On a quasilinear degenerate system arising in semiconductor theory. Part II: Localization of vacuum solutions

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1. Introduction

In solid state physics, the drift-diffusion equations are today the most widely used model to describe semiconductor devices. The drift-diffusion models describe the flow of electrons in the conduction band of the semiconductor material and of holes (or defect electrons) in the valence band of the crystal, influenced by the electric field. Mathematically, they form a system of parabolic equations for the electron density n and the hole density p and the Poisson equation for the electric potential V:

$$\frac{\partial n}{\partial t} - \nabla \cdot (\nabla r(n) - n\nabla V) = -R(n, p), \tag{1}$$

$$\frac{\partial p}{\partial t} - \nabla \cdot (\nabla r(p) + p\nabla V) = -R(n, p), \tag{2}$$

$$\Delta V = n - p - C(x) \quad \text{in } Q_T = \Omega \times (0, T), \tag{3}$$

where $\Omega \subset \mathbb{R}^d$ $(1 \leq d \leq 3)$ is the (bounded) domain occupied by the semiconductor crystal. Here, $C = C(x)$ denotes the doping profile (fixed charged background ions) characterizing the semiconductor under consideration, $r(s)$ is the pressure function, and $R(n, p)$ the recombination–generation rate. The process of transfering an electron of the conduction band of the semiconductor into the lower energetic valence band is called recombination of electron–hole pairs. The inverse process, i.e. the transfer of a valence electron to the conduction band is termed generation of electrons and holes. If recombination of electron–hole pairs exceeds generation then $R(n, p) > 0$, in the opposite case we have $R(n, p) < 0$.

In the standard drift-diffusion model, it holds

$$r(s) = s \quad \text{and} \quad R(n, p) = q(n, p)(np - n_i^2),$$

where $q(n, p)$ is a positive function and $n_i = n_i(x) > 0$ is the so-called intrinsic density [23]. The standard model can be derived from Boltzmann's equation under the assumption that the semiconductor device is in the low injection regime (i.e. for small absolute values of the applied voltage). It is shown in [19] that in the high injection regime, the diffusion terms $\nabla r(n)$, $\nabla r(p)$ are no longer linear, and the function $r(s)$ has to be taken as

$$r(s) = s^\alpha, \quad \alpha = \tfrac{5}{3}.$$

With this pressure function, Eqs. (1) and (2) become of degenerate type, and solutions may exist for which $n = 0$ or $p = 0$ holds locally (so-called *vacuum solutions*).

The function r can be interpreted in the language of gas dynamics. We assume that the particles behave – thermodynamically spoken – as ideal gas such that the gas law $r = nT$ holds (T denotes the particle temperature). In the isothermal case $T = \text{const.}$ the pressure turns out to be linear: $r(n) = n$. In the isentropic case, however, the temperature (only) depends on the concentrations. Then $T(n) = n^{2/3}$ holds for particles without spin in adiabatic and hence for isentropic states [9], which implies $r(n) = n^{5/3}$. (Similar for the holes.)

The equations are supplemented with physically motivated boundary conditions. The boundary $\partial\Omega$ consists of two disjoint subsets Γ_D and Γ_N. The carrier densities and the potential are fixed at Γ_D (Ohmic contacts), whereas Γ_N models the union of insulating boundary segments:

$$n = n_D, \quad p = p_D, \quad V = V_D \quad \text{on } \Gamma_D, \tag{4}$$

$$\nabla r(n) \cdot \nu = \nabla r(p) \cdot \nu = \nabla V \cdot \nu = 0 \quad \text{on } \Gamma_N, \tag{5}$$

where ν denotes the exterior normal vector of $\partial\Omega$ which is assumed to exist a.e. We assume that the densities at time $t = 0$ are known:

$$n(0) = n_I, \quad p(0) = p_I \quad \text{in } \Omega. \tag{6}$$

The standard (low injection) model has been mathematically and numerically investigated in many papers (see [23, 24] and references therein). The existence and uniqueness of weak solutions have been shown. The isentropic (high injection) model is analyzed in [10, 13, 16, 18–20]. The existence of weak solutions (satisfying $\nabla r(n)$, $\nabla r(p) \in L^2$ and $n, p \in L^\infty$) has been proved. The uniqueness of solutions is shown in some special situations [10, 11, 18, 20]. For the derivation of the model we refer to [19, 21, 25].

