ON THE BOUSSINESQ SYSTEM WITH NON LINEAR THERMAL DIFFUSION

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1 INTRODUCTION

The model. The Boussinesq system of hydrodynamics equations [4], [16], arises from a zero order approximation to the coupling between the Navier-Stokes equations and the thermodynamic equation [15]. The presence of density gradients in a fluid means that gravitational potential energy can be converted into motion through the action of bouyant forces. Density differences are induced, for instance, by gradients of temperature arising by heating non uniformely the fluid. In the Boussinesq approximation of a large class of flows problems, thermodynamical coefficients, such as viscosity, specific heat and thermal conductivity, can be assumed as constants leading to a coupled system with linear second order operators in the Navier-Stokes equations and in the heat conduction equation [5], [8] , [10], [11], [9], [12], [17] . However, there are some fluids, such as lubrificants or some plasma flows, for which this is no longer an accurate assumption. In this paper we present a preliminar repport on the study of the well possednes as well as some properties on the supports of solutions to such type of Boussinesq system. For a more detailed work on this topic we refer to the reader to [6]. We start by considering the system derived in [15]

$$\left\{\begin{array}{rcl} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\ (\mu(\theta)\mathbf{D}(\mathbf{u})) + \nabla p & = & \mathbf{F}(\theta), \\ \operatorname{div}\mathbf{u} & = & 0, \\ \mathcal{C}(\theta)_t + \mathbf{u}\cdot\nabla\mathcal{C}(\theta) - \Delta\varphi(\theta) & = & 0. \end{array}\right. \tag{1.1}$$

where $\mathbf{u}$ is the velocity field of the fluid, θ its temperature, p the pressure, $\mu(\theta)$ the viscosity of the fluid, $\mathbf{F}(\theta)$ the buoyancy force, $\mathbf{D}(\mathbf{u}) := \nabla\mathbf{u} + \nabla\mathbf{u}^T$,

$$\mathcal{C}(\theta) := \int_{\theta_0}^{\theta} C(s)ds \qquad \text{and} \qquad \varphi(\theta) := \int_{\theta_0}^{\theta} \kappa(s)ds$$

with $C(\theta)$ and $\kappa(\theta)$ being the specific heat and the conductivity, respectively. Assuming, as usual, $C > 0$ then $\mathcal{C}$ is inversible, and so $\theta = \mathcal{C}^{-1}(\bar\theta)$ for some real argument $\bar\theta$. Then we can define the functions

$$\bar\varphi(\bar\theta) := \varphi\circ\mathcal{C}^{-1}(\bar\theta), \quad \bar{\mathbf{F}}(\bar\theta) := \mathbf{F}\circ\mathcal{C}^{-1}(\bar\theta), \quad \bar\mu(\bar\theta) := \mu\circ\mathcal{C}^{-1}(\bar\theta). \tag{1.2}$$

Substituting these expressions in (1.1) and omitting the bars we get the following formulation of the Boussinesq system

$$\left\{\begin{array}{rcl} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\ (\mu(\theta)\mathbf{D}(\mathbf{u})) + \nabla p & = & \mathbf{F}(\theta), \\ \operatorname{div}\mathbf{u} & = & 0, \\ \theta_t + \mathbf{u}\cdot\nabla\theta - \Delta\varphi(\theta) & = & 0. \end{array}\right. \tag{1.3}$$

We briefly comment some interesting features that characterize this model. There are two paradigmatic situations covered by the model: the *fast* and the *slow* heat diffusion. These cases mathematically correspond to the singular or degenerate character of the heat equation which may occur according the relative behaviour of C and κ. Indeed, since C and κ are non negative, their primitives $\mathcal{C}$ and φ are increasing functions. Suppose that a perturbation of a constant temperature θ_0 causes a small gradient of temperature between the boundary (higher temperature) and the interior (lower temperature) of a neighborhood, and assume that the behaviour of $\mathcal{C}$ and φ near θ_0 can be approximate as

$$\mathcal{C}(s) \sim c_1\,(s-\theta_0) + c_2\,(s-\theta_0)^p\,, \quad \varphi(s) \sim k_1\,(s-\theta_0) + k_2\,(s-\theta_0)^q\,, \tag{1.4}$$

for $s > \theta_0$, where $p, q > 0$. From (1.2) we have that

$$\bar{\varphi}'(\mathcal{C}(s)) = \varphi'(s)(\mathcal{C}^{-1})'(\mathcal{C}(s)) = \frac{\varphi'(s)}{\mathcal{C}'(s)} = \frac{k_1 + k_2 q(s-\theta_0)^{q-1}}{c_1 + c_2 p(s-\theta_0)^{p-1}}.$$

