Turing instability for a nonlinear reaction-diffusion system with cross-diffusion

Gonzalo Galiano

Dpt. of Mathematics -University of Oviedo

Bifurcation: qualitative change in behavior of the equilibrium of a system.

Due to variation of a bifurcation parameter. Leads to a new steady state.

Stability of uniform steady states related to the sign of λ_k of the linearized system. If the steady state is initially stable, $\mathrm{Re}(\lambda_k) < 0$.

At bifurcation, at least one eigenvalue crosses the imaginary axis:

- Turing bifurcation, in which one eigenvalue crosses the origin,
- Hopf bifurcation, where a pair of imaginary eigenvalues crosses the real axis and results in a limit cycle with oscillations.

We analyze the occurence of **Turing bifurcation.**

The contents is extracted from:

1. G. Gambino, M.C. Lombardo, M. Sammartino,
Turing instability and traveling fronts for a nonlinear reaction-diffusion system with cross-diffusion,
Mathematics and Computers in Simulation 82(6) (2012) 1112-1132.

Outline

1. **Linear self-diffusion problem**
   - Conditions for linear instability
   - Linear stability of the competitive Lotka-Volterra system

2. **Cross-diffusion problem**
   - Conditions for linear instability
     - Election of the bifurcation parameter
     - Critical value of the bifurcation parameter
   - Amplitude equations and weakly nonlinear analysis
   - The supercritical case

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Problem:

$$\begin{aligned}
&\partial_t u_1 - d_1 \Delta u_1 = \gamma f_1(u_1, u_2) && \text{in } Q_T = (0,T)\times\Omega,\\
&\partial_t u_2 - d_2 \Delta u_2 = \gamma f_2(u_1, u_2) && \text{in } Q_T,\\
&\nabla u_1 \cdot n = \nabla u_2 \cdot n = 0 && \text{on } \Gamma_T = \partial(0,T)\times\Omega,\\
&u_1(\cdot,0) = u_{10}, \quad u_2(\cdot,0) = u_{20} && \text{in } \Omega.
\end{aligned}$$

We take $\Omega = [0, 2\pi]$, and (f_1, f_2) nonlinear, e.g. competitive Lotka-Volterra.

In vector form:

$$\partial_t \mathbf{u} - d\Delta \mathbf{u} = \gamma \mathbf{f}(\mathbf{u}), \qquad d = diag(d_1, d_2).$$

Non-trivial uniform steady state $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2)$ are constant positive solutions of

$$\mathbf{f}(\tilde{\mathbf{u}}) = \mathbf{0}.$$

Linearization around $\tilde{\mathbf{u}}$ gives, for $\mathbf{w} = \mathbf{u} - \tilde{\mathbf{u}}$,

$$\partial_t \mathbf{w} - d\Delta \mathbf{w} = \gamma D\mathbf{f}(\tilde{\mathbf{u}})\mathbf{w},$$

with

$$D\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \partial_1 f_1(\mathbf{u}) & \partial_2 f_1(\mathbf{u}) \\ \partial_1 f_2(\mathbf{u}) & \partial_2 f_2(\mathbf{u}) \end{pmatrix}.$$

We look for a particular solution of the form

$$\mathbf{w} = \exp(\lambda_k t + ikx)\mathbf{u}_k,$$

where

- $\mathbf{u}_k$ is a constant eigenvector,
- λ_k is an eigenvalue, representing the linear growth rate,
- k is the wavenumber of the perturbation.

Upon substitution, one gets the eigenvalue problem

$$A_k \mathbf{w} = \lambda_k \mathbf{w}, \quad \text{with } A_k = \gamma D\mathbf{f}(\tilde{\mathbf{u}}) - k^2 d.$$

For each k, we obtain the particular solution

$$\left(c_{1k}\mathbf{u}_{1k}e^{\lambda_{1k}t} + c_{2k}\mathbf{u}_{2k}e^{\lambda_{2k}t}\right)e^{ikx},$$

where c_{jk} depend on the initial data.

The general solution can be expressed as

$$\mathbf{w}(t,x) = \sum_k \left(c_{1k}\mathbf{u}_{1k}e^{\lambda_{1k}t} + c_{2k}\mathbf{u}_{2k}e^{\lambda_{2k}t}\right)e^{ikx}.$$

The characteristic equation is, for $A_k = \gamma D\mathbf{f}(\tilde{\mathbf{u}}) - k^2 d$

$$\lambda_k^2 - \mathrm{tr}(A_k)\lambda_k + \det(A_k) = 0,$$

where

$$\begin{aligned}\mathrm{tr}(A_k) &= \gamma(\partial_1 f_1(\tilde{\mathbf{u}}) + \partial_2 f_2(\tilde{\mathbf{u}})) - k^2(d_1 + d_2),\\ \det(A_k) &= d_1 d_2 k^4 - \gamma(d_2\partial_1 f_1(\tilde{\mathbf{u}}) + d_1\partial_2 f_2(\tilde{\mathbf{u}}))k^2 + \gamma^2 \det(D\mathbf{f}(\tilde{\mathbf{u}})),\end{aligned}$$

having the roots

$$\lambda_k = \frac{1}{2}\Big(\mathrm{tr}(A_k) \pm \sqrt{\mathrm{tr}(A_k) - 4\det(A_k)}\Big).$$