In this paper we present results concerning the temporal and spatial localization of the *vacuum sets* $\{n = 0\}$, $\{p = 0\}$. The results can be summarized as follows:

1. *Finite speed of propagation.* If there are vacuum sets initially then there are vacuum sets for small $t > 0$:

$$\text{meas}\{n(0) = 0\} > 0, \quad \text{meas}\{p(0) = 0\} > 0$$

$$\Longrightarrow \text{meas}\{n(t) = 0\} > 0, \quad \text{meas}\{p(t) = 0\} > 0.$$

   This property shows that the speed of propagation of the support of n and p is finite.

2. *Waiting time.* Under some structure condition on $R(n, p)$ and some “flatness” condition on $n(0) = n_I$, there is no dilatation of the initial support:

$$\{n(0) = 0\} \subset \{n(t) = 0\} \quad \text{for small } t > 0.$$

3. *Formation of vacuum.* Under some structure condition on $R(n, p)$ there exists a $T_0 > 0$ such that there is vacuum for $t > T_0$, even if the initial densities are positive:

$$\text{meas}\{n(t) = 0\} > 0.$$

The proof of these results which are formulated below is based on a local energy method for free boundary problems. The idea of the method is to introduce an energy functional (usually given by the norm in the natural energy spaces associated to the equations) and to derive a differential inequality for the energy functional. From this inequality the desired qualitative properties of the solutions can be deduced.

The energy method that we use has two principal features. First, it is a local method, i.e. it operates in subsets of the corresponding domain without need of global informations like boundary conditions or boundedness of the domain. Secondly, it has a very general setting, allowing to consider, for instance, problems in any space dimension or with coefficients depending on the space or time variable. The energy method that we use does not need any monotonicity assumption on the nonlinear functions and it requires no comparison principle.

The method has been introduced by Antontsev [1] and developed by Díaz and Véron [12] and by Antontsev et al. [2–6] for parabolic equations of degenerate type. The energy methods have been extended to equations of arbitrary order [7] and have been applied to equations or systems of equations [8, 22, 14, 18]. We also refer to [5] for an overview of the existing literature.

We now turn to the precise formulation of the localization results. The last result is only valid if the local energy of the density is small enough. The *local energy* $D_n(P)$ of n in a domain $P \subset Q_T = \Omega \times (0, T)$ is defined by

$$D_n(P) = E_n(P) + C_n(P) + b_n(P),$$

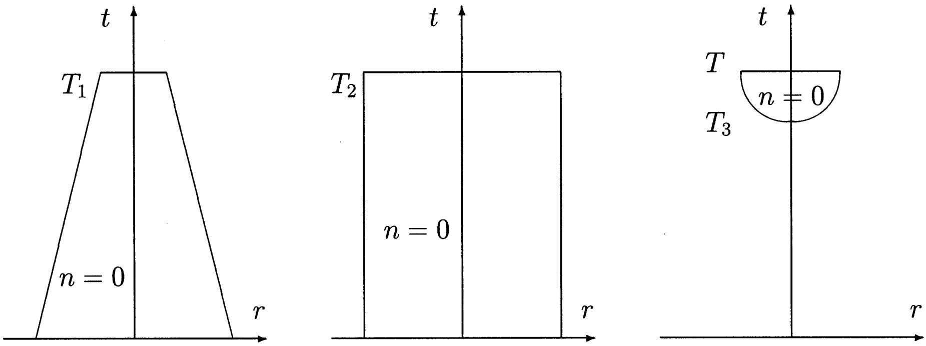

Fig. 1. Localization of the vacuum sets.

where

$$E_n(P) = \int_P |\nabla n^\alpha(x,\tau)|^2 \, dx \, d\tau,$$

$$C_n(P) = \int_P n(x,\tau)^{\alpha+\beta} \, dx \, d\tau,$$

$$b_n(P) = \sup_{s \in (\tilde{t}, t)} \int_{P \cap \{\tau = s\}} n(x,s)^{\alpha+1} \, dx,$$

with $\tilde{t}, t > 0$, and $\beta \in (0,1)$ is a constant to be precised below.

Denote by $W^{s,p}(X)$ the space $W^{s,p}(0,T;X)$ if X is a Banach space. Furthermore, introduce $\mathscr{V} = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$. In the following we assume that

$$r(s) = s^\alpha, \quad \alpha > 1,$$

and that there exists a solution (n, p, V) to Eqs. (1)–(6) satisfying

$$\begin{aligned} &n, p \in L^\infty(Q_T) \cap H^1(0,T;\mathscr{V}^*), \\ &r(n), r(p) \in L^2(0,T;H^1(\Omega)), \\ &V \in L^\infty(0,T;H^1(\Omega)). \end{aligned}$$

The existence of a solution with these regularity properties is shown in [10, 20]. We have the following theorems (also see Fig. 1).