So when $s \to \theta_0$ (and therefore $\mathcal{C}(s) \to 0$) we get one of the following behaviours of $\bar{\varphi}'$ close to zero:

(i) if $p, q > 1$ then $\bar{\varphi}'(0) = k_1/c_1$,

(ii) if $1 > q > p$ either $q > 1 > p$ then $\lim_{\mathcal{C}(s)\to 0} \bar{\varphi}'(\mathcal{C}(s)) = 0$,

(iii) if $p > 1 > q$ either $1 > p > q$ then $\lim_{\mathcal{C}(s)\to 0} \bar{\varphi}'(\mathcal{C}(s)) = +\infty$.

In the first case both linear parts dominate: this case arises, for instance, when conductivity and specific heat are taken as constants, leading to the classical heat equation with a linear diffusion term. In the other two cases the non linear parts dominate and this leads to two different behaviours:

if $p < q$, the specific heat dominates over the conductivity, i.e., when temperature approches θ_0 the fluid *stores more heat and thi sis worstly conduced*. We shall prove that a front of temperature $\theta = \theta_0$ arises. This type of phenomenon is known as slow diffusion: heat spends a positive time to spread over the neighborhood,

if $p > q$ the opposite effect arises: the conductivity dominates over the specific heat. In this case the phenomenon is called fast diffusion. We shall prove that, in fact, $\theta = \theta_0$ in the whole domain when the time is large enough

Assumptions and functional setting. Given $T > 0$ and a bounded domain $\Omega \subset I\!R^N$, $N \geq 2$, and writting $Q_T := \Omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$, we consider the system of equations (1.3) in Q_T, adding the following auxiliary conditions:

$$\left\{ \begin{array}{lll} \mathbf{u} = \mathbf{0}, & \varphi(\theta) = 0 & \text{on } \Sigma_T, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & \theta(x,0) = \theta_0(x) & \text{on } \Omega. \end{array} \right. \tag{1.5}$$

The following conditions will be always assumed along the paper:

1. $\Omega \subset I\!R^N$ is an open, bounded and connected set, with a Lipschitz continuous boundary having finite $(N-1)$ dimensional Haussdorf measure.

2. $\mu : I\!R_+ \to I\!R_+$ is a locally Lipschitz continuous function and there exist $m_1 > m_0 > 0$ such that
$$m_0 \leq \mu(s) \leq m_1 \quad \forall s \in I\!R_+. \tag{1.6}$$

3. $\mathbf{F} : I\!R_+ \to I\!R^N$ is a locally Lipschitz continuous function.

4. $\varphi : I\!R_+ \to I\!R_+$ is a continuous and strictly increasing function, with $\varphi(0) = 0$.

We will work in the usual functional setting of the Navier-Stokes equations by considering the spaces of free divergence (see, e.g., [19]). We introduce the spaces

$$\begin{aligned} \mathcal{C}_\sigma^\infty(\Omega) &:= \left\{ \mathbf{u} \in \mathcal{C}_0^\infty(\Omega; I\!R^N) : \operatorname{div} \mathbf{u} = 0 \right\}, \\ L_\sigma^p(\Omega) &:= \text{closure of } \mathcal{C}_\sigma^\infty(\Omega) \text{ in the norm of } L^p(\Omega; I\!R^N), \\ W_\sigma^{1,p}(\Omega) &:= W_0^{1,p}(\Omega; I\!R^N) \cap L_\sigma^p(\Omega), \end{aligned}$$

and the orthogonal projection

$$P_\sigma : L^2(\Omega; I\!R^N) \to L_\sigma^2(\Omega).$$

Applying the projection to both sides of the Navier-Stokes equations and taking into account that $P_\sigma \nabla p \equiv 0$ and that $\mathbf{u} = P_\sigma \mathbf{u}$ since $\operatorname{div} \mathbf{u} = 0$, we get

$$\begin{cases} \mathbf{u}_t + P_\sigma(\mathbf{u} \cdot \nabla)\mathbf{u} - P_\sigma \operatorname{div}(\mu(\theta)\mathbf{D}(\mathbf{u})) = P_\sigma \mathbf{F}(\theta) & \text{in } Q_T, \\ \theta_t + \mathbf{u} \cdot \nabla\theta - \Delta\varphi(\theta) = 0 & \text{in } Q_T, \\ \mathbf{u} = \mathbf{0}, \quad \varphi(\theta) = 0 & \text{on } \Sigma_T, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x) & \text{on } \Omega, \end{cases} \tag{1.7}$$

which is the final form of the problem we will study.