Conditions for linear instability

We look for diffusion-driven instability: if no spatial variations ($k = 0$) then $\mathrm{Re}(\lambda_{j0}) < 0$. This implies

$$\mathrm{tr}(A_0) = \mathrm{tr}(D\mathbf{f}(\tilde{\mathbf{u}})) < 0, \quad \det(A_0) = \det(D\mathbf{f}(\tilde{\mathbf{u}})) > 0.$$

Returning to spatial-dependent problem: look for changes of sign of $\mathrm{Re}(\lambda_k)$ when varying diffusion coefficients.

We have $\mathrm{tr}(A_k) < 0$. The only way for $\mathrm{Re}(\lambda_k) > 0$ is $\det(A_k) < 0$, with

$$\det(A_k) = d_1 d_2 k^4 - \gamma(d_2 \partial_1 f_1(\tilde{\mathbf{u}}) + d_1 \partial_2 f_2(\tilde{\mathbf{u}}))k^2 + \gamma^2 \det(D\mathbf{f}(\tilde{\mathbf{u}})).$$

The point of minimum is

$$k_c^2 = \gamma \frac{d_2 \partial_1 f_1(\tilde{\mathbf{u}}) + d_1 \partial_2 f_2(\tilde{\mathbf{u}})}{2 d_1 d_2},$$

and the minimum value is

$$h(k_c^2) = \gamma^2 \Big[\det(D\mathbf{f}(\tilde{\mathbf{u}})) - \frac{\big(d_2 \partial_1 f_1(\tilde{\mathbf{u}}) + d_1 \partial_2 f_2(\tilde{\mathbf{u}})\big)^2}{4 d_1 d_2} \Big].$$

We have $h(k_c^2) = 0$ (bifurcation) if d_c is a positive root of

$$\big(\partial_1 f_1(\tilde{\mathbf{u}})\big)^2 d_c^2 + 2\big(2\partial_2 f_1(\tilde{\mathbf{u}})\partial_1 f_2(\tilde{\mathbf{u}}) - \partial_1 f_1(\tilde{\mathbf{u}})\partial_2 f_2(\tilde{\mathbf{u}})\big) d_c + \big(\partial_2 f_2(\tilde{\mathbf{u}})\big)^2 = 0,$$

where d_c is the *critical diffusion ratio*.

If d_c exists, then *critical wavenumber* is obtained from k_c^2, with $d_2^c / d_1^c = d_c$.

For $d^* > d_c$, exists a range of unstable wavenumbers in $[k_1^2, k_2^2]$, where $\det(A_{k_1}) = \det(A_{k_2}) = 0$.

The wavenumbers are discrete and a finite number in $[k_1^2, k_2^2]$.

Within this range, $\mathrm{Re}(\lambda_k)$ is positive and assumes its maximum value for k_c^2.

For large t,

$$\mathbf{w}(t, x) \approx \sum_{k=k_1}^{k_2} \mathbf{u}_k e^{\lambda_k t} e^{ikx}.$$

Linear stability of the competitive Lotka-Volterra system

Consider

$$f_i(\mathbf{u}) = \alpha_i u_i - (\beta_{i1} u_1 + \beta_{i2} u_2) u_i,$$

with $\alpha_i, \beta_{ij} \geq 0$, for $i, j = 1, 2$. The co-existence equilibrium is

$$\tilde{\mathbf{u}} = \Big(\frac{\beta_{22}\alpha_1 - \beta_{12}\alpha_2}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}, \frac{\beta_{11}\alpha_2 - \beta_{21}\alpha_1}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}\Big),$$

with $\tilde{u}_i > 0$, for which

$$D\mathbf{f}(\mathbf{u}) = \begin{pmatrix} -\beta_{11}\tilde{u}_1 & -\beta_{12}\tilde{u}_1 \\ -\beta_{21}\tilde{u}_2 & -\beta_{22}\tilde{u}_2 \end{pmatrix}.$$

$\tilde{\mathbf{u}}$ is stable for the dynamical system if the eigenvalues of $D\mathbf{f}(\mathbf{u})$ are negative,

$$\mu^2 - \text{tr}(D\mathbf{f}(\mathbf{u})) + \det(D\mathbf{f}(\mathbf{u})) = 0.$$

Thus, the conditions are

$$\text{tr}(D\mathbf{f}(\mathbf{u})) < 0, \quad \text{and} \quad \det(D\mathbf{f}(\mathbf{u})) > 0 \qquad \text{(for negative real part),}$$
$$\text{tr}(D\mathbf{f}(\mathbf{u}))^2 - 4\det(D\mathbf{f}(\mathbf{u})) \geq 0 \qquad \text{(for null imaginary part).}$$