Theorem 1.1 (Finite speed of propagation). *Let* $x_0 \in \Omega$, $0 < \rho_0 < \text{dist}(x_0, \partial\Omega)$ *and* $T > 0$. *Assume that*

$$n_I = 0, \quad p_I = 0 \quad \textit{in } B_{\rho_0}(x_0)$$

and

$$R(u,v)(u^\alpha + v^\alpha) \geq -\kappa_R(u^{\alpha+1} + v^{\alpha+1}) \quad \textit{for all } u, v \geq 0, \tag{7}$$

with $\kappa_R \geq 0$ *hold. Then there exist* $T_1 > 0$ *and a non-increasing function* ρ *satisfying* $\rho(\tau) > 0,\ 0 \leq \tau < T_1$, *and* $\rho(0) = \rho_0$ *such that*

$$n(x,t) = 0, \quad p(x,t) = 0 \qquad \text{for a.e. } x \in B_{\rho(t)}(x_0),\ t \in (0, T_1).$$

For the next theorems we need a stronger condition on $R(n, p)$:

$$R(u,v) \geq bu^{\beta} \quad \text{for all } u, v \geq 0, \qquad b > 0,\ \alpha + \beta < 2. \tag{8}$$

Theorem 1.2 (Waiting time). *Let* $x_0 \in \Omega$, $0 < \rho_0 < \rho_1 < \text{dist}(X_0, \partial\Omega)$ *and* $T > 0$. *Assume that Eq.* (8) *and*

$$\int_{B_\rho(x_0)} n_I^{\alpha+1} \leq \varepsilon_0(\rho - \rho_0)_+^{\gamma} \tag{9}$$

for $0 < \rho < \rho_1$ *hold, where* $\varepsilon_0 > 0$ *and*

$$\gamma = \frac{d(\alpha - 1) + 2(\alpha + 1)}{\alpha - 1} > 1.$$

(*Recall that* $d \geq 1$ *is the space dimension.*) *Then there exist* $\varepsilon_1 > 0$ *and* $T_2 \in (0, T)$ *such that if* $\varepsilon_0 \leq \varepsilon_1$ *then*

$$n(x,t) = 0 \quad \text{in } B_{\rho_0}(x_0) \times (0, T_2).$$

Theorem 1.3 (Formation of vacuum). *Let* $x_0 \in \Omega$ *and* $T > 0$. *Assume that Eq.* (8) *holds. Then there exist* $M > 0$, $T_3 \in (0, T)$, *and* $\gamma, \kappa \in (0,1)$ *such that if* $D_n(Q_T) \leq M$ *then*

$$n(x,t) = 0 \quad \text{for a.e. } x \in B_{\rho(t)}(x_0),\ t \in (T_3, T),$$

where $\rho(t) = \gamma(t - T_3)^{\kappa}$.

The proofs of these theorems are presented in Section 2. The difficulties in proving the above results are due to the coupling of Eqs. (1)–(3) and in particular, due to the drift terms $\text{div}(n\nabla V)$, $-\text{div}(p\nabla V)$. Indeed, the electric field $-\nabla V$ induces (or prevents) a flow of electrons or holes in some direction influencing the support of the densities.

Condition (8) which is also needed in [6] is almost optimal in the following sense. Let $R(u,v) \leq bu^{\beta}$ for all $u, v \geq 0$ satisfying $\alpha + \beta > 2$, and let the initial and boundary densities be strictly positive in Ω, $\Omega \times (0, \infty)$, respectively. Then, choosing $\beta \geq 1$ (such that $\alpha > 1$), there exists a solution (n, p, V) to Eqs. (1)–(6) satisfying

$$n(t) > 0, \quad p(t) > 0 \quad \text{in } \Omega,\ 0 < t < \infty$$

(see [20]). In this situation, no vacuum occurs.

The three localization results are illustrated by numerical examples in one space dimension in Section 3. For the discretization we use an exponentially fitted mixed finite element method as in [17]. Modeling a one-dimensional forward biased *pn* junction diode, the presented properties can be verified.