2 EXISTENCE AND UNIQUENESS OF SOLUTIONS

Existence of solutions. The existence of weak solutions for the system (1.7) is a consequence of previous results on the Navier-Stokes equations and on nonlinear diffusion equations. We give here a sketch of the proof, based in Galerkin method and Semigroups theory techniques, although other strategies are also possible (see, for instance, the formulation made in [17] as a suitable variational inequality).

THEOREM **2.1** Assume 1- 4. Then if $\mathbf{u}_0 \in L_\sigma^2(\Omega)$ and $\theta_0 \in L^\infty(\Omega)$ problem (1.7) has at least one solution $(\mathbf{u}, \theta)$ in the sense of distributions, with the following regularity:

$$\begin{aligned} &\mathbf{u} \in L^\infty(0,T; L_\sigma^2(\Omega)) \cap L^2(0,T; W_\sigma^{1,2}(\Omega)), \\ &\theta \in L^\infty(Q_T), \\ &\varphi(\theta) \in L^2(0,T; H_0^1(\Omega)). \end{aligned}$$

Moreover, if $\theta_0 \geq 0$ a.e. in Ω then $\theta \geq 0$ in Q_T.

The proof (see [6]) consists on the following steps:

- uncoupling $\mathbf{u}$ and θ in the system by introducing a suitable iterative scheme,
- proving existence of solutions for the Navier-Stokes system with a non constant viscosity and prescribed θ by using a Galerkin method ([13],[19]),
- proving existence of solutions of the nonlinear heat equation with prescribed $\mathbf{u}$ by applying Semigroups theory ([3]),
- passing to the limit in the iterative process by means of some *a priori* estimates.

Uniqueness of solutions. The main difficulty in the study of the uniqueness of solutions for this kind of systems is due to the presence in the thermal equation of a coupling convection-diffusion where the convective term has a different homogeinity than the one of the diffusion term. Here we shall restrict ourselves to the non degenerate case.

THEOREM **2.2** Assume **1 – 4**, $N = 2$ and $\varphi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ locally Lipschitz continuous. Then, at most there exists a solution $(\mathbf{u}, \theta)$ of (1.7) such that $\mathbf{u} \in L^\infty(0, T, W^{1,\infty}_\sigma(\Omega))$.

Proof. Suppose there exist two solutions, $(\mathbf{u}_1, \theta_1), (\mathbf{u}_2, \theta_2)$ and define $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$, $\theta := \theta_1 - \theta_2$, $\mu_i := \mu(\theta_i)$, and $\mathbf{F}_i := \mathbf{F}(\theta_i)$. Then $(\mathbf{u}, \theta)$ satisfy:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}_1 \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}_2 - \operatorname{div}(\mu_1 D(\mathbf{u}) + (\mu_1 - \mu_2) D(\mathbf{u}_2)) = \mathbf{F}_1 - \mathbf{F}_2 & \text{in } Q_T, \\ \theta_t + \mathbf{u}_1 \cdot \nabla\theta + \mathbf{u} \cdot \nabla\theta_2 - \Delta\left(\varphi(\theta_1) - \varphi(\theta_2)\right) = 0 & \text{in } Q_T, \\ \mathbf{u} = \mathbf{0}, \qquad \varphi(\theta) = 0 & \text{on } \Sigma_T, \\ \mathbf{u}(x, 0) = \mathbf{0} \qquad \theta(x, 0) = 0 & \text{in } \Omega. \end{cases} \tag{2.8}$$

Then if $\mathbf{w}, \xi$ are test functions, with $\operatorname{div} \mathbf{w} = 0$, adding the two resulting integral identities we get

$$\begin{aligned} \int_\Omega (\mathbf{u}(T) \cdot \mathbf{w}(T) + \theta(T)\xi(T)) &= \int_{Q_T} \mathbf{u} \cdot [\mathbf{w}_t + (\mathbf{u}_1 \cdot \nabla)\mathbf{w} + \operatorname{div}(\mu_1 D(\mathbf{w}))] + \\ &+ \mathbf{u}_2 \cdot (\mathbf{u} \cdot \nabla)\mathbf{w} - \int_{Q_T} [(\mu_1 - \mu_2) D(\mathbf{u}_2) : \nabla\mathbf{w} + (\mathbf{F}_1 - \mathbf{F}_2) \cdot \mathbf{w}] + \\ &+ \int_{Q_T} (\theta\,(\xi_t + \mathbf{u}_1 \cdot \nabla\xi) + \theta_2 \mathbf{u} \cdot \nabla\xi + (\varphi(\theta_1) - \varphi(\theta_2))\,\Delta\xi). \end{aligned}$$