The second condition is equivalent to

$$(\beta_{11}\tilde{u}_1 - \beta_{22}\tilde{u}_2)^2 + 4\beta_{12}\beta_{21}\tilde{u}_1\tilde{u}_2 \geq 0.$$

Both conditions are satisfied if $\beta_{ij} \geq 0$, and

$$\text{tr}(B) > 0, \quad \text{and} \quad \det(B) > 0, \quad \text{with} \quad B = (\beta_{ij}).$$

Returning to the spatial-dependent problem and writing

$$\big(\partial_1 f_1(\tilde{\mathbf{u}})\big)^2 d_c^2 + 2\big(2\partial_2 f_1(\tilde{\mathbf{u}})\partial_1 f_2(\tilde{\mathbf{u}}) - \partial_1 f_1(\tilde{\mathbf{u}})\partial_2 f_2(\tilde{\mathbf{u}})\big) d_c + \big(\partial_2 f_2(\tilde{\mathbf{u}})\big)^2 = 0,$$

as $ad_c^2 + bd_c + c = 0$, the solutions are $d_c = \frac{1}{2}(-b \pm \sqrt{b^2 - 4ac})$. For real and positive solutions

$$b^2 - 4ac > 0 \quad \text{and} \quad b < 0.$$

After some computations,

$$b^2 - 4ac > 0 \iff \beta_{12}\beta_{21} > \beta_{11}\beta_{22},$$

contradicts the stability assumption $\det(B) > 0$ for the dynamical system.

Thus $\tilde{\mathbf{u}}$ is linearly stable for any choice of the diffusion coefficients.

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May the Lotka-Volterra system be instable in more complex situations?

We study the SKT cross-diffusion case:

$$
\begin{aligned}
&\partial_t u_1 - \operatorname{div} J_1(\mathbf{u}) = \gamma f_1(u_1, u_2) && \text{in } Q_T,\\
&\partial_t u_2 - \operatorname{div} J_2(\mathbf{u}) = \gamma f_2(u_1, u_2) && \text{in } Q_T,\\
&J_1(\mathbf{u}) \cdot n = J_2(\mathbf{u}) \cdot n = 0 && \text{on } \Gamma_T,\\
&u_1(\cdot, 0) = u_{10}, \quad u_2(\cdot, 0) = u_{20} && \text{in } \Omega,
\end{aligned}
$$

with flows and reaction terms

$$
\begin{aligned}
J_i(\mathbf{u}) &= \nabla\big(u_i(d_1 + a_{i1}\nabla u_1 + a_{i2}\nabla u_2)\big)\\
f_i(\mathbf{u}) &= \alpha_i u_i - (\beta_{i1} u_1 + \beta_{i2} u_2) u_i,
\end{aligned}
$$

with the coefficients $B = (\beta_{ij})$ satisfying the kinetic stability conditions

$$
\operatorname{tr}(B) > 0, \quad \text{and} \quad \det(B) > 0.
$$

We study the co-existence homogeneous stationary state

$$\tilde{\mathbf{u}} = \Big(\frac{\beta_{22}\alpha_1 - \beta_{12}\alpha_2}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}, \frac{\beta_{11}\alpha_2 - \beta_{21}\alpha_1}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}}\Big),$$

with $\tilde{u}_i > 0$, for which,

$$K := D\mathbf{f}(\tilde{\mathbf{u}}) = \begin{pmatrix} -\beta_{11}\tilde{u}_1 & -\beta_{12}\tilde{u}_1 \\ -\beta_{21}\tilde{u}_2 & -\beta_{22}\tilde{u}_2 \end{pmatrix}.$$

Linearization around $\tilde{\mathbf{u}}$ gives the following system for $\mathbf{w} = \mathbf{u} - \tilde{\mathbf{u}}$

$$\partial_t \mathbf{w} - D\Delta\mathbf{w} = \gamma K \mathbf{w},$$

with

$$D = \begin{pmatrix} d_1 + 2a_{11}\tilde{u}_1 + a_{12}\tilde{u}_2 & a_{12}\tilde{u}_1 \\ a_{21}\tilde{u}_2 & d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2 \end{pmatrix}.$$

The corresponding eigenvalue problem leads to

$$\lambda_k^2 - \mathrm{tr}(A_k)\lambda_k + h(k^2) = 0,$$

with $A_k = \gamma K - k^2 D$, and

$$h(k^2) = \det(A_k) = \det(D)k^4 + \gamma q k^2 + \gamma^2 \det(K),$$

being

$$\begin{aligned} q =& \beta_{11}\tilde{u}_1(2a_{22}\tilde{u}_2 + d_2) + \beta_{22}\tilde{u}_2(2a_{11}\tilde{u}_1 + d_1) \\ &+ a_{12}\tilde{u}_2(\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1) + a_{21}\tilde{u}_1(\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2). \end{aligned}$$

Conditions for linear instability

Spatial patterns arise for $\mathrm{Re}(\lambda_k) > 0$.

