_n)\nabla u^k_{\varepsilon,n} \rightharpoonup a_\varepsilon(v)\nabla z \quad \text{weakly in } L^2(\Omega).$$

Thus, we get , as $n \to \infty$,

$$\frac{1}{\tau}\int_\Omega z\varphi + \int_\Omega a_\varepsilon(v)\nabla z \cdot \nabla\varphi = \sigma \int_\Omega f_\varepsilon(v)\varphi + \frac{\sigma}{\tau}\int_\Omega u_\varepsilon^{k-1}\varphi,$$

so z is a weak solution corresponding to v.

Moreover, the limit z is unique because the solution of the limit problem may be obtained by Lax-Milgram's lemma.

Therefore, we deduce that the whole sequence converges, this is, $z = S(v,\sigma)$.

Point 4: uniform bound of the fixed points of S.

Assume $v = u_\varepsilon^{k,\sigma}$ and let us prove $\|u_\varepsilon^{k,\sigma}\|_{L^2} \leq C$, for all $\sigma \in [0,1]$.

$u_\varepsilon^{k,\sigma}$ satisfies

$$\frac{1}{\tau} u_\varepsilon^{k,\sigma} - \mathrm{div}(a_\varepsilon(u_\varepsilon^{k,\sigma}) \nabla u_\varepsilon^{k,\sigma})) = \sigma\Big(f_\varepsilon(u_\varepsilon^{k,\sigma}) + \frac{1}{\tau} u_\varepsilon^{k-1}\Big) \qquad \text{in } \Omega,$$

$$a_\varepsilon(u_\varepsilon^{k,\sigma}) \nabla u_\varepsilon^{k,\sigma} \cdot n = 0 \qquad \text{on } \partial\Omega.$$

Using $\varphi = u_\varepsilon^{k,\sigma} \in H^1(\Omega)$ we obtain (as before)

$$\int_\Omega |u_\varepsilon^{k,\sigma}|^2 + \tau\varepsilon \int_\Omega |\nabla u_\varepsilon^{k,\sigma}|^2 \leq C(1 + \tau^2 \varepsilon^{-4} \sigma^2) \leq C(1 + \tau^2 \varepsilon^{-4}).$$

We deduce the existence of a fixed point of $S(v, 1)$, which we denote by u_ε^k, which is a solution of the nonlinear time-discrete problem.

Further estimates for the nonlinear time-discrete problem

Until now, we have shown the existence of a solution

$$\frac{1}{\tau}\int_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})\varphi + \int_\Omega a_\varepsilon(u_\varepsilon^k)\nabla u_\varepsilon^k \cdot \nabla\varphi = \int_\Omega f_\varepsilon(u_\varepsilon^k)\varphi, \quad \text{for all } \varphi \in H^1(\Omega).$$

Now, we deduce uniform estimates with respect to ε:

Taking $\varphi = F_\varepsilon'(u_\varepsilon^k)$

$$\frac{1}{\tau}\int_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})F_\varepsilon'(u_\varepsilon^k) + \int_\Omega |\nabla u_\varepsilon^k|^2 = \int_\Omega f_\varepsilon(u_\varepsilon^k)F_\varepsilon'(u_\varepsilon^k).$$

For the first term of the LHS, we use the convexity estimate

$$(s-t)F_\varepsilon'(s) \geq F_\varepsilon(s) - F_\varepsilon(t), \quad \text{for all} \quad s,t \in \mathbb{R}.$$

For the term at the right hand side, we use

$$F_\varepsilon(s) \geq \frac{\varepsilon}{2}s^2 - 2 \quad \text{for all } s \geq 0, \quad F_\varepsilon(s) \geq \frac{s^2}{2\varepsilon} \quad \text{for all } s \leq 0,$$
$$\max\{a_\varepsilon(s), sF_\varepsilon'(s)\} \leq 2F_\varepsilon(s) + 1 \quad \text{for all } s \in \mathbb{R},$$
$$a_\varepsilon(s)F_\varepsilon'(s) \geq s - 1 \quad \text{for all } s \in \mathbb{R},$$
$$F_\varepsilon(a_\varepsilon(s)) \leq F_\varepsilon(s) \quad \text{for all } s \in \mathbb{R},$$
$$[1 - s]_+ \leq 1 + [s]_-$$

to deduce

$$f_\varepsilon(s)F_\varepsilon'(s) \leq (2\alpha + 4\beta)F_\varepsilon(s) + \alpha + 3\beta.$$

Therefore,

$$(1 - \omega\tau)\int_\Omega F_\varepsilon(u_\varepsilon^k) + \tau\int_\Omega |\nabla u_\varepsilon^k|^2 \leq C\tau + \int_\Omega F_\varepsilon(u_\varepsilon^{k-1}),$$

with $\omega = 2(\alpha + \beta)$. Here, we impose $\tau < \omega^{-1}$.

Therefore, we have

$$(1-\omega\tau)\int_\Omega F_\varepsilon(u_\varepsilon^k)+\tau\int_\Omega|\nabla u_\varepsilon^k|^2\leq C\tau+\int_\Omega F_\varepsilon(u_\varepsilon^{k-1}),$$

Estimate of the entropy.

$$\max_{k=1,\ldots,K}\int_\Omega F_\varepsilon(u_\varepsilon^k)\leq e^{\omega T/(1-\omega\tau)}\Big(CT+\int_\Omega F_\varepsilon(u_0)\Big)\leq C.$$

Estimate of the gradient. Summing in k

$$\int_\Omega F_\varepsilon(u_\varepsilon^k)+\tau\sum_{k=1}^K\int_\Omega|\nabla u_\varepsilon^k|^2\leq C\tau K+\int_\Omega F_\varepsilon(u_0)+\omega\tau\sum_{k=1}^K\int_\Omega F_\varepsilon(u_\varepsilon^k)$$
$$\leq CT+\int_\Omega F_\varepsilon(u_0)+\omega T\max_{k=1,\ldots,K}\int_\Omega F_\varepsilon(u_\varepsilon^k),$$

implying

$$\tau\sum_{k=1}^K\int_\Omega|\nabla u_\varepsilon^k|^2\leq C.$$

Extenct of negativity of u_ε^k. From

$$F_\varepsilon(s) \geq \frac{s^2}{2\varepsilon} \quad \text{for all } s \leq 0$$

we get

$$\begin{aligned}\frac{1}{2\varepsilon}\int_\Omega |[u_\varepsilon^k]_-|^2 &\leq \int_\Omega F_\varepsilon([u_\varepsilon^k]_-) = \int_{u_\varepsilon^k \leq 0} F_\varepsilon(u_\varepsilon^k) \\ &\leq \int_{u_\varepsilon^k \leq 0} F_\varepsilon(u_\varepsilon^k) + \int_{u_\varepsilon^k \geq 0} F_\varepsilon(u_\varepsilon^k) = \int_\Omega F_\varepsilon(u_\varepsilon^k) \leq C.\end{aligned}$$

Thus,

$$\max_{k=1,\ldots,K} \int_\Omega |[u_\varepsilon^k]_-|^2 \leq C\varepsilon.$$

L^1 estimate.

Using $\varphi = 1$

$$(1-\alpha\tau)\int_\Omega u_\varepsilon^k \leq \int_\Omega u_\varepsilon^{k-1}, \quad \text{implying} \quad \max_{k=1,\dots,K}\int_\Omega u_\varepsilon^k \leq C.$$

Young's inequality,

$$\int_\Omega |u_\varepsilon^k| = \int_\Omega([u_\varepsilon^k]_+ + [u_\varepsilon^k]_-) = \int_\Omega u_\varepsilon^k + 2\int_\Omega [u_\varepsilon^k]_- \leq C\Big(1+\int_\Omega |[u_\varepsilon^k]_-|^2\Big),$$

and then,

$$\max_{k=1,\dots,K}\int_\Omega |u_\varepsilon^k| \leq C.$$

Summarizing, we have obtained the bound

$$\max_{k=1,\ldots,K}\Big(\int_\Omega F_\varepsilon(u_\varepsilon^k)+\int_\Omega |u_\varepsilon^k|+\frac{1}{\varepsilon}\int_\Omega([u_\varepsilon^k]_-)^2\Big)+\tau\sum_{k=1}^K\int_\Omega|\nabla u_\varepsilon^k|^2\leq C.$$

Back to the evolution problem

Take $u_\varepsilon^{(\tau)}(t,x) = u_\varepsilon^k(x)$, $\quad \tilde{u}_\varepsilon^{(\tau)}(t,x) = u_\varepsilon^k(x) + \frac{t_k - t}{\tau}(u_\varepsilon^{k-1}(x) - u_\varepsilon^k(x))$,
satisfying the identity

$$\int_0^T \partial_t \tilde{u}_\varepsilon^{(\tau)} \varphi + \int_{Q_T} a_\varepsilon(u_\varepsilon^{(\tau)}) \nabla u_\varepsilon^{(\tau)} \cdot \nabla \varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon^{(\tau)}) \varphi,$$

for all $\varphi \in V$, to be chosen such that $V \subset L^2(0,T;H^1(\Omega))$.