Adding and substracting the terms $k(\mathbf{u} \cdot \mathbf{w} + \theta\xi)$, $M\Delta\mathbf{w}$ and h_L, where k, M are positive constants to be fixed and $h_L \equiv h_L(x, t)$ to be given later, and using the formula of variation of the parameters we obtain

$$\begin{aligned} \int_\Omega (\mathbf{u}(T) \cdot \mathbf{w}(T) + \theta(T)\xi(T)) &= \int_0^T e^{k(T-t)} \left\{ \int_\Omega \mathbf{u} \cdot [\mathbf{w}_t - k\mathbf{w} + M\Delta\mathbf{w}] + \right. \\ &+ \left. \int_\Omega \theta\,(\xi_t - k\xi + h_L\Delta\xi) + (h - h_L)\,\theta\Delta\xi + I \right\}, \end{aligned}$$

where, given $L \in \mathbb{R}$

$$h_L = \begin{cases} h & \text{if } h \le L, \\ L & \text{if } h > L, \end{cases}$$

with

$$h = \frac{\varphi(\theta_1) - \varphi(\theta_2)}{\theta},$$

and

$$\begin{aligned} I &:= \int_\Omega [\mathbf{u} \cdot (\mathbf{u}_1 \cdot \nabla)\mathbf{w} + \mathbf{u} \cdot [\operatorname{div}(\mu_1 D(\mathbf{w})) - M\Delta\mathbf{w}] + \mathbf{u}_2(\mathbf{u} \cdot \nabla)\mathbf{w} - (\mu_1 - \mu_2)\, D(\mathbf{u}_2) : \nabla\mathbf{w}] \\ &+ \int_\Omega (\mathbf{F}_1 - \mathbf{F}_2) \cdot \mathbf{w} + \theta\mathbf{u}_1 \cdot \nabla\xi + \theta_2 \mathbf{u} \cdot \nabla\xi. \end{aligned}$$

Notice that as φ^{-1} is increasing and Lipschitz continuous then both h and h_L are bounded from below by a positive constant h_0. Now we choose the test functions as solutions of the following uncoupled system:

$$\begin{cases} \mathbf{w}_t - k\mathbf{w} + M\Delta\mathbf{w} = \mathbf{u} & \text{in } Q_T, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } Q_T, \\ \xi_t - k\xi + h_L\Delta\xi = \theta h_L & \text{in } Q_T, \\ \mathbf{w} = \mathbf{0} \qquad \xi = 0 & \text{on } \Sigma_T, \\ \mathbf{w}(T) = \mathbf{0}, \qquad \xi(T) = 0 & \text{on } \Omega. \end{cases} \tag{2.9}$$

Since $\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}_\sigma(\Omega))$, $\theta h_L \in L^\infty(Q_T)$ and $0 < h_0 < h_L \le L$, the existence of regular solutions to (2.9) holds by well-known results (see e.g. [14]). Moreover, multiplying by $\Delta\xi$ in the third equation of (2.9) we get the following estimate

$$\int_{Q_T} |\Delta\xi|^2 \le \frac{4}{h_0^2}\int_{Q_T} |\varphi(\theta_1) - \varphi(\theta_2)|^2 . \tag{2.10}$$

Using such $\mathbf{w}$ and ξ as test functions we obtain

$$\begin{aligned} 0 \;&\ge\; \int_0^T e^{k(T-t)}\Big\{\int_\Omega -M\frac{d}{dt}|\nabla \mathbf{w}|^2 + 2kM|\nabla\mathbf{w}|^2 + (M\Delta\mathbf{w})^2 + \frac{h_0}{2}\theta^2 - \frac{1}{2}\frac{d}{dt}|\nabla\xi|^2 + \\ &+\; \int_\Omega k|\nabla\xi|^2 + \theta(h-h_L)\Delta\xi + I\Big\}. \end{aligned}$$