Since $\tilde{\mathbf{u}}$ is stable for the kinetics then $\mathrm{tr}(A_k) < 0$.

Therefore, the only way to have $\mathrm{Re}(\lambda_k) > 0$ is $h(k^2) < 0$.

Condition for marginal stability

$$\min(h(k_c^2)) = 0.$$

The minimum of h is attained for

$$k_c^2 = -\frac{\gamma q}{2\det(D)},$$

which requires $q < 0$.

The only potential destabilizing mechanism in q is the cross-diffusion.

Election of the bifurcation parameter

$$q = \beta_{11}\tilde{u}_1(2a_{22}\tilde{u}_2 + d_2) + \beta_{22}\tilde{u}_2(2a_{11}\tilde{u}_1 + d_1) + a_{12}\tilde{u}_2(\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1) + a_{21}\tilde{u}_1(\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2).$$

Conditions on the positiveness and stability of $\tilde{\mathbf{u}}$ imply that

$$\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1 < 0 \quad \text{OR} \quad \beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2 < 0.$$

When a_{12} destabilizes then a_{21} stabilizes and vice versa.

We choose $\beta_{22}\tilde{u}_2 - \beta_{21}\tilde{u}_1 < 0$ and

$b := a_{12}$ as the bifurcation parameter.

Critical value of the bifurcation parameter

Since $h(k^2)$ depends on b, one gets the bifurcation value from

$$\min(h(k_c^2)) = 0.$$

Consider

$$m_1 = \tilde{u}_2(\beta_{21}\tilde{u}_1 - \beta_{22}\tilde{u}_2) \geq 0,$$
$$m_2 = \beta_{11}\tilde{u}_1(2a_{22}\tilde{u}_2 + d_2) + \beta_{22}\tilde{u}_2(2a_{11}\tilde{u}_1 + d_1) + a_{21}\tilde{u}_1(\beta_{11}\tilde{u}_1 - \beta_{12}\tilde{u}_2) \geq 0,$$

so $q = -m_1 b + m_2$. The minimum value of $h(k^2)$ is

$$\min(h(k_c^2)) = \gamma^2\Big(\det(K) - \frac{(-m_1 b + m_2)^2}{4\det(D)}\Big).$$

Let $\xi \in \mathbb{R}$, to be determined, and set $b = m_2/m_1 + \xi$. We get the marginal stability condition

$$\frac{m_1^2}{4\det(K)}\xi^2 - \det(D) = 0.$$

Replacing $a_{12} \equiv b = m_2/m_1 + \xi$ in D, we get

$$\det(D) = \tilde{u}_2(d_2 + 2a_{22}\tilde{u}_2)\xi + \Big(\frac{m_2}{m_1}\tilde{u}_2(d_2 + 2a_{22}\tilde{u}_2) + (d_1 + 2a_{11}\tilde{u}_1)(d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2)\Big).$$

Therefore, $\frac{m_1^2}{4\det(K)}\xi^2 - \det(D) = 0$ has a positive root, denoted by ξ^+.

The critical value for bifurcation is

$$b^c = \frac{m_2}{m_1} + \xi^+.$$

Observe that $q := -m_1 b + m_2 < 0$ is guaranteed.

For $b > b^c$ the system has a finite k pattern-forming stationary instability.

Unstable wavenumbers are between the roots of $h(k^2)$, denoted by k_1^2 and k_2^2.

It is straightforward to check that these roots are proportional to γ.

For pattern formation, γ must be big enough so that at least one of the modes allowed by the boundary conditions is in $[k_1^2, k_2^2]$.

Amplitude equations and weakly nonlinear analysis

Linear stability theory is useful for understanding pattern formation:

- Diffusion is the key mechanism.
- Determine conditions on system parameters.
- Gives length scale of pattern formation, $1/k_c$.

But,

- the exponentially growing solutions are physically meaningless.

To predict the amplitude and the form, nonlinear terms must be included.

We perform a weakly nonlinear analysis based on multiple scales.

Nonlinear expansion

In Turing bifurcation,

- Vlose to the bifurcation, $\mathrm{Re}(\lambda_k) < 0$.
- The linear instability must be preceded by $\mathrm{Re}(\lambda_k) = 0$.

Therefore, the pattern evolves on a slow temporal scale, like $e^{\lambda_k t}$, with $\lambda_k \approx 0$.

- New, scaled, magnitudes are considered, and treated as separate variables.
- We fix a control parameter

$$\varepsilon^2 = \frac{b - b_c}{b_c},$$

and write the solution of the original system as a expansion in terms of ε^2.