2. Proofs of the main results

For the proofs of Theorems 1.1–1.3 we have to estimate the local energies in the domain

$$P = \{(x,\tau) \in \mathbb{R}^d \times [0,\infty) : |x - x_0| \le r(\tau),\ \tau \in (\tilde{t}, t)\},$$

where $\tilde{t}, t \in [0,T]$, $\tilde{t} < t$, $x_0 \in \Omega$, and $r \in C^1(\tilde{t}, t)$. In this section, r always denotes a radius (function). Since the pressure function $r(s)$ is taken to be s^α and does not appear in this section, there should be no confusion of the meaning of r. The lateral surface of P is given by

$$\partial_l P = \{(x,\tau) : |x - x_0| = r(\tau),\ \tau \in (\tilde{t}, t)\},$$

and the outer unit normal $\nu = (\nu_x, \nu_\tau)$ of P has the components

$$\nu_x = \frac{e_x}{\sqrt{1 + r'(\tau)^2}}, \qquad \nu_\tau = \frac{-r'(\tau)}{\sqrt{1 + r'(\tau)^2}},$$

where e_x is the unit vector in the direction of ν_x. We choose the parameters $\tilde{t}$, t and the function $r(\tau)$ as follows:

(1) Theorem 1.1: P is a truncated cone with $r(\tau) = \rho - M\tau$, $0 < \varepsilon < \rho \le \rho_0$, $0 < \tau < t$ and $M > 0$.
(2) Theorem 1.2: P is a cylinder $B_\rho(x_0) \times (0,\tau)$ with $0 < \rho \le \rho_0$ and $0 < \tau < T$.
(3) Theorem 1.3: P is a paraboloid with $r(\tau) = \gamma(\tau - t)^\kappa$, $t < \tau < T$ and $\gamma, \kappa \in (0,1)$.

Proof of Theorem 1.1. Using local elliptic regularity theory (cf., e.g., [15]) and noting that $n, p \in L^\infty(B_{\rho_0}(x_0))$, we see that $\nabla V \in L^\infty(B_{\rho_0}(x_0) \times (0,T))$. Let

$$M = \|\nabla V\|_{0,\infty,B_{\rho_0}(x_0)\times(0,T)}, \quad \varepsilon \in (0,\rho_0), \quad t_1 = \varepsilon/2M,$$

and consider the cone

$$P = P(\rho, t) = \{(x,\tau) : x \in B_r(x_0),\ \tau \in (0,t)\},$$

where $\rho \in (\varepsilon, \rho_0)$, $t \in (0, t_1)$, and $r = r(\rho,\tau) = \rho - M\tau$. For almost all ρ and τ it holds

$$\begin{aligned}\int_P n\nabla V \cdot \nabla n^\alpha \,\mathrm{d}x\,\mathrm{d}\tau &= \frac{\alpha}{\alpha+1}\int_P \nabla V \cdot \nabla n^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau \\ &= -\frac{\alpha}{\alpha+1}\int_P \Delta V n^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\alpha}{\alpha+1}\int_{\partial_l P}(\nabla V \cdot \nu_x) n^{\alpha+1}\,\mathrm{d}\sigma\,\mathrm{d}\tau,\end{aligned}$$

and therefore, using the local integration by parts formula (see Lemma A.1 of the appendix):

$$\frac{1}{\alpha+1}\int_{P\cap\{\tau=t\}} n(t)^{\alpha+1}\,\mathrm{d}x + \int_P |\nabla n^\alpha|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$\le \frac{1}{\alpha+1}\int_{P\cap\{\tau=0\}} n(0)^{\alpha+1}\,\mathrm{d}x + \int_{\partial_l P}(\nabla n^\alpha \cdot \nu_x) n^\alpha\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$
\begin{aligned}
&-\frac{1}{\alpha+1}\int_{\partial_l P}(v_\tau+\nabla V\cdot v_x)n^{\alpha+1}\,\mathrm{d}\sigma\,\mathrm{d}\tau-\frac{\alpha}{\alpha+1}\int_P \Delta V n^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau\\
&-\int_P R(n,p)n^\alpha\,\mathrm{d}x\,\mathrm{d}\tau\\
=\ & I_1+\cdots+I_5. \qquad (10)
\end{aligned}
$$