For passing to the limits $\tau \to 0$ and $\varepsilon \to 0$ we need:

1. **Time derivative:** weak convergence of $\partial_t \tilde{u}_\varepsilon^{(\tau)}$ in some *large* space.
2. **Diffusive term:** strong convergence of $a_\varepsilon(u_\varepsilon^{(\tau)})$, and weak convergence of $\nabla u_\varepsilon^{(\tau)}$. Since the latter will be in $L^2(Q_T)$, we need to investigate the larger space in which $a_\varepsilon(u_\varepsilon^{(\tau)})$ converges strongly to fix the space of test functions.
3. **Reaction term:** strong convergence of $u_\varepsilon^{(\tau)}$ in some $L^p(Q_T)$.
4. **Identification:** The limits of $u_\varepsilon^{(\tau)}$ and $\tilde{u}_\varepsilon^{(\tau)}$ are the same function.

Uniform estimates in ε and τ

The estimates for the sequence of time-independent problems lead to

$$\max_{t\in(0,T)}\Big(\int_\Omega F_\varepsilon(u_\varepsilon^{(\tau)}(t)) + \int_\Omega |u_\varepsilon^{(\tau)}(t)| + \frac{1}{\varepsilon}\int_\Omega([u_\varepsilon^{(\tau)}(t)]_-)^2\Big) + \int_{Q_T}|\nabla u_\varepsilon^{(\tau)}|^2 \leq C.$$

Theorem (Poincaré-Wirtinger's inequality)

Let Ω be a connected open set of class C^1 and let $1\leq p\leq\infty$. Then, for all $u\in W^{1,p}(\Omega)$, there exists a constant C such that

$$\|u-u_\Omega\|_{L^p}\leq C\|\nabla u\|_{L^p}, \quad \textit{where } u_\Omega = \frac{1}{|\Omega|}\int_\Omega u.$$

Therefore

$$\|u_\varepsilon^{(\tau)}\|_{L^2}^2 \leq \frac{1}{|\Omega|}\|u_\varepsilon^{(\tau)}\|_{L^1}^2 + C\|\nabla u_\varepsilon^{(\tau)}\|_{L^2}^2 \leq C \implies \|u_\varepsilon^{(\tau)}\|_{L^2(H^1)}\leq C.$$

We also have, for $\sigma_\tau u_\varepsilon^{(\tau)}(t) = u_\varepsilon^{k-1}$ if $t \in (t_{k-1}, t_k]$,

$$\|\tilde{u}_\varepsilon^{(\tau)}\|_{L^2} \leq 2\|u_\varepsilon^{(\tau)}\|_{L^2} + \|\sigma_\tau u_\varepsilon^{(\tau)}\|_{L^2} \leq C,$$

$$\|\tilde{u}_\varepsilon^{(\tau)}\|_{L^2(H^1)} \leq 2\|u_\varepsilon^{(\tau)}\|_{L^2(H^1)} + \|\sigma_\tau u_\varepsilon^{(\tau)}\|_{L^2(H^1)} \leq C.$$

Time derivative estimate

$$\int_0^T \langle \partial_t \tilde{u}_\varepsilon^{(\tau)}, \varphi \rangle \leq \int_{Q_T} |a_\varepsilon(u_\varepsilon^{(\tau)})| |\nabla u_\varepsilon^{(\tau)}| |\nabla \varphi| + \int_{Q_T} |f_\varepsilon(u_\varepsilon^{(\tau)})| |\varphi|$$
$$\leq \|a_\varepsilon(u_\varepsilon^{(\tau)})\|_{L^\infty} \|\nabla u_\varepsilon^{(\tau)}\|_{L^2} \|\nabla \varphi\|_{L^2} + \alpha \|u_\varepsilon^{(\tau)}\|_{L^2} \|\varphi\|_{L^2}$$
$$+\beta \|a_\varepsilon(u_\varepsilon^{(\tau)})\|^2_{L^\infty} \|\varphi\|_{L^1},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product in $(H^1(\Omega))' \times H^1(\Omega)$.
Then,

$$\int_0^T \langle \partial_t \tilde{u}_\varepsilon^{(\tau)}, \varphi \rangle \leq C\varepsilon^{-1} \|\nabla \varphi\|_{L^2} + \left(C + \varepsilon^{-2}\right) \|\varphi\|_{L^2} \leq C\varepsilon^{-2} \|\varphi\|_{L^2(H^1)},$$

and thus,

$$\|\partial_t \tilde{u}_\varepsilon^{(\tau)}\|_{L^2((H^1)')} \leq C\varepsilon^{-2}.$$

The limit $\tau \to 0$

From the bounds we deduce the existence of $u_\varepsilon, z_\varepsilon \in L^2(0,T;H^1(\Omega))$ and of subsequences of $u_\varepsilon^{(\tau)}$ and $\tilde{u}_\varepsilon^{(\tau)}$ (not relabeled) such that, as $\tau \to 0$,

$$
\begin{aligned}
u_\varepsilon^{(\tau)} &\rightharpoonup u_\varepsilon && \text{weakly in } L^2(0,T;H^1(\Omega)),\\
u_\varepsilon^{(\tau)} &\rightharpoonup u_\varepsilon && \text{weakly in } L^2(Q_T),\\
\tilde{u}_\varepsilon^{(\tau)} &\rightharpoonup z_\varepsilon && \text{weakly in } L^2(0,T;H^1(\Omega)),\\
\tilde{u}_\varepsilon^{(\tau)} &\rightharpoonup z_\varepsilon && \text{weakly in } L^2(Q_T),\\
\partial_t \tilde{u}_\varepsilon^{(\tau)} &\rightharpoonup \partial_t z_\varepsilon && \text{weakly in } L^2(0,T;(H^1(\Omega))').
\end{aligned}
$$

Identification $\mathbf{z} = \mathbf{u}$. Like in the linear case,

$$\|\tilde{u}_\varepsilon^{(\tau)} - u_\varepsilon^{(\tau)}\|_{L^2((H^1)')} \leq \tau \|\partial_t \tilde{u}_\varepsilon^{(\tau)}\|_{L^2((H^1)')} \to 0 \quad \text{as } \tau \to 0,$$

and hence $z_\varepsilon = u_\varepsilon$.

