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On the uniqueness of solutions of a nonlinear elliptic problem arising in the confinement of a plasma in a Stellarator device

We obtain the uniqueness of solutions of a nonlocal elliptic problem when the nonlinear terms at the right hand side are assumed to be prescribed. The problem arises in the study of the magnetic confinement of a plasma in a Stellarator device.

1. Introduction

The main goal of this communicztion is to prove the uniqueness of the solution of a two dimensional free boundary problem modeling the magnetic confinement of a plasma in a Stellarator device. The model consists of a second order partial differential equation of elliptic type, obtained from the 3-D ideal MHD system by Hender and Carreras [4] by using toroidal averaging arguments and a suitable system of coordinates. This problem has recently been studied by Díaz [1] who introduced the following formulation in the form of a *free boundary problem.* Let Ω be an open, bounded, regular and connected set contained in $\mathbb{R}^2$, and let

$$\lambda > 0, F_v > 0, a, b \in L^\infty(\Omega), a \geq 0, b > 0 \text{ a.e. in } \Omega.$$

Given $\gamma \in \mathbb{R}_- := \{t \in \mathbb{R} : t < 0\}$, the problem is to find

$$u : \Omega \to \mathbb{R} \text{ and } F : \mathbb{R} \to \mathbb{R}_+$$

such that $F(s) = F_v$ for any $s \leq 0$ and the following conditions hold:

$$(\mathcal{P})\begin{cases} -\Delta u &= aF(u) + F(u)F'(u) + \lambda b u_+ \quad \text{in } \Omega \\ u &= \gamma \quad \text{on } \partial\Omega \\ 0 &= \int_{\{u>t\}} \{F(u)F'(u) + \lambda u_+ b\} \quad \forall t \in [\text{essinf}u,\ \text{esssup}u] \end{cases}$$

In the sequel we will refer to the family of integral identities stated in $(\mathcal{P})$ as the *Stellarator Condition.*

In order to characterize the unknown function F, the above problem was reformulated in Díaz [1] using the notion of *relative rearrangement.* There, he proved that if (u, F) is a solution of $(\mathcal{P})$ such that $u \in \mathcal{U}$ where

$$\mathcal{U} = \{u \in W^{2,p}(\Omega), \text{ for any } 1 \leq p < \infty : meas\,\{x \in \Omega : \nabla u(x) = 0\} = 0\}$$

then u satisfies the following uncoupled non local equation

$$-\Delta u = a\left[F_v^2 - 2\lambda \int_0^{u_+(x)} \sigma b_{*u}(|u > \sigma|)d\sigma\right]_+^{1/2} + \lambda u_+[b - b_{*u}(|u > u(x)|)] \quad \text{in } \Omega \tag{1}$$

where we denote $meas\,\{x \in \Omega : u(x) > t\}$ by $|u > t|$, u_* represents the decreasing rearrangement of u and b_{*u} is the relative rearrangement of b with respect to u (the definition of these functions and some of their properties can be found, for instance, in [6] and in its references). Moreover, if u satisfies (1) then the function $F = F^u$ is given by

$$F^u(t) = \left[F_v^2 - 2\lambda \int_0^{t_+} \sigma b_{*u}(|u > \sigma|)d\sigma\right]_+^{1/2} \quad \text{for any } t \in [\text{essinf}u,\ \text{esssup}u].$$

The existence of u, solution of $(\mathcal{P})$ in the class of functions $\mathcal{U}$ was proved by Díaz and Rakotoson [2, 3]under some additional assumptions.

Here we give a partial result to the uniqueness question. Notice that the equation of $(\mathcal{P})$ involves nonlinear terms which do not need to be convex neither concave functions. Our proof uses some a priori estimates, some properties of the relative rearrangement and the study of a suitable weighted eigenvalue problem. The idea of using an auxiliary linear eigenvalue problem is inspired by the technique used in Puel [5] to establish the uniqueness of solution of a different free boundary problem arising in the study of the plasma confinement in Tokamak devices.

2. The main result

To state the uniqueness result we shall need to refer to the weighted eigenvalue linear problem

$$(\mathcal{P}_g^\mu)\begin{cases} -\Delta w = \mu g(x) w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

as well as to a suitable positive constant $\lambda_0 + \lambda_0(a, b, F_v, |\Omega|, \mu_2)$ which depends on the data of the problem a, b, F_v, on the constants of Poincaré and of a Sobolev's Imbedding and on μ_2, the second eigenvalue of $(\mathcal{P}_g^\mu)$.

Theorem Let (u, F) with $u \in \mathcal{U}$ be a solution of $(\mathcal{P})$. Suppose that FF' is Lipschitz on $I\!R$, i.e.,

$$|F(t)F'(t) - F(\hat{t})F'(\hat{t})| \leq \lambda K |t - \hat{t}|$$

for every $t, \hat{t} \in I\!R$ and for some positive constant K. Suppose also that the parameter $\lambda > 0$ is such that

$$\lambda < \lambda_0 \ ,$$

where λ_0 is the above mentioned constant and g is defined by

$$g(x) := C_2 \|b\|_{L^\infty(\Omega)} a(x) + b(x) + K$$

for some known constant $C_2 > 0$. Then, if (v, F) is an other solution of $(\mathcal{P})$, then, necessarily, $v \equiv u$. .

Proof. Suppose that there exist two solutions u, v of $(\mathcal{P})$. The proof consists in two main steps:

- To verify that for small values of the parameter λ $(\lambda < \lambda_0)$ the solutions are necessarily ordered, for instance, $u \leq v$. To do this we adapt a technique used by Puel [5] as well as some technical results on the regularity and positiveness of the function F.
- To derive a contradiction: if $u \geq v$ in Ω then, as $u = v$ on $\partial\Omega$ we obtain $\nabla u \cdot n \leq \nabla v \cdot n$. But integrating the equation of $\mathcal{P}$ in Ω and using the the Stellarator Condition , the Divergence theorem and the strictly decreasing character of F we arrive to

$$\int_{\partial\Omega} \nabla u \cdot n > \int_{\partial\Omega} \nabla v \cdot n$$

leading then to a contradiction.

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