Since $n(0)$ vanishes in $B_{\rho_0}(x_0)$, we have $I_1=0$. For the estimate of I_2 observe that in spherical coordinates with center x_0 (cf. [6, 12])

$$
\begin{aligned}
\frac{\partial E_n}{\partial\rho}(\rho,t) &= \frac{\partial}{\partial\rho}\int_P |\nabla n^\alpha|^2\,\mathrm{d}x\,\mathrm{d}\tau=\frac{\partial}{\partial\rho}\int_0^t\int_0^{r(\rho,t)}\int_{S^{d-1}}|\nabla n^\alpha|^2\tilde r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}\tilde r\,\mathrm{d}\tau\\
&=\int_{\partial_l P}|\nabla n^\alpha|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau.
\end{aligned}
$$

Hence

$$
I_2\le\left(\int_{\partial_l P}|\nabla n^\alpha|^2\right)^{1/2}\left(\int_{\partial_l P}n^{2\alpha}\right)^{1/2}=\left(\frac{\partial E_n}{\partial\rho}\right)^{1/2}\|n^\alpha\|_{0,2,\partial_l P}.
$$

We use the interpolation-trace Lemma A.2 (see the appendix) with $p=q=2$ and $r=s=1+1/\alpha$:

$$
\|n^\alpha\|_{0,2,\partial B_r}\le c_0(\|\nabla n^\alpha\|_{0,2,B_r}+r^{-\delta}\|n^\alpha\|_{0,1+1/\alpha,B_r})^\theta\|n^\alpha\|_{0,1+1/\alpha,B_r}^{1-\theta}, \qquad (11)
$$

where

$$
\theta=\frac{d(\alpha-1)+(\alpha+1)}{d(\alpha-1)+2(\alpha+1)}\in(0,1),\qquad \delta=\frac{2(\alpha+1)+d(\alpha-1)}{2(\alpha+1)}>1. \qquad (12)
$$

By the definition of r, we have

$$
r^{-\delta}=(\rho-M\tau)^{-\delta}\le(\varepsilon-Mt_1)^{-\delta}=(2/\varepsilon)^\delta.
$$

Thus, applying Hölder's inequality with exponent $1/\theta$ and setting $K_1=c_0^2\max(1,(2/\varepsilon)^\delta)$, we obtain

$$
\begin{aligned}
\int_0^t\|n^\alpha\|_{0,2,\partial B_r}^2\,\mathrm{d}\tau&\le 2K_1\int_0^t(\|\nabla n^\alpha\|_{0,2,B_r}^2+\|n^\alpha\|_{0,1+1/\alpha,B_r}^2)^\theta\|n^\alpha\|_{0,1+1/\alpha,B_r}^{2(1-\theta)}\,\mathrm{d}\tau\\
&\le 2K_1\left(\int_0^t\|\nabla n^\alpha\|_{0,2,B_r}^2\,\mathrm{d}\tau+\int_0^t\|n^\alpha\|_{0,1+1/\alpha,B_r}^2\,\mathrm{d}\tau\right)^\theta\left(\int_0^t\|n^\alpha\|_{0,1+1/\alpha,B_r}^2\,\mathrm{d}\tau\right)^{1-\theta}\\
&\le 2K_1t^{1-\theta}(E_n(\rho,t)+t_1b_n(\rho_0,t_1)^{(\alpha-1)/(\alpha+1)}b_n(\rho,t))^\theta b_n(\rho,t)^{2\alpha(1-\theta)/(\alpha+1)},
\end{aligned}
$$

where

$$
b_n(\rho,t)=\sup_{\tau\in(0,t)}\int_{B_{r(\rho,\tau)}(x_0)}n(x,\tau)^{\alpha+1}\,\mathrm{d}x.
$$

This yields

$$\|n^\alpha\|_{0,2,\partial_l P} \le K_2 t^{(1-\theta)/2}(E_n(\rho,t)+b_n(\rho,t))^{\theta/2} b_n(\rho,t)^{\alpha(1-\theta)/(\alpha+1)}$$
$$\le K_2 t^{(1-\theta)/2}(E_n(\rho,t)+b_n(\rho,t))^{\mu},$$

where $K_2^2 = 2K_1 \max(1, t_1 b_n(\rho_0,t_1)^{(\alpha-1)/(\alpha+1)})$ and

$$\mu = \frac{\theta}{2} + \frac{\alpha}{\alpha+1}(1-\theta) \in (\tfrac{1}{2}, 1).$$

We conclude

$$I_2 \le K_2 t^{(1-\theta)/2} \left(\frac{\partial E_n}{\partial \rho}\right)^{1/2} (E_n(\rho,t)+b_n(\rho,t))^{\mu}.$$

Thanks to the special structure of $r = r(\rho,\tau)$ and the definition of M, we have

$$v_\tau + \nabla V \cdot v_x = \frac{M + \nabla V \cdot e_x}{\sqrt{1+M^2}} \ge 0,$$

so that $I_3 \le 0$. Furthermore,

$$I_4 \le K_3 \int_P n^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau,$$

where $K_3 = (\alpha/(\alpha+1))\|\Delta V\|_{0,\infty,Q_T}$.