Compactness and strong convergence for $\tilde{u}_\varepsilon^{(\tau)}$. The compactness Aubin-Lions-Simon's lemma, gives

$$\tilde{u}_\varepsilon^{(\tau)} \to u_\varepsilon \quad \text{strongly in } L^2(Q_T), \quad \text{and a.e. in } Q_T.$$

Lemma

Let $(H, \|\cdot\|_H)$ be a Hilbert space and let $V \subset H$ be a proper linear subspace dense in H. Assume that $(V, \|\cdot\|_V)$ is a Banach space and, under the identification $H = H'$, consider the triplet $V \subset H \subset V'$. Then

$$\langle f, v\rangle_{V'\times V} = (f, v)_H, \quad \textit{for all } f \in H, v \in V.$$

In particular, for all $v \in V$,

$$\|v\|_H^2 = \langle v, v\rangle_{V'\times V} \le \|v\|_{V'}\|v\|_V.$$

Strong convergence for $u_\varepsilon^{(\tau)}$. Setting $V = L^2(0, T; H^1(\Omega))$, $H = L^2(Q_T)$

$$\begin{aligned}\|u_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2} &\le \|u_\varepsilon^{(\tau)} - \tilde u_\varepsilon^{(\tau)}\|_{L^2} + \|\tilde u_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2}\\ &\le \|u_\varepsilon^{(\tau)} - \tilde u_\varepsilon^{(\tau)}\|_{L^2((H^1)')}^{1/2}\|u_\varepsilon^{(\tau)} - \tilde u_\varepsilon^{(\tau)}\|_{L^2(H^1)}^{1/2} + \|\tilde u_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2},\end{aligned}$$

and then

$$\|u_\varepsilon^{(\tau)} - u_\varepsilon\|_{L^2} \le C\|u_\varepsilon^{(\tau)} - \tilde u_\varepsilon^{(\tau)}\|_{L^2((H^1)')}^{1/2} + \|\tilde u_\varepsilon^{(\tau)} - u\|_{L^2} \to 0 \quad \text{as } \tau \to 0.$$

Convergence. We have to pass to the limit $\tau \to 0$ in the expression

$$\int_0^T \langle \partial_t \tilde{u}_\varepsilon^{(\tau)}, \varphi \rangle + \int_{Q_T} a_\varepsilon(u_\varepsilon^{(\tau)}) \nabla u_\varepsilon^{(\tau)} \cdot \nabla \varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon^{(\tau)}) \varphi, \quad \text{for } \varphi \in L^2(0,T;H^1(\Omega))$$

The time derivative term, recalling $z_\varepsilon = u_\varepsilon$, passes to the limit without any additional reasoning. The linear part of the reaction term, also passes to the limit, thanks to, e.g., the $L^2(Q_T)$ strong convergence.

For the convergence of $a_\varepsilon(u_\varepsilon^{(\tau)})$ we use the dominated convergence theorem:

- By continuity of a_ε, we have $a_\varepsilon(u_\varepsilon^{(\tau)}) \to a_\varepsilon(u_\varepsilon)$ a.e. in Q_T as $\tau \to 0$.
- $\|a_\varepsilon(u_\varepsilon^{(\tau)})\|_{L^\infty} \leq \varepsilon^{-1}$.

Thus (DCT), as $\tau \to 0$,

$$a_\varepsilon(u_\varepsilon^{(\tau)}) \to a_\varepsilon(u_\varepsilon) \quad \text{strongly in } L^p(Q_T) \text{ for any } 1 \leq p < \infty.$$

Quadratic term of the reaction function.

$$\int_{Q_T} \left| a_\varepsilon(u_\varepsilon^{(\tau)})^2 - a_\varepsilon(u_\varepsilon)^2 \right|^2 = \int_{Q_T} \left| a_\varepsilon(u_\varepsilon^{(\tau)}) - a_\varepsilon(u_\varepsilon) \right|^2 \left| a_\varepsilon(u_\varepsilon^{(\tau)}) + a_\varepsilon(u_\varepsilon) \right|^2$$
$$\leq \| a_\varepsilon(u_\varepsilon^{(\tau)}) - a_\varepsilon(u_\varepsilon) \|_{L^4}^2 \| a_\varepsilon(u_\varepsilon^{(\tau)}) + a_\varepsilon(u_\varepsilon) \|_{L^4}^2,$$

and therefore

$$a_\varepsilon(u_\varepsilon^{(\tau)})^2 \to a_\varepsilon(u_\varepsilon)^2 \quad \text{strongly in } L^2(Q_T).$$

For the diffusion term, we have

$$a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)} \rightharpoonup u_\varepsilon \nabla u_\varepsilon \quad \text{weakly in } L^q(Q_T) \text{ for any } q<2.$$

However, we also have

$$\|a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)}\|_{L^2} \leq \|a_\varepsilon(u_\varepsilon^{(\tau)})\|_{L^\infty}\|\nabla u_\varepsilon^{(\tau)}\|_{L^2} \leq C\varepsilon^{-1},$$

implying

$$a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)} \rightharpoonup u_\varepsilon \nabla u_\varepsilon \quad \text{weakly in } L^2(Q_T).$$

Therefore, we may pass to the limit to obtain that
$u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$ satisfies

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi\rangle + \int_{Q_T} a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon)\varphi, \quad \text{for all } \varphi \in L^2(0,T;H^1(\Omega)).$$

The limit $\varepsilon \to 0$

Since the uniform bound of a_ε is lost in the limit $\varepsilon \to 0$, we can not expect

$$\lim_{\varepsilon\to 0} \int_{Q_T} a_\varepsilon(u_\varepsilon^{(\tau)})\nabla u_\varepsilon^{(\tau)} \cdot \nabla\varphi$$

to be well defined for test functions $\varphi \in L^2(0, T; H^1(\Omega))$.

We have to investigate in which L^p space may $a_\varepsilon(u_\varepsilon^{(\tau)})$ converge strongly, and then seek for a **suitable space of test functions** in which this limit may be performed.

In addition, **the time derivative bounds we obtained are dependent of the regularity of the other terms** (through the argument for the duality $\langle \partial_t \tilde{u}, \varphi\rangle$).

Thus, if the other terms are less regular, the time derivative will be less regular too, and we shall therefore need to impose more regularity of φ in both the space and the time variables.

Uniform estimates in ε and weak convergences.

Taking the limit $\tau \to 0$ we get

$$\max_{t\in(0,T)} \Big(\int_\Omega F_\varepsilon(u_\varepsilon(t)) + \int_\Omega |u_\varepsilon(t)| + \frac{1}{\varepsilon}\int_\Omega ([u_\varepsilon(t)]_-)^2 \Big) + \int_{Q_T} |\nabla u_\varepsilon|^2 \leq C.$$

and then

$$\|u_\varepsilon\|_{L^2(H^1)} \leq C.$$

Theorem (Gagliardo-Niremberg's interpolation inequality)

Let $\Omega \subset \mathbb{R}^N$ be a regular open bounded set, and let $u \in L^q(\Omega) \cap W^{m,r}(\Omega)$, with $1 \leq p, q \leq \infty$, and $m \in \mathbb{N}$. Then $u \in W^{j,p}(\Omega)$, and

$$\|D^j u\|_{L^p} \leq C \|D^m u\|_{L^r}^{\theta} \|u\|_{L^q}^{1-\theta},$$

where

$$\frac{1}{p} = \frac{j}{n} + \Big(\frac{1}{r} - \frac{m}{n}\Big)\theta + \frac{1-\theta}{q}, \quad \text{and} \quad \frac{j}{m} \leq \theta \leq 1.$$

Taking $p = (2N+2)/N$, $\theta = 2N(p-1)/(p(N+2))$, and thus $\theta p = 2$, yields

$$\|u_\varepsilon\|_{L^p} \leq \Big(\int_0^T \|u_\varepsilon\|_{L^1(\Omega)}^{(1-\theta)p} \|u_\varepsilon\|_{H^1(\Omega)}^{\theta p}\Big)^{1/p} \leq \|u_\varepsilon\|_{L^\infty(L^1)}^{1-\theta} \|u_\varepsilon\|_{L^2(H^1)}^{\theta} \leq C.$$

Time derivative estimate.

Let $r' = r/(r-1)$ to be determined. Using $p > 2$,

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle \leq \int_{Q_T} |a_\varepsilon(u_\varepsilon)||\nabla u_\varepsilon||\nabla \varphi| + \int_{Q_T} |f_\varepsilon(u_\varepsilon)||\varphi|$$
$$\leq \|a_\varepsilon(u_\varepsilon)\|_{L^p}\|\nabla u_\varepsilon\|_{L^2}\|\nabla \varphi\|_{L^{r'}} + \alpha\|u_\varepsilon\|_{L^p}\|\varphi\|_{L^{p'}}$$
$$+\beta\|a_\varepsilon(u_\varepsilon)\|^2_{L^p}\|\varphi\|_{L^{(p/2)'}},$$

where $\langle \cdot, \cdot \rangle$ is for $(W^{1,r'}(\Omega))' \times W^{1,r'}(\Omega)$.

Here, r' is such that

$$1 = \frac{1}{p} + \frac{1}{2} + \frac{1}{r'} \implies r' = 2(N+1).$$

Then, we take $L^{r'}(0, T; W^{1,r'}(\Omega))$ as the new (smaller, more regular) space of test funtions.