The integral I can be estimated by using Poincaré, Holdër and Young's inequalities. We get

$$-I \le d_1\|\mathbf{u}\|_2^2 + \frac{\delta}{k}\|\mathbf{u}\|_{1,2}^2 + d_2k\|\mathbf{w}\|_{1,2}^2 + M^2\|\Delta\mathbf{w}\|_2^2 + \frac{d_3}{k}\|\theta\|_2^2 + \frac{k}{4}\|\xi\|_{1,2}^2$$

where d_i are some positive constants and $\delta > 0$ will be fixed later on (here we used the notation $\|f\|_p := \|f\|_{L^p(\Omega)}$ and $\|f\|_{p,q} := \|f\|_{W^{p,q}(\Omega)}$). We have thus that

$$\begin{aligned} 0 \;&\ge\; \int_0^T e^{k(T-t)}\Big\{-M\frac{d}{dt}\left(\|\mathbf{w}\|_{1,2}^2\right) - d_1\|\mathbf{u}\|_2^2 - \frac{\delta}{k}\|\mathbf{u}\|_{1,2}^2 + (2M-d_2)k\|\mathbf{w}\|_{1,2}^2 + \\ &+\; (\frac{h_0}{2} - \frac{d_3}{k})\|\theta\|_2^2 - \frac{1}{2}\frac{d}{dt}(\|\xi\|_{1,2}^2) + \frac{3k}{4}\|\xi\|_{1,2} + \int_\Omega (h-h_L)\theta\Delta\xi\Big\}. \end{aligned}$$

Integrating by parts (in time) and using the following estimate

$$\begin{aligned} \|\mathbf{u}\|_{1,2}^2 \;&\le\; -\frac{1}{m_1}\frac{d}{dt}\|\mathbf{u}\|_2^2 + \left(\frac{2}{m_1} + \frac{8}{m_1^2}\|\mathbf{u}_2\|_{1,2}^2\right)\|\mathbf{u}\|_2^2 + \\ &+\; \left(\frac{2}{m_1}\|\mathbf{F}\|_{Lips}^2 + \frac{8m}{m_1^2}\|D(\mathbf{u}_2)\|_\infty^2\right)\|\theta\|_2^2 , \end{aligned}$$

which is deduced by multiplying the first equation of (2.8) by $\mathbf{u}$ (and where the assumption $N = 2$ is used), we get

$$\begin{aligned} 0 \;&\ge\; \int_0^T e^{k(T-t)}\Big\{\left[\frac{\delta}{m_1} - d_1 - c_1\frac{\delta}{k}\right]\|\mathbf{u}\|_2^2 + k(M-d_2)\|\mathbf{w}\|_{1,2}^2 + \\ &+\; \left(\frac{h_0}{2} - \frac{1}{k}(d_3 - c_3)\right)\|\theta\|_2^2 + \tfrac{k}{4}\|\xi\|_{1,2}^2 + M\|\mathbf{w}(0)\|_{1,2}^2 + \frac{1}{2}\|\xi(0)\|_{1,2}^2 + \\ &+\; \frac{\delta}{km_1}\|\mathbf{u}(T)\|_2^2 + \int_\Omega \theta(h-h_L)\Delta\xi\Big\}, \end{aligned}$$

where c_i are suitable positive constants. Notice that the conclusion holds if we can select the positive constants, k and δ, such that

$$\begin{aligned} &\frac{\delta}{m_1} - d_1 - c_1\frac{\delta}{k} > 0, \\ &\frac{h_0}{2} - \frac{1}{k}(d_3 - c_2) > 0, \end{aligned}$$

and if

$$\int_0^T e^{k(T-t)}\int_\Omega \theta(h-h_L)\Delta\xi \to 0 \quad \text{as} \quad L\to\infty, \tag{2.11}$$

since in that case, making $M = d_2$, we get

$$0 \geq \int_0^T e^{k(T-t)} \left(\|\mathbf{u}\|_2^2 + \|\theta\|_2^2 \right).$$

The required inequalities are satisfied by choosing

$$k = \max\left\{ \frac{2}{h_0}(d_3 - c_2), c_1 \right\} \quad \text{and} \quad \delta = \frac{d_1 m_1 k}{k - c_1} + 1.$$

The property (2.11) follows from the inequality

$$\int_0^T e^{k(T-t)} \int_\Omega (h - h_L)\theta \Delta\xi \leq e^{kT} \|\Delta\xi\|_{L^2(Q_T)} \|(h - h_L)\theta\|_{L^2(Q_T)},$$

the uniform estimate on $\|\Delta\xi\|_{L^2(Q_T)}$ given in (2.10) and the a.e. convergence of h_L to h. Indeed, by inequality

$$|\theta(h - h_L)| \leq 2\,|\varphi(\theta_1) + \varphi(\theta_2)|,$$

we can apply the dominate convergence theorem which implies (2.11).

REMARK The Lipschitz continuity of φ^{-1} plays an essential role in the derivation of the uniform bound on $\Delta\xi$. Without this assumption the limit problem satisfied by ξ, when $L \to \infty$, becomes degenerate and hence the estimate, in general, does not hold.