For p we get an analogous inequality to Eq. (10) and similar estimates involving the local energies E_p and b_p defined by

$$E_p(\rho,t) = \int_{P(\rho,t)} |\nabla p^\alpha|^2\,\mathrm{d}x\,\mathrm{d}\tau, \qquad b_p(\rho,t) = \sup_{\tau\in(0,t)} \int_{B_{r(\rho,\tau)}(X_0)} p(x,\tau)^{\alpha+1}\,\mathrm{d}x.$$

Therefore, we have the estimate

$$\frac{1}{\alpha+1}\int_{B_r(x_0)} (n(t)^{\alpha+1} + p(t)^{\alpha+1})\,\mathrm{d}x + \int_P (|\nabla n^\alpha|^2 + |\nabla p^\alpha|^2)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le K_4 t^{(1-\theta)/2}\left(\left(\frac{\partial E_n}{\partial\rho}\right)^{1/2}(E_n+b_n)^\mu + \left(\frac{\partial E_p}{\partial\rho}\right)^{1/2}(E_p+b_p)^\mu\right)$$
$$+ K_3\int_P (n^{\alpha+1} + p^{\alpha+1})\,\mathrm{d}x\,\mathrm{d}\tau - \int_P R(n,p)(n^\alpha + p^\alpha)\,\mathrm{d}x\,\mathrm{d}\tau,$$

where

$$K_4^2 = 2K_1 \max(1, t_1 b_n(\rho_0,t_1)^{(\alpha-1)/(\alpha+1)}, t_1 b_p(\rho_0,t_1)^{(\alpha-1)/(\alpha+1)}).$$

Introduce

$$E = E_n + E_p, \qquad b = b_n + b_p.$$

Employing the assumption on $R(n, p)$ gives

$$\begin{aligned}
&\frac{1}{\alpha+1}\int_{B_r(x_0)}(n(t)^{\alpha+1}+p(t)^{\alpha+1})\,\mathrm{d}x+E(\rho,t)\\
&\quad\le 2K_4t^{(1-\theta)/2}\left(\frac{\partial E}{\partial\rho}\right)^{1/2}(E+b)^{\mu}+(K_3+\kappa_R)\int_P(n^{\alpha+1}+p^{\alpha+1})\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\le 2K_4t^{(1-\theta)/2}\left(\frac{\partial E}{\partial\rho}\right)^{1/2}(E+b)(\rho,t)^{\mu}+tK_5b(\rho,t),
\end{aligned}$$

with $K_5=K_3+\kappa_R$. Since the right-hand side of the above inequality in non-decreasing in t, we can write

$$(E+b)(\rho,t)\le 2(\alpha+1)K_4t^{(1-\theta)/2}\left(\frac{\partial E}{\partial\rho}\right)^{1/2}(b+E)^{\mu}+(\alpha+1)K_5tb.$$

Choosing $t<t_2=\min(t_1,(2(\alpha+1)K_5)^{-1})$, we get

$$b+E\le 4(\alpha+1)K_4t^{(1-\theta)/2}\left(\frac{\partial E}{\partial\rho}\right)^{1/2}(b+E)^{\mu} \tag{13}$$

and

$$E(\rho,t)^{2(1-\mu)}\le(b+E)(\rho,t)^{2(1-\mu)}\le K_6t^{1-\theta}\frac{\partial}{\partial\rho}E(\rho,t),$$

where $K_6=16(\alpha+1)^2K_4^2$. Integrating this differential inequality for E in (ρ,ρ_0) gives (note that $\mu>\frac{1}{2}$)

$$E(\rho,t)^{2\mu-1}\le E(\rho_0,t)^{2\mu-1}-K_6^{-1}t^{\theta-1}(\rho_0-\rho).$$