In addition, $r' \geq \max\{p', (p/2)'\}$, and thus the norms of the reaction term are also well defined.

Therefore, noting that $a_\varepsilon(s) \leq \varepsilon + s$, we find

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle \leq (C + \|u_\varepsilon\|_{L^p}) \|\nabla u_\varepsilon\|_{L^2} \|\nabla \varphi\|_{L^{r'}} + \left(C + \|u_\varepsilon\|_{L^p} + \|u_\varepsilon\|_{L^p}^2\right) \|\varphi\|_{L^{r'}}$$
$$\leq C \|\varphi\|_{L^{r'}(W^{1,r'})},$$

and thus, for $r = (2N + 2)/(2N + 1)$,

$$\|\partial_t u_\varepsilon\|_{L^r((W^{1,r'})')} \leq C.$$

From the general estimate, we also deduce

$$\|[u_\varepsilon]_-\|_{L^\infty(L^2)} \leq C\sqrt{\varepsilon}.$$

Thus we have the existence of $u, z \in L^2(0, T; H^1(\Omega))$ and of subsequences of u_ε (not relabeled) such that

$$\begin{aligned}
u_\varepsilon &\rightharpoonup u && \text{weakly in } L^2(0, T; H^1(\Omega)), \\
u_\varepsilon &\rightharpoonup u && \text{weakly in } L^p(Q_T), \\
\partial_t u_\varepsilon &\rightharpoonup \partial_t u && \text{weakly in } L^r(0, T; (W^{1,r'}(\Omega))'), \\
[u_\varepsilon]_- &\rightharpoonup 0 && \text{weakly*-weakly in } L^\infty(0, T; L^2(\Omega)).
\end{aligned}$$

Compactness and strong convergences.

We again use the compactness Aubin-Lions-Simon's lemma, to get

$$u_\varepsilon \to u \quad \text{strongly in } L^\gamma(0,T;L^2(\Omega)), \text{ for any } \gamma < 2, \quad \text{and a.e. in } Q_T.$$

Lemma

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let f_n be a sequence in $L^p(\Omega) \cap L^\gamma(\Omega)$, with $p > \gamma$, and $f \in L^\gamma(\Omega)$. Assume that

$$f_n \to f \quad \text{strongly in } L^\gamma(\Omega) \text{ and } \|f_n\|_{L^p} \le C.$$

Then $f \in L^q(\Omega)$ and $f_n \to f$ strongly in $L^q(\Omega)$ for all $\gamma \le q < p$.

Using the bound $\|u_\varepsilon\|_{L^p} \le C$, we get

$$u_\varepsilon \to u \quad \text{strongly in } L^q(Q_T), \text{ for any } \gamma \le q < p.$$

In particular, we may choose $2 \le q < p = (2N+2)/N$.
This further implies, using $\|[u_\varepsilon]_-\|_{L^p} \le \|u_\varepsilon\|_{L^p}$,

$$[u_\varepsilon]_- \to 0 \quad \text{strongly in } L^q(Q_T) \text{ and a.e. in } Q_T, \text{ that is } u \ge 0 \text{ a.e. in } Q_T.$$

Convergence of $a_\varepsilon(u_\varepsilon)$.

$$\|u - a_\varepsilon(u_\varepsilon)\|_{L^q} \le \|u - \tilde{a}_\varepsilon(u)\|_{L^q} + \|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)\|_{L^q} + \|\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)\|_{L^q},$$

$$\tilde{a}_\varepsilon(s) := \begin{cases} s & \text{if } s \le \varepsilon^{-1}, \\ \varepsilon^{-1} & \text{if } s \ge \varepsilon^{-1}. \end{cases}$$

Theorem (Monotone convergence theorem)

Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $f_n \in L^1(\Omega)$ be a sequence of functions satisfying

1. $f_1 \le f_2 \le \cdots$ *a.e. in* Ω,
2. $\sup_n \int_\Omega f_n < \infty$.

Then there exists $f \in L^1(\Omega)$ such that $f_n \to f$ strongly in $L^1(\Omega)$ and a.e. in Ω.

$\tilde{a}_\varepsilon(s)$ is monotone increasing, $\tilde{a}_\varepsilon(s) \le s$ for all $s \in \mathbb{R}$, and $u \in L^1(Q_T)$ imply (MCT) $\tilde{a}_\varepsilon(u) \to a(u)$ strongly in $L^1(Q_T)$.

Using the uniform bound of $\|u\|_{L^q}$ we deduce $\|u - \tilde{a}_\varepsilon(u)\|_{L^q} \to 0$ as $\varepsilon \to 0$.

$$\|u - a_\varepsilon(u_\varepsilon)\|_{L^q} \leq \|u - \tilde{a}_\varepsilon(u)\|_{L^q} + \|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)\|_{L^q} + \|\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)\|_{L^q},$$

Since $\tilde{a}_\varepsilon$ is Lipschitz continuous, we get

$$|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)| \leq |u - u_\varepsilon|$$

and then the strong convergence $u_\varepsilon \to u$ implies

$$\|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)\|_{L^q} \to 0 \quad \text{as } \varepsilon \to 0.$$

$$\|u - a_\varepsilon(u_\varepsilon)\|_{L^q} \leq \|u - \tilde{a}_\varepsilon(u)\|_{L^q} + \|\tilde{a}_\varepsilon(u) - \tilde{a}_\varepsilon(u_\varepsilon)\|_{L^q} + \|\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)\|_{L^q},$$

We have

$$|\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)| = |u_\varepsilon - \varepsilon| 1_{u_\varepsilon \leq \varepsilon} = (\varepsilon - u_\varepsilon) 1_{0 \leq u_\varepsilon \leq \varepsilon} + (|u_\varepsilon| + \varepsilon) 1_{u_\varepsilon < 0}.$$

The first term of the RHS is bounded by $\varepsilon|Q_T|$, while the second is equal to $[u_\varepsilon]_- + \varepsilon 1_{u_\varepsilon < 0}$. Thus

$$\int_{Q_T} |\tilde{a}_\varepsilon(u_\varepsilon) - a_\varepsilon(u_\varepsilon)|^q \leq C\Big(\varepsilon^q + \int_{Q_T} |[u_\varepsilon]_-|^q\Big) \to 0$$

as $\varepsilon \to 0$. Therefore

$$a_\varepsilon(u_\varepsilon) \to u \quad \text{strongly in } L^q(Q_T).$$

We have to pass to the limit in the expression

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle + \int_{Q_T} a_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi = \int_{Q_T} f_\varepsilon(u_\varepsilon)\varphi, \quad \text{for all } \varphi \in L^{r'}(0,T; W^{1,r'}(\Omega)).$$

The **time derivative** and the **linear part of the reaction** term pass to the limit without any additional reasoning.

For the **quadratic part of the reaction term**, we have

$$\begin{aligned}\int_{Q_T} \left| a_\varepsilon(u_\varepsilon)^2 - u^2 \right|^{q/2} &\leq \Big(\int_{Q_T} |a_\varepsilon(u_\varepsilon) - u|^q \Big)^{1/2} \Big(\int_{Q_T} |a_\varepsilon(u_\varepsilon) + u|^q \Big)^{1/2} \\ &\leq \| a_\varepsilon(u_\varepsilon) - u \|_{L^q}^{q/2} \| a_\varepsilon(u_\varepsilon) + u \|_{L^q}^{q/2} \to 0,\end{aligned}$$

as $\varepsilon \to 0$. Thus, since

$$\frac{2}{q} + \frac{1}{r'} \leq 1 \quad \text{if we choose} \quad q \geq \frac{4(N+1)}{2N+1},$$

which is possible, we deduce

$$\int_{Q_T} a_\varepsilon(u_\varepsilon)^2 \varphi \to \int_{Q_T} u^2 \varphi.$$

For the **diffusion term**, we have the same reasoning than before: $a_\varepsilon(u_\varepsilon) \to u$ strongly in $L^q(Q_T)$ and $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(Q_T)$, implly

$$a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \rightharpoonup u\nabla u \quad \text{weakly in } L^\gamma(Q_T),$$

with $\gamma = 2q/(2+q)$, which is smaller than r. However, we also have

$$\|a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\|_{L^{r'}} \leq \|a_\varepsilon(u_\varepsilon)\|_{L^p}\|\nabla u_\varepsilon\|_{L^2} \leq C,$$

implying

$$a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \rightharpoonup u\nabla u \quad \text{weakly in } L^r(Q_T).$$

Finally, observe that due to the convergence of $[u_\varepsilon]_- \to 0$ in $L^q(Q_T)$, we deduce $u \geq 0$ a.e. in Q_T.