Considering a random perturbation, $\mathbf{w}$, around the steady state, we recast the original nonlinear system as

$$\partial_t \mathbf{w} = \mathcal{L}^b \mathbf{w} + \mathcal{N}^b \mathbf{w},$$

where $\mathcal{L}^b = \gamma K + D^b \Delta$ is the linear part, and $\mathcal{N}^b$ contains 2nd order terms,

$$\mathcal{N}^b = \frac{1}{2} \mathcal{Q}_K(\mathbf{w}, \mathbf{w}) + \frac{1}{2} \Delta \mathcal{Q}_D^b(\mathbf{w}, \mathbf{w}),$$

with the bilinear forms

$$\mathcal{Q}_K(\mathbf{x}, \mathbf{y}) = \gamma \begin{pmatrix} -2\beta_{11} x^1 y^1 - \beta_{12}(x^1 y^2 + x^2 y^1) \\ -2\beta_{22} x^2 y^2 - \beta_{21}(x^1 y^2 + x^2 y^1) \end{pmatrix},$$

$$\mathcal{Q}_D^b(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 2a_{11} x^1 y^1 + b(x^1 y^2 + x^2 y^1) \\ 2a_{22} x^2 y^2 + a_{21}(x^1 y^2 + x^2 y^1) \end{pmatrix}.$$

The idea is:
Expand $\mathbf{w}$ in ε, so the leading term is $A(t)e^{ik_c x}$, with A slowly varying.

We consider

$$\begin{aligned} b &= b^c + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 b_3 + O(\varepsilon^4), \\ \mathbf{w} &= \varepsilon \mathbf{w}_1 + \varepsilon^2 \mathbf{w}_2 + \varepsilon^3 \mathbf{w}_3 + O(\varepsilon^4), \\ \partial_t &= \varepsilon \partial_{T_1} + \varepsilon^2 \partial_{T_2} + \varepsilon^3 \partial_{T_3} + O(\varepsilon^4). \end{aligned}$$

Then,

$$\begin{aligned} D^b = &\begin{pmatrix} d_1 + 2a_{11}\tilde{u}_1 + b\tilde{u}_2 & b\tilde{u}_1 \\ a_{21}\tilde{u}_2 & d_2 + a_{21}\tilde{u}_1 + 2a_{22}\tilde{u}_2 \end{pmatrix} \\ = &D^{b^c} + \sum_{j=1}^{3} \varepsilon^j \begin{pmatrix} b_j \tilde{u}_2 & b_j \tilde{u}_1 \\ 0 & 0 \end{pmatrix} + O(\varepsilon^4). \end{aligned}$$

$\mathcal{L}^b = \gamma K + D^b \Delta$ takes the form

$$\mathcal{L}^b = \mathcal{L}^{b^c} + \sum_{j=1}^{3} \varepsilon^j \begin{pmatrix} b_j \tilde{u}_2 & b_j \tilde{u}_1 \\ 0 & 0 \end{pmatrix} \Delta + O(\varepsilon^4), \quad \text{with } \mathcal{L}^{b^c} = \gamma K + D^{b^c} \Delta.$$

For the quadratic terms

$$\mathcal{Q}_K(\mathbf{w}, \mathbf{w}) = \varepsilon^2 \mathcal{Q}_K(\mathbf{w}_1, \mathbf{w}_1) + 2\varepsilon^3 \mathcal{Q}_K(\mathbf{w}_1, \mathbf{w}_2) + O(\varepsilon^4),$$
$$\mathcal{Q}_D^b(\mathbf{w}, \mathbf{w}) = \varepsilon^2 \mathcal{Q}_D^{b^c}(\mathbf{w}_1, \mathbf{w}_1) + 2\varepsilon^3 \Big(Q_D^{b^c}(\mathbf{w}_1, \mathbf{w}_2) + (b_1 w_1^1 w_1^2, 0)^t \Big) + O(\varepsilon^4).$$

For the time derivative expansion,

$$\partial_t \mathbf{w} = \varepsilon^2 \partial_{T_1} \mathbf{w}_1 + \varepsilon^3 (\partial_{T_1} \mathbf{w}_2 + \partial_{T_2} \mathbf{w}_1) + O(\varepsilon^4).$$