Let

$$\tilde{\rho}(t)=\rho_0-K_6t^{1-\theta}E(\rho_0,t)^{2\mu-1}.$$

Then $\tilde{\rho}(0)=\rho_0$ and $\tilde{\rho}$ is non-increasing. Choose $T_1\in(0,t_2)$ such that $\tilde{\rho}(T_1)>\varepsilon$. Then, for $t\in(0,T_1)$ and $\rho\in(\varepsilon,\tilde{\rho}(t)]$,

$$\begin{aligned}
E(\rho,t)^{2\mu-1}&\le E(\tilde{\rho}(t),t)^{2\mu-1}\\
&\le E(\rho_0,t)^{2\mu-1}-K_6^{-1}t^{\theta-1}(\rho_0-\tilde{\rho}(t))=0.
\end{aligned}$$

Thus (see (13)), for $\rho=\tilde{\rho}(t)$,

$$n(x,t)=p(x,t)=0\quad\text{for a.e. } t\in(0,T_1),\ x\in B_{\tilde{\rho}(t)}(x_0).$$

The conclusion follows. □

The proof of Theorem 1.3 contains an estimate used in the proof of Theorem 1.2 and is therefore given before.

Proof of Theorem 1.3. We take the paraboloid

$$P = P(t) = \{(x,\tau)\colon\ x \in B_r(x_0),\ \tau \in (t,T)\},$$

where $t \in (0,T)$, $r = r(\tau,t) = \gamma(\tau - t)^\kappa$, $\gamma,\kappa \in (0,1)$. Choose $\gamma > 0$ small enough such that $2\gamma \max(1,T) \le \operatorname{dist}(x_0,\partial\Omega)$. Then $r(\tau,t) \le \gamma T^\kappa \le \operatorname{dist}(x_0,\partial\Omega)/2$ and $B_{r(\tau,t)}(x_0) \subset \omega$ for some domain $\omega \subset\subset \Omega$, for $t \in (0,T)$ and $\tau \in (t,T)$. We get from Eq. (10)

$$\begin{aligned}
&\frac{1}{\alpha+1}\int_{P\cap\{\tau=T\}} n(T)^{\alpha+1}\,\mathrm{d}x + \int_P |\nabla n^\alpha|^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \frac{1}{\alpha+1}\int_{P\cap\{\tau=t\}} n(t)^{\alpha+1}\,\mathrm{d}x + \int_{\partial_l P} (\nabla n^\alpha \cdot v_x) n^\alpha\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad - \frac{1}{\alpha+1}\int_{\partial_l P} (v_\tau + \nabla V \cdot v_x) n^{\alpha+1}\,\mathrm{d}\sigma\,\mathrm{d}\tau - \frac{\alpha}{\alpha+1}\int_P \Delta V n^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad - \int_P R(n,p) n^\alpha\,\mathrm{d}x\,\mathrm{d}\tau \\
&= I_1 + \cdots + I_5.
\end{aligned}$$

Since $\operatorname{meas}(P\cap\{\tau=t\}) = 0$, $I_1 = 0$ holds. For the estimate of I_2 we proceed as follows:

$$I_2 \le \left(\int_{\partial_l P} \left|\frac{\partial r}{\partial t}\right| |\nabla n^\alpha|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2} \left(\int_{\partial_l P} \left|\frac{\partial r}{\partial t}\right|^{-1} |v_x| n^{2\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2}.$$

Taking into account $|v_x| \le 1$ and (with spherical coordinates $(\tilde r,\omega)$)

$$\begin{aligned}
-\frac{\mathrm{d}E_n}{\mathrm{d}t}(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_t^T \int_0^{r(\tau,t)} \int_{S^{d-1}} |\nabla n(\tau)^\alpha|^2 \tilde r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}\tilde r\,\mathrm{d}\tau \\
&= \int_0^{r(t,t)} \int_{S^{d-1}} |\nabla n(t)^\alpha|^2 \tilde r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}\tilde r \\
&\quad - \int_t^T \frac{\partial r}{\partial t}(\tau,t) \int_{S^{d-1}} |\nabla n(\tau)^\alpha|^2 \tilde r^{d-1}\,\mathrm{d}\omega|_{\tilde r = r(\tau,t)}\,\mathrm{d}\tau \\
&= \int_{\partial_l P} \left|\frac{\partial r}{\partial t}\right| |\nabla n^\alpha|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau,
\end{aligned}$$

we obtain

$$I_2 \le \frac{1}{\sqrt{\gamma\mu}}\left(-\frac{\mathrm{d}E_n}{\mathrm{d}t}\right)^{1/2} \left(\int_t^T |\tau - t|^{1-\kappa} \int_{\partial B_r(x_0)} n^{2\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2}.$$