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz continuous boundary, and let $T > 0$. Suppose that $u_0 \in L^2(\Omega)$. Then, there exists $u \geq 0$ in Q_T with

$$u \in L^2(0,T;H^1(\Omega)) \cap L^p(Q_T) \cap W^{1,r}(0,T;(W^{1,r'}(\Omega))'),$$

where $p = 2(N+1)/N$, $r = 2(N+1)/(2N+1)$, and $r' = 2(N+1)$, satisfying, for all $\varphi \in L^{r'}(0,T;W^{1,r'}(\Omega))$,

$$\int_0^T < \partial_t u, \varphi > + \int_{Q_T} u \nabla u \cdot \nabla \varphi = \int_{Q_T} f(u)\,\varphi,$$

with $< \cdot, \cdot >$ denoting the duality product between $W^{1,r'}(\Omega)$ and its dual $(W^{1,r'}(\Omega))'$, being the initial data satisfied in the sense

$$\int_0^T < \partial_t u, \psi > + \int_{Q_T} (u - u_0)\partial_t \psi = 0,$$

for all $\psi \in L^{r'}(0,T;W^{1,r'}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $\psi(T) = 0$ a.e. in Ω.

Outline

Problem: Find $u_1, u_2 : (0,T) \times \Omega \to \mathbb{R}$ such that, using the notation $\mathbf{u} = (u_1, u_2)$,

$$\begin{aligned} &\partial_t u_1 - \operatorname{div} J_1(\mathbf{u}) = f_1(\mathbf{u}) && \text{in } Q_T, \\ &\partial_t u_2 - \operatorname{div} J_2(\mathbf{u}) = f_2(\mathbf{u}) && \text{in } Q_T, \\ &J_1(\mathbf{u}) \cdot n = J_2(\mathbf{u}) \cdot n = 0 && \text{on } \Gamma_T, \\ &\mathbf{u}(\cdot, 0) = \mathbf{u}_0 && \text{in } \Omega. \end{aligned}$$

The reaction terms are of the competitive Lotka-Volterra type

$$f_i(\mathbf{u}) = u_i\big(\alpha_i - (\beta_{i1} u_1 + \beta_{i2} u_2)\big), \quad \alpha_i, \beta_{ij} \geq 0 \quad \text{for } i = 1, 2.$$

The flows are of the Bousenberg-Travis (BT) model

$$J_i(\mathbf{u}) = a_{i0} \nabla u_i + u_i(a_{i1} \nabla u_1 + a_{i2} \nabla u_2) - b_i u_i \nabla \Phi, \quad a_{ij} \geq 0, b_i \geq 0.$$

The Shigesada-Kawasaki-Teramoto (SKT) model, for which

$$J_i^{SKT}(\mathbf{u}) = \nabla\big(u_i(a_{i0} + a_{i1} u_1 + a_{i2} u_2)\big) - b_i u_i \nabla \Phi,$$

may be treated in a similarly way.

In terms of population dynamics,

- The populations diffuses partly randomly, and partly to avoid overcrowding caused by both populations.
- The populations are drifted to the minima of the environmental potential Φ, representing the best environmental locations.
- The newborns are proportional to the existent population, but there is a growth limit given in terms of the intra- and inter-specific competence between populations. The corresponding kinetics ($\partial_t u_i = f_i(\mathbf{u})$) has stable equilibria at

$$\left(\frac{\alpha_1}{\beta_{11}}, 0\right), \quad \left(0, \frac{\alpha_2}{\beta_{22}}\right), \quad \left(\frac{\alpha_1\beta_{22} - \alpha_2\beta_{12}}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}, \frac{\alpha_2\beta_{11} - \alpha_1\beta_{21}}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}\right),$$

  depending on the relationship between the coefficients. However, **due to the cross-diffusion, these equilibria are not always the steady state solutions of the problem.**

Introducing the rescaling[1] $U_1 = a_{21} u_1$ and $U_2 = a_{12} u_2$, we write

$$J_i(\mathbf{u}) = a_{i0} \nabla u_i + u_i (a_i \nabla u_i + \nabla u_j) - b_i u_i \nabla \Phi, \quad \text{for } i, j = 1, 2, \text{ with } j \neq i.$$

We shall follow the line of the proof of existence of weak solutions developed for an scalar equation.

[1] Here, we assume $a_{12} \neq 0$ and $a_{21} \neq 0$. Otherwise, the system is triangular (instead of full), and the problem is simpler.

Formal estimates

- Multiplying by $\ln(u_i)$, we get, for $F(s) = s(\ln(s) - 1) + 1 \geq 0$,

$$\sum_{i=1}^{2} \int_{\Omega} F(u_i(T)) + \sum_{i=1}^{2} \int_{Q_T} \frac{a_{0i}}{u_i} |\nabla u_i|^2 + \int_{Q_T} \Big(a_1 |\nabla u_1|^2 + a_2 |\nabla u_2|^2 + 2 \nabla u_1 \cdot \nabla u_2 \Big)$$

$$= \sum_{i=1}^{2} \int_{\Omega} F(u_{i0}) + \sum_{i=1}^{2} \int_{Q_T} f_i(\mathbf{u}) \ln(u_i) + \sum_{i=1}^{2} \int_{Q_T} \nabla \Phi \cdot \nabla u_i.$$

We have $\frac{1}{u_i} |\nabla u_i|^2 = 4 |\nabla \sqrt{u_i}|^2$, and, if $a_1 a_2 > 1$ ($\det(a) > 0$, before rescaling),

$$a_1 |\nabla u_1|^2 + a_2 |\nabla u_2|^2 + 2 \nabla u_1 \cdot \nabla u_2 \geq a_0 (|\nabla u_1|^2 + |\nabla u_2|^2),$$

for some $a_0 > 0$. Thus, if the RHS may be controled by the LHS, we get

$$\sum_{i=1}^{2} \int_{\Omega} F(u_i(T)) + \int_{Q_T} (|\nabla u_1|^2 + |\nabla u_2|^2) \leq C.$$

- Integrating the equations we get (if $u_i \geq 0$)

$$\int_\Omega (u_1(T) + u_2(T)) \leq \int_\Omega (u_{10} + u_{20}) + \tilde{\alpha} \int_{Q_T} (u_1 + u_2),$$

with $\tilde{\alpha} = \max\{\alpha_1, \alpha_2\}$, and then Gronwall's lemma implies

$$\int_\Omega (u_1(T) + u_2(T)) \leq e^{\tilde{\alpha} T} \int_\Omega (u_{10} + u_{20}) \leq C.$$

We then deduce from these two estimates that $\|u_i\|_{L^2(H^1)} \leq C$, like in the scalar case.

Symmetrization

We simplify: $a_{i0} = 0$ and $\Phi = 0$, and write

$$\partial_t \mathbf{u} - \mathrm{div}(a(\mathbf{u})\nabla \mathbf{u}) = \mathbf{f}(\mathbf{u}),$$

where $a(\mathbf{u})$ is the non-symmetric matrix given by

$$a(\mathbf{u}) = \begin{pmatrix} a_1 u_1 & u_1 \\ u_2 & a_2 u_2 \end{pmatrix}.$$

We used the notation

$$\mathrm{div}(a(\mathbf{u})\nabla \mathbf{u}) = \begin{pmatrix} \mathrm{div}(a_1 u_1 \nabla u_1 + u_1 \nabla u_2) \\ \mathrm{div}(u_2 \nabla u_1 + a_2 u_2 \nabla u_2) \end{pmatrix}.$$

Following the line of previous sections, we first discretize in time,

$$\frac{1}{\tau}(\mathbf{u}^k - \mathbf{u}^{k-1}) - \mathrm{div}(a(\mathbf{u}^k)\nabla \mathbf{u}^k) = \mathbf{f}(\mathbf{u}^k),$$

and then (approximate) and linearize to use Lax-Milgram's lemma

$$\frac{1}{\tau}(\mathbf{u}^k - \mathbf{u}^{k-1}) - \mathrm{div}(a(\mathbf{v})\nabla \mathbf{u}^k) = \mathbf{f}(\mathbf{v}).$$

Since a is non-symmetric, the corresponding bilinear form

$$A(\mathbf{u},\mathbf{u}) = \int_\Omega \left(a_1 v_1 |\nabla u_1|^2 + a_2 v_2 |\nabla u_2|^2 + (v_1 + v_2)\nabla u_1 \cdot \nabla u_2\right),$$

is not, in general, coercive since the condition for this form to be coercive is that the matrix

$$\begin{pmatrix} a_1 v_1 & \frac{1}{2}(v_1 + v_2) \\ \frac{1}{2}(v_1 + v_2) & a_2 v_2 \end{pmatrix}$$

is positive definite, that is, $4a_1 a_2 v_1 v_2 > (v_1 + v_2)^2$, which is not true in general.