Introducing these expansions in $\partial_t \mathbf{w} = \mathcal{L}^b \mathbf{w} + \mathcal{N}^b \mathbf{w}$, leads to

$$O(\varepsilon): \quad \mathcal{L}^{b^c} \mathbf{w}_1 = 0,$$

$$O(\varepsilon^2): \quad \mathcal{L}^{b^c} \mathbf{w}_2 = \partial_{T_1} \mathbf{w}_1 - \frac{1}{2}\left(\mathcal{Q}_K(\mathbf{w}_1, \mathbf{w}_1) + \Delta \mathcal{Q}_D^{b^c}(\mathbf{w}_1, \mathbf{w}_1)\right) - b_1 \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} \Delta \mathbf{w}_1 =: \mathbf{F},$$

$$O(\varepsilon^3): \quad \mathcal{L}^{b^c} \mathbf{w}_3 = \partial_{T_1} \mathbf{w}_2 + \partial_{T_2} \mathbf{w}_1 - \mathcal{Q}_K(\mathbf{w}_1, \mathbf{w}_2) - \Delta \mathcal{Q}_D^{b^c}(\mathbf{w}_1, \mathbf{w}_2) - b_1 \Delta \begin{pmatrix} w_1^1 w_1^2 \\ 0 \end{pmatrix} - \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} (b_1 \Delta \mathbf{w}_2 + b_2 \Delta \mathbf{w}_1) =: \mathbf{G},$$

with

$$\mathcal{L}^{b^c} = \gamma K + D^{b^c} \Delta.$$

Studying the orders of the expansion

We solve in $x \in (0, 2\pi/k_c)$, and later adapt to $\Omega = (0, 2\pi)$.

Order ε: The solution of the linear problem $\mathcal{L}^{b^c}\mathbf{w}_1 = 0$ in $(0, 2\pi/k_c)$ with Neumann boundary conditions is

$$\mathbf{w}_1 = A(T_1, T_2)\boldsymbol{\rho}\cos(k_c x), \quad \text{with } \boldsymbol{\rho} \in \ker(\gamma K - k_c^2 D^{b^c}),$$

where A is still arbitrary.

$\boldsymbol{\rho}$ is defined up to a multiplicative constant, we normalize

$$\boldsymbol{\rho} = (1, M)^t, \quad \text{with } M = \frac{-\gamma K_{21} + D_{21}^{b^c} k_c^2}{\gamma K_{22} - D_{22}^{b^c} k_c^2},$$

where $K_{ij}, D_{ij}^{b^c}$ are the i, j-entries of the matrices K and D^{b^c}.

Order ε^2: $\mathcal{L}^{b^c}\mathbf{w}_2 = \mathbf{F}$.

Observing that

$$\mathcal{Q}_K(\mathbf{w}_1,\mathbf{w}_1) = A(T_1,T_2)^2\cos^2(k_c x)\mathcal{Q}_K(\boldsymbol{\rho},\boldsymbol{\rho}),$$
$$\mathcal{Q}_D^{b^c}(\mathbf{w}_1,\mathbf{w}_1) = A(T_1,T_2)^2\cos^2(k_c x)\mathcal{Q}_D^{b^c}(\boldsymbol{\rho},\boldsymbol{\rho}),$$

and using standard trigonometric identities, we find

$$\begin{aligned}\frac{1}{2}\left(\mathcal{Q}_K(\mathbf{w}_1,\mathbf{w}_1)+\Delta\mathcal{Q}_D^{b^c}(\mathbf{w}_1,\mathbf{w}_1)\right)\\ &=\frac{1}{4}A(T_1,T_2)^2\Big(\mathcal{Q}_K(\boldsymbol{\rho},\boldsymbol{\rho})+\left(\mathcal{Q}_K(\boldsymbol{\rho},\boldsymbol{\rho})-4k_c^2\mathcal{Q}_D^{b^c}(\boldsymbol{\rho},\boldsymbol{\rho})\right)\cos(2k_c x)\Big)\\ &=\frac{1}{4}A(T_1,T_2)^2\sum_{j=0,2}\mathcal{M}_j(\boldsymbol{\rho},\boldsymbol{\rho})\cos(jk_c x),\end{aligned}$$

with

$$\mathcal{M}_j = \mathcal{Q}_K - j^2k_c^2\mathcal{Q}_D^{b^c}.$$

Therefore,

$$\mathbf{F} = -\frac{1}{4}A(T_1, T_2)^2 \sum_{j=0,2} \mathcal{M}_j(\boldsymbol{\rho}, \boldsymbol{\rho}) \cos(jk_c x) + \left(\partial_{T_1} A_1 \boldsymbol{\rho} + b_1 k_c^2 A_1 (\tilde{u}_2 + \tilde{u}_1 M, 0)^t\right) \cos(k_c x).$$

Fredholm alternative: since $\ker \mathcal{L}^{b^c} = span(\rho)$, a solution if and only if

$$\langle \mathbf{F}, \boldsymbol{\psi} \rangle = 0 \quad \text{for} \quad \boldsymbol{\psi} \in \ker((\mathcal{L}^{b^c})^*),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(0, 2\pi/k_c)$, and

$$\boldsymbol{\psi} = (1, M^*)^t \cos(k_c x), \quad \text{with } M^* = \frac{-\gamma K_{12} + D_{12}^{b^c} k_c^2}{\gamma K_{22} - D_{22}^{b^c} k_c^2}.$$

$$0 = \langle \mathbf{F}, \psi \rangle = -\frac{1}{4}A^2 \sum_{j=0,2} \mathcal{M}_j(\boldsymbol{\rho}, \boldsymbol{\rho})(1, M^*)^t \int_0^{\frac{2\pi}{k_c}} \cos(jk_c x)\cos(k_c x)dx$$

$$+\Big(\partial_{T_1} A\boldsymbol{\rho} + b_1 k_c^2 A(\tilde{u}_2 + \tilde{u}_1 M, 0)^t\Big)(1, M^*)^t \int_0^{\frac{2\pi}{k_c}} \cos^2(k_c x)dx.$$