3 STUDY OF THE SUPPORT OF THE TEMPERATURE

In this section we shall show two different properties on the support of temperature component of the Boussinesq system according to the balance between specific heat and thermal conductivity mentioned in the introduction:

(i) the localization in time of the support (extinction in finite time property), and

(ii) the localization in space of the support (the finite speed of propagation property).

These kind of properties have been intensively studied in the recent years by a number of authors, and several techniques have been developed to this end (see [2], [1] and the references therein). We use here two Energy Methods which consist, in both cases, in the derivation of an ordinary differential inequality for some energy norm of the temperature. A study of the respective inequalities proves the ocurrence of the mentioned phenomena.

Property (i) occurs in the fast diffusion case, meanwhile property (ii) corresponds to the slow diffusion case. More precisely,

DEFINITION **3.1** Given a solution $(\mathbf{u}, \theta)$ of (1.7), we say that θ satisfies the *extinction in finite time* property if there exists $t_f > 0$ such that $\theta(\cdot, t) \equiv 0$ in Ω for all $t \geq t_f$.

DEFINITION **3.2** Consider $x_0 \in \Omega$ and a ball $B_{\rho_0}(x_0) = \left\{ x \in \mathbb{R}^N : |x - x_0| < \rho_0 \right\}$ where $\theta_0(x) = 0$. Given $(\mathbf{u}, \theta)$ solution of (1.7) we say that θ satisfies the *finite speed of propagation* property if there exist $t_0 > 0$ and a function $\rho : [0, t_0] \to [0, \rho_0]$ with $\rho_0 := \rho(0)$ such that $\theta(x, t) = 0$ a.e. in $B_{\rho(t)}(x_0)$, for all $t \in [0, t_0]$.

A sufficient condition for property (i) is given in the next result:

THEOREM **3.1** Assume that

$$\varphi'(s) \geq c_1 s^{m-1} \qquad \text{for some} \quad m \in (0,1), \tag{3.12}$$

and $c_1 > 0$. Then any solution $(\mathbf{u}, \theta)$ of (1.7) is such that θ has the property of extinction in finite time.

Proof. Multiplying the temperature equation of (1.7) *formally* by θ^p with $p > 0$ (which can be justified by regularizing techniques) and integrating in Ω we get

$$\frac{d}{dt}\int_\Omega \theta^{p+1} + \frac{p+1}{p} c_1 \int_\Omega \theta^{m-1} \nabla\theta \cdot \nabla\theta^p \leq 0. \tag{3.13}$$

Here we used (3.12) and that

$$\int_\Omega (\mathbf{u}\cdot\nabla)\theta \cdot \theta^p = 0$$

as a consequence of the divergence theorem and that $\mathbf{u} \in L^2(0,T; W_\sigma^{1,2}(\Omega))$. The rest of the proof follows as in the case of pure diffusion equations (see, e.g., [1], [2] and their references). We define the energy

$$E(t) := \|\theta(t)\|_{L^{p+1}(\Omega)}^{p+1}.$$

Using (3.13) and the Sobolev's imbedding theorem we get the differential inequality

$$\begin{cases} \dfrac{dE}{dt}(t) + CE(t)^\alpha \leq 0, \\ E(0) = \|\theta_0\|_{L^{p+1}(\Omega)}^{p+1} := E_0 \geq 0, \end{cases} \tag{3.14}$$

where $\alpha = \frac{p+m+1}{p+1}$ and $C > 0$. As $m \in (0,1)$ it follows that $\alpha \in (0,1)$. We easily get the upper bound

$$E(t) \leq \left(E_0^{1-\alpha} - C(1-\alpha)t\right)_+^{\frac{1}{1-\alpha}},$$

and the result holds with $t_f := \dfrac{E_0^{1-\alpha}}{C(1-\alpha)}$.

Finally, we briefly mention one of the results obtained for the spatial localization for solutions of the Boussinesq system.

THEOREM **3.2** Assume

$$0 \leq \varphi'(s) \leq c_2 s^{m-1} \qquad \text{for some} \quad m \in (1,\infty) \tag{3.15}$$

and $c_2 > 0$. Let $(\mathbf{u}, \theta)$ be any solution of (1.7). Assume $\mathbf{u}$ locally Lipschitz continuous in Q_T. Then θ satisfies the finite speed of propagation property.