The entropy estimate of the nonlinear problem is not inherited by the linear approximation, as it happened for the scalar problem.

The existence of an entropy estimate is usually accompanied by a change of unknowns which symmetrizes the problem: for $w_i = F'(u_i) = \ln(u_i)$

$$\partial_t \begin{pmatrix} e^{w_1} \\ e^{w_2} \end{pmatrix} - \mathrm{div}(b(\mathbf{w})\nabla \mathbf{w}) = \mathbf{f}(e^{w_1}, e^{w_2}),$$

being $b(\mathbf{w})$ the symmetric matrix

$$b(\mathbf{w}) = \begin{pmatrix} a_1 e^{2w_1} & e^{w_1+w_2} \\ e^{w_1+w_2} & a_2 e^{2w_2} \end{pmatrix}.$$

Strategy: solve for $\mathbf{w}$, and justify the equivalency.
This is not straightforward. For instance, since

$$\nabla u_i = \nabla e^{w_i} = e^{w_i} \nabla w_i,$$

if we obtain $\nabla w_i \in L^2$, this regularity does not inmmediately translates to ∇u_i, unless $w_i \in L^\infty$, which is not expected.

The formal calculations of the previous section may be also done in terms of the approximation to the logarithm given by F'_ε.

Since F'_ε is increasing in $\mathbb{R}$, its inverse is well defined (approximating the exponential). We introduce the notation

$$g_\varepsilon = (F'_\varepsilon)^{-1}, \quad \text{satisfying} \quad g'_\varepsilon = a_\varepsilon \circ g_\varepsilon.$$

Then, for $\sigma \in [0,1]$, we set the problem: Given $\mathbf{w}_\varepsilon^{k-1} \in L^2(\Omega)^2$, with $F_\varepsilon(g_\varepsilon(w_{i,\varepsilon}^{k-1})) \in L^1(\Omega)$, find $\mathbf{w}_\varepsilon^k : \Omega \to \mathbb{R}^2$ such that

$$\begin{aligned}
&\frac{\sigma}{\tau}(g_\varepsilon(w_{i,\varepsilon}^k) - g_\varepsilon(w_{i,\varepsilon}^{k-1})) - \operatorname{div} G_i^\varepsilon(\mathbf{w}_\varepsilon^k) + \varepsilon w_{i,\varepsilon}^k = \sigma h_i^\varepsilon(\mathbf{w}_\varepsilon^k) && \text{in } \Omega,\\
&G_i^\varepsilon(\mathbf{w}_\varepsilon^k) \cdot n = 0 && \text{on } \partial\Omega,
\end{aligned}$$

with, for $i, j = 1, 2$ and $i \neq j$,

$$\begin{aligned}
G_i^\varepsilon(\mathbf{w}) &= g'_\varepsilon(w_i)(a_i g'_\varepsilon(w_i)\nabla w_i + g'_\varepsilon(w_j)\nabla w_j),\\
h_i^\varepsilon(\mathbf{w}) &= \alpha_i g_\varepsilon(w_i) - g'_\varepsilon(w_i)\big(\beta_{i1} g'_\varepsilon(w_1) + \beta_{i2} g'_\varepsilon(w_2)\big).
\end{aligned}$$

Lax-Milgram. $A : H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{R}$ and $F : L^2(\Omega)^2 \times L^2(\Omega)^2 \to \mathbb{R}$ defined by, for $\mathbf{v} \in L^2(\Omega)^2$ and $\sigma \in [0,1]$,

$$A(\mathbf{w}, \boldsymbol{\varphi}) = \sum_{i=1}^{2} \Big(\int_\Omega \varepsilon w_i \varphi_i + \sum_{\substack{i,j=1 \\ j \neq i}}^{2} \int_\Omega g'_\varepsilon(v_i) \big(a_i g'_\varepsilon(v_i) \nabla w_i + g'_\varepsilon(v_j) \nabla w_j \big) \cdot \nabla \varphi_i \Big),$$

$$F(\boldsymbol{\varphi}) = \sigma \sum_{i=1}^{2} \Big(\int_\Omega \big(\alpha_i g_\varepsilon(v_i) - g'_\varepsilon(v_i) \big(\beta_{i1} g'_\varepsilon(v_1) + \beta_{i2} g'_\varepsilon(v_2) \big) \big) \varphi_i - \frac{1}{\tau} \int_\Omega (g_\varepsilon(v_i) - g_\varepsilon(w_{i,\varepsilon}^{k-1})) \varphi_i \Big).$$

with $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in H^1(\Omega)^2$. We have, using $a_1 a_2 > 1$,

$$A(\mathbf{w}, \mathbf{w}) \geq \sum_{i=1}^{2} \Big(\varepsilon \int_\Omega w_i^2 + a_0 c(\varepsilon) \int_\Omega |\nabla w_i|^2 \Big),$$

with $c(\varepsilon) = \min_{s \in \mathbb{R}} (g'_\varepsilon(s))^2 > \varepsilon^2$. Thus, A is coercive, and both A and F are clearly continuous.

Lax-Milgram's lemma ensures the existence of a unique weak solution, $\mathbf{w}^k_{\varepsilon,\sigma} \in H^1(\Omega)^2$, of

$$\begin{aligned} &\frac{\sigma}{\tau}(g_\varepsilon(v_i) - g_\varepsilon(w^{k-1}_{i,\varepsilon})) - \operatorname{div} G^\varepsilon_i(\mathbf{w}^k_{\varepsilon,\sigma}, \mathbf{v}) + \varepsilon w^k_{i,\varepsilon,\sigma} = \sigma h^\varepsilon_i(\mathbf{w}^k_{\varepsilon,\sigma}, \mathbf{v}) \qquad &&\text{in } \Omega, \\ &G^\varepsilon_i(\mathbf{w}^k_{\varepsilon,\sigma}, \mathbf{v}) \cdot n = 0 &&\text{on } \partial\Omega, \end{aligned}$$

with

$$\begin{aligned} G^\varepsilon_i(\mathbf{w}, \mathbf{v}) &= g'_\varepsilon(v_i)\big(a_i g'_\varepsilon(v_i)\nabla w_i + g'_\varepsilon(v_j)\nabla w_j\big), \\ h^\varepsilon_i(\mathbf{w}, \mathbf{v}) &= \alpha_i g_\varepsilon(v_i) - g'_\varepsilon(v_i)\big(\beta_{i1} g'_\varepsilon(v_1) + \beta_{i2} g'_\varepsilon(v_2)\big). \end{aligned}$$

Fixed point. Define the map $S : L^2(\Omega)^2 \times [0,1] \to L^2(\Omega)^2$ given by $S(\mathbf{v}, \sigma) = \mathbf{w}^k_{\varepsilon,\sigma}$.

To apply the Leray-Schauder's theorem, we have to check the following:

1. Continuity and compactness of S. The arguments are similar to the case of a scalar equation.
2. $S(\mathbf{v}, 0) = 0$, which is inmmediate.
3. If $\mathbf{v} = S(\mathbf{v}, \sigma)$ for $(\mathbf{v}, \sigma) \in L^2(\Omega)^2 \times [0,1]$ then $\|\mathbf{v}\|_{L^2} \leq C$.

Let us prove the last point.