The first integrand at the right hand side vanishes, so we obtain

$$\partial_{T_1} A(T_1, T_2) = \chi A(T_1, T_2), \quad \text{with } \chi = -\frac{b_1 k_c(\tilde{u}_2 + \tilde{u}_1 M)}{1 + MM^*}.$$

No indication on the asymptotic behavior, $T_i \to \infty$, of the amplitude!

Suppress secular terms in **F** setting $T_1 = b_1 = 0$.

Then the compatibility condition is satisfied, and

$$\mathbf{F} = -\frac{1}{4}A(T_2)^2 \sum_{j=0,2} \mathcal{M}_j(\boldsymbol{\rho}, \boldsymbol{\rho})\cos(jk_c x).$$

The solution of $\mathcal{L}^{b^c}\mathbf{w}_2 = \mathbf{F}$ is explicitly computed: Introducing

$$\mathbf{w}_2 = A(T_2)^2 \sum_{j=0,2} \mathbf{w}_{2j} \cos(jk_c x),$$

we get

$$\mathcal{L}^{b^c}\mathbf{w}_2 = (\gamma K + D^{b^c}\Delta)\mathbf{w}_2 = A(T_2)^2 \sum_{j=0,2} \left(\gamma K - (jk_c)^2 D^{b^c}\right)\mathbf{w}_{2j} \cos(jk_c x).$$

$\mathbf{w}_{2j}$ must satisfy the linear systems

$$L_j \mathbf{w}_{2j} = -\frac{1}{4}\mathcal{M}_j(\boldsymbol{\rho}, \boldsymbol{\rho}), \quad \text{for } j = 0, 2,$$

with $L_j = \gamma K - j^2 k_c^2 D^{b^c}$.

Order ε^3: $\mathcal{L}^{b^c}\mathbf{w}_3 = \mathbf{G}$.

Since $T_1 = b_1 = 0$,

$$\mathbf{G} = \partial_{T_2}\mathbf{w}_1 - \mathcal{Q}_K(\mathbf{w}_1, \mathbf{w}_2) - \Delta\mathcal{Q}_D^{b^c}(\mathbf{w}_1, \mathbf{w}_2) - \begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} b_2 \Delta \mathbf{w}_1,$$

where, for $\boldsymbol{\rho} = (\tilde{u}_2 + \tilde{u}_1 M, 0)^t$,

$$\mathbf{w}_1 = A(T_2)\boldsymbol{\rho}\cos(k_c x), \qquad \mathbf{w}_2 = A(T_2)^2(\mathbf{w}_{20} + \mathbf{w}_{22}\cos(2k_c x)).$$

Then

$$\partial_{T_2}\mathbf{w}_1 = \boldsymbol{\rho}\cos(k_c x)\partial_{T_2}A(T_2),$$

$$-\begin{pmatrix} \tilde{u}_2 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} b_2 \Delta \mathbf{w}_1 = A(T_2)k_c^2\cos(k_c x)b_2(\tilde{u}_2 + \tilde{u}_1 M, 0)^t.$$

Using that $\mathcal{Q}_K$ and $\mathcal{Q}_D^{b^c}$ are bilinear, and

$$2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y),$$

and recalling

$$\mathbf{w}_1 = A(T_2)\boldsymbol{\rho}\cos(k_c x), \qquad \mathbf{w}_2 = A(T_2)^2(\mathbf{w}_{20} + \mathbf{w}_{22}\cos(2k_c x)).$$

we get

$$\begin{aligned}
\mathcal{Q}_K(\mathbf{w}_1,\mathbf{w}_2) &= A(T_2)^2\mathcal{Q}_K(\mathbf{w}_1,\mathbf{w}_{20}) + A(T_2)^2\cos(2k_c x)\mathcal{Q}_K(\mathbf{w}_1,\mathbf{w}_{22}) \\
&= A(T_2)^3\cos(k_c x)\mathcal{Q}_K(\boldsymbol{\rho},\mathbf{w}_{20}) \\
&\quad + A(T_2)^3\cos(2k_c x)\cos(k_c x)\mathcal{Q}_K(\boldsymbol{\rho},\mathbf{w}_{22}) \\
&= A(T_2)^3\Big(\cos(k_c x)\big(\mathcal{Q}_K(\boldsymbol{\rho},\mathbf{w}_{20}) \\
&\quad + \frac{1}{2}\mathcal{Q}_K(\boldsymbol{\rho},\mathbf{w}_{22})\big) + \frac{1}{2}\cos(3k_c x)\mathcal{Q}_K(\boldsymbol{\rho},\mathbf{w}_{22})\Big).
\end{aligned}$$