We shall give a sketch of the proof. First we introduce the change of unknown $\varphi(\theta) := v$ and we write $\psi := \varphi^{-1}$. Then v satisfies the following equation:

$$\psi(v)_t + \mathbf{u}\cdot\nabla\psi(v) - \Delta v = 0.$$

We define the characteristics of the flow by

$$\begin{cases} \dfrac{d}{dt}\mathbf{X}(\mathbf{x},t) = \mathbf{u}(\mathbf{X}(\mathbf{x},t),t) \quad \text{in } (0,T), \\ \mathbf{X}(\mathbf{x},0) = \mathbf{x}. \end{cases}$$

Multiplying (formally) by $v(\mathbf{x},t)$, for $t>0$ fixed and integrating over the set

$$B_\rho(\mathbf{x}_0)_t := \left\{\mathbf{y}\in I\!R^N : \mathbf{y}=\mathbf{X}(\mathbf{x},t) \text{ for some } \mathbf{x}\in B_\rho(\mathbf{x}_0)\right\}$$

we get

$$\int_{B_\rho(\mathbf{x}_0)_t}\frac{\partial}{\partial t}\Psi(v)+\int_{B_\rho(\mathbf{x}_0)_t}\mathbf{u}\cdot\nabla\Psi(v)+\int_{B_\rho(\mathbf{x}_0)_t}|\nabla v|^2=\int_{\partial B_\rho(\mathbf{x}_0)_t}v\nabla v\cdot\nu.$$

where

$$\Psi(s) := s\psi(s)-\int_0^s\psi(\sigma)d\sigma.$$

Notice that (3.15) implies that $\Psi(s)\geq cs^{p+1}$ with $p:=\frac{1}{m}$. From the Reynolds Transport Lemma

$$\int_{B_\rho(\mathbf{x}_0)_t}\frac{\partial}{\partial t}\psi(v(\mathbf{y},t))\,d\mathbf{y}=\frac{d}{dt}\int_{B_\rho(\mathbf{x}_0)_t}\psi(v(\mathbf{y},t))\,d\mathbf{y}-\int_{B_\rho(\mathbf{x}_0)_t}\mathbf{u}\cdot\nabla\psi(v(\mathbf{y},t))\,d\mathbf{y}.$$

Thus, integrating in $(0,t)$

$$\int_{B_\rho(\mathbf{x}_0)_t}\Psi(v(\mathbf{y},t))\,d\mathbf{y}+\int_0^t\int_{B_\rho(\mathbf{x}_0)_t}|\nabla v|^2\,d\mathbf{y}=\int_0^t\int_{\partial B_\rho(\mathbf{x}_0)_t}v\nabla v\cdot\nu+\int_{B_\rho(\mathbf{x}_0)_t}\Psi(v(\mathbf{y},0))\,d\mathbf{y}.$$

Defining the energies

$$b(\rho,t) := \sup_{0\leq\tau\leq t}\int_{B_\rho(\mathbf{x}_0)_t}v(\mathbf{y},\tau)^{p+1}d\mathbf{y} \text{ and } E(\rho,t) := \int_0^t\int_{B_\rho(\mathbf{x}_0)_t}|\nabla v|^2\,d\mathbf{y}$$

and using that

$$\partial(B_\rho(\mathbf{x}_0)_t)\equiv\partial(\mathbf{X}(B_\rho(\mathbf{x}_0),t))=\mathbf{X}(\partial B_\rho(\mathbf{x}_0),t)$$

we prove that

$$\frac{\partial E}{\partial\rho}(\rho,t)=\int_{\partial B_\rho(\mathbf{x}_0)_t}|\nabla v|^2 \text{ for a.e. } \rho>0.$$

By the Hölder inequality

$$\int_0^t\int_{\partial B_\rho(\mathbf{x}_0)_t}v\nabla v\cdot\nu\leq\left(\frac{\partial E}{\partial\rho}\right)^{1/2}\left(\int_0^t\int_{\partial B_\rho(\mathbf{x}_0)_t}|v|^2\right)^{1/2}.$$

Using a variant of the interpolation-trace trace inequality obtained in [7] we get

$$\left(\int_0^t\int_{\partial B_\rho(\mathbf{x}_0)_t}v^2\right)^{\frac{1}{2}}\leq K(t)(E+b)^\kappa,$$

where $K(t)$ is a superlinear positive funtion of t,

$$\kappa=\omega/2+(1-\omega)/(p+1),$$

and

$$\omega=\frac{N(1-p)+p+1}{N(1-p)+2p+2}.$$

Gathering the different estimates we obtain

$$b+E\leq K\left(\frac{\partial E}{\partial\rho}\right)^{1/2}(b+E)^\kappa$$

and by the Young's inequality

$$b + E \leq \hat{K}\left(\frac{\partial E}{\partial \rho}\right)^{1/2}(b+E)^{1/\beta}$$

where $\beta := 2(1-\kappa) \in (0,1)$. It is important to remark here that $\kappa > 1/2$ if and only if $p < 1$. Therefore the arguments that follow can only be applied if $p < 1$. This is the point in the reasoning where the degeneracy of the nonlinearity φ is essential. A direct integration of this inequality shows that $E(\rho,t) = 0$ for (ρ,t) such that $\rho \leq \rho_0 - K(t)E^{1-\beta}(\rho_0,T)$ from which we conclude that $v(\mathbf{x},t) \equiv 0$ on $B_\rho(\mathbf{x}_0)_t$ for $t \in (0,t^*)$ with $t^* := K^{-1}(\frac{\rho_0}{E^{1-\beta}(\rho_0,T)})$.