We assume that $\mathbf{v} = \mathbf{w}^k_{\varepsilon,\sigma}$, and we have to show an uniform bound, with respect to $\sigma \in [0,1]$, of $\|\mathbf{w}^k_{\varepsilon,\sigma}\|_{L^2}$.

For clarity, we replace $\mathbf{w}^k_{\varepsilon,\sigma}$ by $\mathbf{w}$, and $\mathbf{w}^{k-1}_{\varepsilon}$ by $\tilde{\mathbf{w}}$.

We have that, by assumption, $\mathbf{w}$ solves

$$\frac{\sigma}{\tau}(g_\varepsilon(w_i) - g_\varepsilon(\tilde{w}_i)) - \operatorname{div} G_i^\varepsilon(\mathbf{w}) + \varepsilon w_i = \sigma h_i^\varepsilon(\mathbf{w}) \qquad \text{in } \Omega,$$

$$G_i^\varepsilon(\mathbf{w}) \cdot n = 0 \qquad \text{on } \partial\Omega.$$

Using $\varphi = w_i$ we get, similarly to the deduction of the coercivity of A,

$$\sum_{i=1}^{2} \Big(\sigma \int_\Omega (g_\varepsilon(w_i) - g_\varepsilon(\tilde{w}_i)) w_i + \tau\varepsilon \int_\Omega w_i^2 + \tau a_0 c(\varepsilon) \int_\Omega |\nabla w_i|^2 \Big) \leq \tau\sigma \sum_{i=1}^{2} \int_\Omega h_i^\varepsilon(\mathbf{w}) w_i$$

The convexity of F_ε implies $F_\varepsilon(x) - F_\varepsilon(y) \leq F'_\varepsilon(x)(x-y)$.

Choosing $x = g_\varepsilon(w_i)$ and $y = g_\varepsilon(\tilde{w}_i)$, and noticing $g_\varepsilon^{-1} = F'_\varepsilon$, we deduce

$$\int_\Omega (g_\varepsilon(w_i) - g_\varepsilon(\tilde{w}_i)) w_i \geq \int_\Omega (F_\varepsilon(g_\varepsilon(w_i)) - F_\varepsilon(g_\varepsilon(\tilde{w}_i))).$$

For the RHS term, we claim (and it's true!) that, for $i = 1, 2$,

$$f_i^\varepsilon(s_1, s_2) F_\varepsilon'(s_i) \leq C(1 + F_\varepsilon(s_1) + F_\varepsilon(s_2)) \quad \text{for all } s_1, s_2 \in \mathbb{R},$$

with $f_i^\varepsilon(s_1, s_2)) = \alpha_i s_i - a_\varepsilon(s_i)\big(\beta_{i1} a_\varepsilon(s_1) + \beta_{i2} a_\varepsilon(s_2)\big)$.

Taking $s_i = g_\varepsilon(w_i)$, we get

$$\sum_{i=1}^{2} \int_\Omega h_i^\varepsilon(\mathbf{w}) w_i \leq C \sum_{i=1}^{2} \int_\Omega (1 + F_\varepsilon(g_\varepsilon(w_i))).$$

Therefore, we obtain ($\tau < 1/C$ and $\sigma \leq 1$)

$$\sum_{i=1}^{2} \Big(\sigma(1 - C\tau) \int_\Omega F_\varepsilon(g_\varepsilon(w_i)) + \tau\varepsilon \int_\Omega w_i^2 + \tau a_0 c(\varepsilon) \int_\Omega |\nabla w_i|^2 \Big)$$
$$\leq C\sigma\tau + \sigma \sum_{i=1}^{2} \int_\Omega \int_\Omega F_\varepsilon(g_\varepsilon(\tilde{w}_i)).$$

Since, by assumption, $F_\varepsilon(g_\varepsilon(w_{i,\varepsilon}^{k-1})) \in L^1(\Omega)$, we deduce the $\sigma-$uniform estimate for $\mathbf{w}_{\varepsilon,\sigma}^k$.

Back to the original unknowns

We define $u_{i,\varepsilon}^k = g_\varepsilon(w_{i,\varepsilon}^k)$ and notice that $u_{i,\varepsilon}^k \in H^1(\Omega)$, since

$$\nabla u_{i,\varepsilon}^k = g_\varepsilon'(w_{i,\varepsilon}^k)\nabla w_{i,\varepsilon}^k = a_\varepsilon(g_\varepsilon(w_{i,\varepsilon}^k))\nabla w_{i,\varepsilon}^k.$$

Introducing this change of unknowns in the equations for $\mathbf{w}_\varepsilon^k$, with $\sigma = 1$, we see that $\mathbf{u}_\varepsilon^k$ satisfies,

$$\begin{aligned} &\frac{1}{\tau}(u_{i,\varepsilon}^k - u_{i,\varepsilon}^{k-1}) - \operatorname{div} J_i^\varepsilon(\mathbf{u}_\varepsilon^k) + \varepsilon F_\varepsilon'(u_{i,\varepsilon}^k) = f_i^\varepsilon(\mathbf{u}_\varepsilon^k) && \text{in } \Omega, \\ &J_i^\varepsilon(\mathbf{u}_\varepsilon^k)\cdot n = 0 && \text{on } \partial\Omega, \end{aligned}$$

for given $\mathbf{u}_\varepsilon^{k-1} \in L^2(\Omega)^2$ with $F_\varepsilon(u_{i,\varepsilon}^{k-1}) \in L^1(\Omega)$, with

$$\begin{aligned} J_i^\varepsilon(\mathbf{u}) &= a_\varepsilon(u_i)(a_i\nabla u_i + \nabla u_j), \\ f_i^\varepsilon(\mathbf{u}) &= \alpha_i u_i - a_\varepsilon(u_i)\big(\beta_{i1}a_\varepsilon(u_1) + \beta_{i2}a_\varepsilon(u_2)\big). \end{aligned}$$

Moreover, using $\varphi = F'_\varepsilon(u^k_{i,\varepsilon})$ leads to

$$\sum_{i=1}^{2} \max_{k=1,\ldots,K} \Big(\int_\Omega F_\varepsilon(u^k_{i,\varepsilon}) + \int_\Omega |u^k_{i,\varepsilon}| + \frac{1}{\varepsilon} \int_\Omega ([u^k_{i,\varepsilon}]_-)^2 + \tau\varepsilon \int_\Omega |F'_\varepsilon(u^k_{i,\varepsilon})|^2 \Big) +$$
$$\tau \sum_{i=1}^{2} \sum_{k=1}^{K} \int_\Omega |\nabla u^k_{i,\varepsilon}|^2 \leq C.$$

Back to the evolution problem

Consider

$$u_{i,\varepsilon}^{(\tau)}(t,x) = u_{i,\varepsilon}^{k}(x), \quad \tilde{u}_{i,\varepsilon}^{(\tau)}(t,x) = u_{i,\varepsilon}^{k}(x) + \frac{t_k - t}{\tau}(u_{i,\varepsilon}^{k-1}(x) - u_{i,\varepsilon}^{k}(x)).$$

Replacing these functions in the weak formulation

$$\int_0^T \partial_t \tilde{u}_{i,\varepsilon}^{(\tau)} \varphi + \int_{Q_T} J_i^{\varepsilon}(\mathbf{u}_{\varepsilon}^{(\tau)}) \cdot \nabla \varphi + \varepsilon \int_{Q_T} F_{\varepsilon}'(u_{i,\varepsilon}^{(\tau)})\varphi = \int_{Q_T} f_i^{\varepsilon}(\mathbf{u}_{\varepsilon}^{(\tau)})\varphi,$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$, and the discrete energy estimate gives

$$\max_{t\in(0,T)} \Big(\int_{\Omega} F_{\varepsilon}(u_{i,\varepsilon}^{(\tau)}(t)) + \int_{\Omega} |u_{i,\varepsilon}^{(\tau)}(t)| + \frac{1}{\varepsilon}\int_{\Omega}([u_{i,\varepsilon}^{(\tau)}(t)]_-)^2 \Big) + \varepsilon \int_{Q_T} |F_{\varepsilon}'(u_{i,\varepsilon}^{(\tau)})|^2 + \int_{Q_T} |\nabla u_{i,\varepsilon}^{(\tau)}|^2 \leq C.$$

Poincaré-Wirtinger's inequality implies

$$\|u_{i,\varepsilon}^{(\tau)}\|_{L^2(H^1)} \leq C, \quad \|\tilde{u}_{i,\varepsilon}^{(\tau)}\|_{L^2(H^1)} \leq C.$$

Time derivative estimate.