Similarly

$$\Delta \mathcal{Q}_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) = A(T_2)^3\Big(-k_c^2\cos(k_c x)\big(\mathcal{Q}_D^{b^c}(\boldsymbol{\rho},\mathbf{w}_{20}) + \frac{1}{2}\mathcal{Q}_D^{b^c}(\boldsymbol{\rho},\mathbf{w}_{22})\big) - \frac{9}{2}k_c^2\cos(3k_c x)\mathcal{Q}_D^{b^c}(\boldsymbol{\rho},\mathbf{w}_{22})\Big).$$

Recalling the definition $\mathcal{M}_j = \mathcal{Q}_K - j^2 k_c^2 \mathcal{Q}_D^{b^c}$,

$$\mathcal{Q}_K(\mathbf{w}_1,\mathbf{w}_2) + \Delta\mathcal{Q}_D^{b^c}(\mathbf{w}_1,\mathbf{w}_2) = A(T_2)^3\Big(\cos(k_c x)\big(\mathcal{M}_1(\boldsymbol{\rho},\mathbf{w}_{20}) + \frac{1}{2}\mathcal{M}_1(\boldsymbol{\rho},\mathbf{w}_{22})\big) + \frac{1}{2}\cos(3k_c x)\mathcal{M}_3(\boldsymbol{\rho},\mathbf{w}_{22})\Big).$$

Thus,

$$\mathbf{G} = \Big(\rho\partial_{T_2}A + \mathbf{G}_1^{(1)}A + \mathbf{G}_1^{(3)}A^3\Big)\cos(k_c x) + \mathbf{G}_3 A^3\cos(3k_c x),$$

$$\mathbf{G}_1^{(1)} = (\tilde{u}_2 + \tilde{u}_1 M)k_c^2 b_2 (1,0)^t,$$

$$\mathbf{G}_1^{(3)} = -\big(\mathcal{M}_1(\boldsymbol{\rho},\mathbf{w}_{20}) + \frac{1}{2}\mathcal{M}_1(\boldsymbol{\rho},\mathbf{w}_{22})\big),$$

$$\mathbf{G}_3 = -\frac{1}{2}\mathcal{M}_3(\boldsymbol{\rho},\mathbf{w}_{22}).$$

The solvability condition is $\langle \mathbf{G}, \psi \rangle = 0$, leading to

$$\langle \boldsymbol{\rho} \cos(k_c x), \psi \rangle \partial_{T_2} A + \langle \mathbf{G}_1^{(1)} \cos(k_c x), \psi \rangle A + \langle \mathbf{G}_1^{(3)} \cos(k_c x), \psi \rangle A^3 = 0.$$

Thus, recalling the definition of $\psi = (1, M^*)^t \cos(k_c x)$, and defining

$$\sigma = \frac{\mathbf{G}_1^{(1)} \cdot \boldsymbol{\eta}}{\boldsymbol{\rho} \cdot \boldsymbol{\eta}}, \qquad L = \frac{\mathbf{G}_3^{(1)} \cdot \boldsymbol{\eta}}{\boldsymbol{\rho} \cdot \boldsymbol{\eta}},$$

for $\boldsymbol{\eta} = (1, M^*)^t$, the resulting **Stuart-Landau equation** is

$$\partial_{T_2} A = \sigma A - L A^3.$$

We may check that $\sigma > 0$. Two qualitatively cases depending on the sign of L:

- the supercritical case, for $L > 0$,
- the subcritical case, for $L < 0$.

The supercritical case

If $\sigma, L > 0$ in $\partial_{T_2} A = \sigma A - LA^3$ then $A_\infty = \sqrt{\sigma/L}$.

The amplitude and the form of the asymptotic pattern is

$$\widetilde{\mathbf{w}} = \varepsilon \boldsymbol{\rho} \sqrt{\frac{\sigma}{L}} \cos(k_c x) + \varepsilon^2 \frac{\sigma}{L} (\mathbf{w}_{20} + \mathbf{w}_{22} \cos(2k_c x)) + O(\varepsilon^3).$$

In general, this solution is not compatible with the Neumann boundary conditions in $\Omega = [0, 2\pi]$, that require k_c to be integer or semi-integer.

We define $\bar{k}_c$ as the first integer or semi-integer to become unstable when $b > b_c$, and take

$$\widetilde{\mathbf{w}} = \varepsilon \boldsymbol{\rho} \sqrt{\frac{\sigma}{L}} \cos(\bar{k}_c x) + \varepsilon^2 \frac{\sigma}{L} (\mathbf{w}_{20} + \mathbf{w}_{22} \cos(2\bar{k}_c x)) + O(\varepsilon^3).$$