REMARK A more detailed exposition containing more general assumptions on $\mathbf{u}$, as well as the waiting time property, will be published elsewhere.

REFERENCES

1. ANTONTSEV, A. & DIAZ, J.I., Space and time localization in the flow of two inmiscible fluids through a porous medium: energy methods applied to systems. *Nonlinear Anal., Theory, Meth. & App.*, Vol 16, No 4, 299-313, 1991.
2. ANTONTSEV, A. & DIAZ, J.I., *Energy Methods for Free Boundary Problems in Continuum Mechanics*. To appear in Birkhäuser.
3. BENILAN, P. *Equations d'evolution dans un space de Banach quelconque et applications*, Thesis, Univ. de Paris Sud, Orsay, 1972.
4. BOUSSINESQ, J., *Theorie analytique de la chaleur*, Vol 2, Gauthier-Villars, Paris 1903.
5. CASAS, E., The Navier-Stokes equations coupled with the heat equation: analysis and control. *Control and Cybernetics*. Vol 23, No 4, 605-620, 1994.
6. DIAZ, J.I. & GALIANO, G., To appear in *Topological Methods in Nonlinear Analysis*, (Journal of the Juliusz Schauder Center), Warsaw, Poland, 1997.
7. DIAZ, J.I. & VERON, L., Local vanishing properties of solutions to elliptic and parabolic equations. *Trans. Am. Math. Soc.*, 290, 787-814, 1985.
8. DIAZ, J.I. & VRABIE, I.I., Compactness of the Green Operator of Nonlinear Diffusion Equations: applications to Boussinesq type systems in Fluid Dynamics. *Topological Methods in Nonlinear Analysis*, (Journal of the Juliusz Schauder Center), Vol 4, 1994, 399-416.
9. FOIAS, C., , MANLEY, O. and TEMAM, R., Attractors for the Benard Problem: existence and physical bounds on their fractal dimension. *Nonlinear Analysis*, Vol 11, No 8, 939-967, 1987.
10. GONTSCHAROWA, O., Solvability of the nonstationary problem for the free convection equation with the temperature dependent viscosity. *Dinamika sploshnoi sredy*, No 96, 1990.
11. GONTSCHAROWA, O., About the uniqueness of the solution of the two-dimensional nonstationary problem for the equations of free convection with viscousity depending on temperature. *Red Sib.mat.j.* No 260, V92, 1990.
12. JOSEPH, D.D., *Stability of fluid motions I and II. Springer Tracts in Natural Philosophy*, Vol 28, Berlin 1976.
13. LADYZHENSKAYA, O.A., *The Mathematical Theory of Viscous Incompressible Flow*. 2nd Edition, Gordon & Breach, New York, 1969.
14. LADYZHENSKAYA, O.A., SOLONNIKOV, V.A. & URAL'CEVA, N.N., *Linear and Quasilinear Equations of Parabolic type*. Transl. Math. Monographs, Vol 23, Amer. Math. Soc., Providence, 1968.
15. MILHALJAN, J.M. A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid. *Astrophysics J.*, Vol 136, 1126, 1962.
16. OBERBECK, A., Uber die Wärmeleitung der Flussigkeiten bei der Berücksichtigung der Strömungen infolge von Temperaturdifferenzen. *Annalen der Physik und Chemie* 7, 271 (1879).
17. RODRIGUES, J.F., Weak solutions for thermoconvective flows of Boussinesq-Stefan type in *Mathematical Topics in Fluid Mechanics*, J.F. Rodrigues and A. Sequeira eds., Pitman Research Notes in Mathematics Series, No 274, 93-116, Harlow, 1992.
18. RULLA, J.. Weak solutions to Stefan problems with prescribed convection. *SIAM J. Math. Anal.*, Vol 18, No 6, 1784-1800, 1987.
19. TEMAM, R., *Navier-Stokes Equations, Theory and Numerical Analysis*, 3rd revised edition, North-Holland, Amsterdam, 1984.