We have, for $\varphi \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T \langle \partial_t \tilde{u}_{i,\varepsilon}^{(\tau)}, \varphi\rangle \leq a_i \int_{Q_T} |a_\varepsilon(u_{i,\varepsilon}^{(\tau)})| \big(|\nabla u_{i,\varepsilon}^{(\tau)}| + |\nabla u_{j,\varepsilon}^{(\tau)}|\big)|\nabla\varphi| + \int_{Q_T} |f_i^\varepsilon(\mathbf{u}_\varepsilon^{(\tau)})||\varphi|$$
$$+\varepsilon \int_{Q_T} |F_\varepsilon'(u_{i,\varepsilon}^{(\tau)})||\varphi| \leq C\varepsilon^{-2}\|\varphi\|_{L^2(H^1)},$$

and thus

$$\|\partial_t \tilde{u}_{i,\varepsilon}^{(\tau)}\|_{L^2((H^1)')} \leq C\varepsilon^{-2}.$$

The limit $\tau \to 0$

From the previous bounds, we deduce, as $\tau \to 0$,

$$
\begin{aligned}
\mathbf{u}_\varepsilon^{(\tau)} &\rightharpoonup \mathbf{u}_\varepsilon && \text{weakly in } L^2(0,T;H^1(\Omega))^2,\\
\mathbf{u}_\varepsilon^{(\tau)} &\rightharpoonup \mathbf{u}_\varepsilon && \text{weakly in } L^2(Q_T)^2,\\
\tilde{\mathbf{u}}_\varepsilon^{(\tau)} &\rightharpoonup \mathbf{z}_\varepsilon && \text{weakly in } L^2(0,T;H^1(\Omega))^2,\\
\tilde{\mathbf{u}}_\varepsilon^{(\tau)} &\rightharpoonup \mathbf{z}_\varepsilon && \text{weakly in } L^2(Q_T)^2,\\
\partial_t\tilde{\mathbf{u}}_\varepsilon^{(\tau)} &\rightharpoonup \partial_t\mathbf{z}_\varepsilon && \text{weakly in } L^2(0,T;(H^1(\Omega))')^2,
\end{aligned}
$$

the identification $\mathbf{z}_\varepsilon = \mathbf{u}_\varepsilon$ being deduced like in the scalar case.

Compactness and strong convergences.

Aubin-Lions-Simon's lemma, gives

$$\tilde{\mathbf{u}}_\varepsilon^{(\tau)} \to \mathbf{u}_\varepsilon \quad \text{strongly in } L^2(Q_T)^2, \text{ and a.e. in } Q_T.$$

In particular, like in the scalar case, we also obtain

$$\mathbf{u}_\varepsilon^{(\tau)} \to \mathbf{u}_\varepsilon \quad \text{strongly in } L^2(Q_T), \text{ and a.e. in } Q_T.$$

Convergence.

Passing to the limit $\tau \to 0$ is justified like in the previous cases.

We obtain that $\mathbf{u}_\varepsilon \in L^2(0,T;H^1(\Omega))^2 \cap H^1(0,T;(H^1(\Omega))')^2$ satisfies,

$$\int_0^T \langle \partial_t u_{i,\varepsilon}, \varphi \rangle + \int_{Q_T} J_i^\varepsilon(\mathbf{u}_\varepsilon) \cdot \nabla \varphi + \varepsilon \int_{Q_T} F'_\varepsilon(u_{i,\varepsilon})\varphi = \int_{Q_T} f_i^\varepsilon(\mathbf{u}_\varepsilon)\varphi,$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$,

The limit $\varepsilon \to 0$

Taking the limit $\tau \to 0$ we get

$$\max_{t\in(0,T)} \Big(\int_\Omega F_\varepsilon(u_\varepsilon(t)) + \int_\Omega |u_\varepsilon(t)| + \frac{1}{\varepsilon}\int_\Omega ([u_\varepsilon(t)]_-)^2 \Big) + \varepsilon \int_{Q_T} |F'_\varepsilon(u_{i,\varepsilon})|^2 + \int_{Q_T} |\nabla u_\varepsilon|^2 \leq C.$$

- $\|u_\varepsilon\|_{L^2(H^1)} \leq C$.
- Gagliardo-Nirenberg: $\|u_\varepsilon\|_{L^p} \leq C$, for $p = (2N+2)/N$,
- Time derivative: $\|\partial_t u_\varepsilon\|_{L^r((W^{1,r'})')} \leq C$, for $r = (2N+2)/(2N+1)$.
- Also, $\|[u_{i,\varepsilon}]_-\|_{L^\infty(L^2)} \leq C\sqrt{\varepsilon}$ and $\sqrt{\varepsilon}\|F'_\varepsilon(u_{i,\varepsilon})\|_{L^2} \leq C$.

From these bounds, we get

$$
\begin{aligned}
\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly in } L^2(0,T;H^1(\Omega))^2,\\
\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly in } L^p(Q_T)^2,\\
\partial_t \mathbf{u}_\varepsilon &\rightharpoonup \partial_t \mathbf{u} && \text{weakly in } L^r(0,T;(W^{1,r'}(\Omega))')^2,\\
[\mathbf{u}_\varepsilon]_- &\rightharpoonup 0 && \text{weakly*-weakly in } L^\infty(0,T;L^2(\Omega))^2
\end{aligned}
$$

Compactness and strong convergences.

Aubin-Lions-Simon's lemma, and similar reasonings than in the scalar case:

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{strongly in } L^q(Q_T)^2, \text{ for any } 1 \le q < p.$$

$$[\mathbf{u}_\varepsilon]_- \to 0 \quad \text{strongly in } L^q(Q_T)^2 \text{ and a.e. in } Q_T, \text{ that is } u_i \ge 0 \text{ a.e. in } Q_T,$$

$$a_\varepsilon(u_{i,\varepsilon}) \to u_i \quad \text{strongly in } L^q(Q_T).$$

with $2 \le q < p = (2N+2)/N$.

Convergence.

Except for the term involving $F'_\varepsilon(u_{i,\varepsilon})$, the passing to the limit of the rest of terms are justified like in the scalar case. For the former,

$$\varepsilon \int_{Q_T} F'_\varepsilon(u_{i,\varepsilon})\varphi \le \varepsilon \|F'_\varepsilon(u_{i,\varepsilon})\|_{L^2} \|\varphi\|_{L^2} \le C\sqrt{\varepsilon} \to 0.$$

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz continuous boundary, and let $T > 0$. Suppose that $u_{i0} \in L^2(\Omega)$ are non-negative, for $i = 1, 2$, and define

$$\begin{aligned} J_i(u_1, u_2) &= u_i(a_i \nabla u_i + \nabla u_j), \quad \text{with } a_i > 0,\ a_1 a_2 > 1, \\ f_i(u_1, u_2) &= u_i\big(\alpha_i - (\beta_{i1} u_1 + \beta_{i2} u_2)\big), \end{aligned}$$

for $i, j = 1, 2$ and $i \neq j$. Then, there exists non-negative (u_1, u_2), with

$$u_i \in L^2(0, T; H^1(\Omega)) \cap L^p(Q_T) \cap W^{1,r}(0, T; (W^{1,r'}(\Omega))'),$$

where $p = 2(N+1)/N$, $r = 2(N+1)/(2N+1)$, and $r' = 2(N+1)$, satisfying, for all $\varphi \in L^{r'}(0, T; W^{1,r'}(\Omega))$ and $i = 1, 2$,

$$\int_0^T < \partial_t u_i, \varphi > + \int_{Q_T} J_i(u_1, u_2) \cdot \nabla \varphi = \int_{Q_T} f_i(u_1, u_2)\, \varphi,$$

with $\langle \cdot, \cdot \rangle$ denoting the duality product between $W^{1,r'}(\Omega)$ and its dual $(W^{1,r'}(\Omega))'$. The initial data is satisfied in the sense of $(W^{1,r'}(\Omega))'$.

Thank you!