Using the interpolation-trace inequality (11), the boundary integral can be estimated by

$$
\begin{aligned}
\gamma\kappa \int_{\partial_l P} \left|\frac{\partial r}{\partial t}\right|^{-1} |v_x| n^{2\alpha} \,\mathrm{d}\sigma\,\mathrm{d}\tau &\le \int_t^T |\tau - t|^{1-\kappa} \int_{\partial B_r(x_0)} n^{2\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\le c_0^2 \int_t^T |\tau - t|^{1-\kappa} \max(1, r^{-2\delta\theta})(\|\nabla n^\alpha\|_{0,2,B_r(x_0)} + \|n^\alpha\|_{0,1+1/\alpha,B_r(x_0)})^{2\theta} \\
&\quad \times \|n^\alpha\|_{0,1+1/\alpha,B_r(x_0)}^{2(1-\theta)}\,\mathrm{d}\tau \\
&\le K_1(t)\left(\int_t^T \|\nabla n^\alpha\|_{0,2,B_r(x_0)}^2\,\mathrm{d}\tau + \int_t^T \|n^\alpha\|_{0,1+1/\alpha,B_r(x_0)}^2\,\mathrm{d}\tau\right)^\theta \\
&\quad \times \left(\int_t^T \|n^\alpha\|_{0,1+1/\alpha,B_r(x_0)}^2\,\mathrm{d}\tau\right)^{1-\theta},
\end{aligned}
$$

where we have used Hölder's inequality with exponent $1/\theta$, the inequality $(a+b)^2 \le 2(a^2+b^2)$ and the definition

$$
K_1(t) = 2c_0^2 \gamma^{-2\delta\theta} \sup_{\tau \in (t,T)} \max(|\tau - t|^{1-\kappa}, |\tau - t|^{1-\kappa-2\kappa\delta\theta}).
$$

The constant $K_1(t)$ is finite if we choose $\kappa \le 1/(1+2\delta\theta)$. Furthermore, $K_1(t) \le K_2^2 \stackrel{\text{def}}{=} 2c_0^2\gamma^{-2\delta\theta}\max(T^{1-\kappa}, T^{1-\kappa-2\kappa\delta\theta})$. Therefore

$$
\begin{aligned}
&\sqrt{\gamma\kappa}\left(\int_{\partial_l P} \left|\frac{\partial r}{\partial t}\right|^{-1} |v_x| n^{2\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2} \\
&\le K_2 \left(\int_P |\nabla n^\alpha|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_t^T \left(\int_\Omega n^{\alpha+1}\,\mathrm{d}x\right)^{2\alpha/(\alpha+1)}\mathrm{d}\tau\right)^{\theta/2} \\
&\quad \times \left[\int_t^T \left(\int_\Omega n^{\alpha+1}\,\mathrm{d}x\right)^{2\alpha/(\alpha+1)}\right]^{(1-\theta)/2} \\
&\le K_2(E_n(t) + (T-t) b_n(m,T)^{(\alpha-1)/(\alpha+1)} b_n(r,t))^{\theta/2} \\
&\quad \times (T-t)^{(1-\theta)/2} b_n(r,t)^{\alpha(1-\theta)/(\alpha+1)} \\
&\le K_2 (T-t)^{(1-\theta)/2} \max(1, (T-t) b_n(m,T)^{(\alpha-1)/(\alpha+1)})^{\theta/2} \\
&\quad \times (E_n(t) + b_n(r,t))^{\theta/2} b_n(r,t)^{\alpha(1-\theta)/(\alpha+1)} \\
&\le K_3 (E_n(t) + b_n(r,t))^\mu,
\end{aligned}
$$

where

$$
b_n(t) = b_n(r,t) = \sup_{\tau \in (t,T)} \int_{B_{r(\tau,t)}(x_0)} n(\tau)^{\alpha+1}\,\mathrm{d}x,
$$

$m = \gamma \max(1, T)$, $\mu = \theta/2 + \alpha(1-\theta)/(\alpha+1) \in (\frac{1}{2}, 1)$, and

$$K_3 = K_2 T^{(1-\theta)/2} \max(1, Tb_n(m,T)^{(\alpha-1)/(\alpha+1)})^{\theta/2}.$$

We conclude

$$I_2 \le \frac{K_3}{\sqrt{\gamma\mu}} \left(-\frac{\mathrm{d}E_n(t)}{\mathrm{d}t}\right)^{1/2} (E_n(t) + b_n(r,t))^{\mu}.$$

Now we estimate the integral I_3. Here we need the assumption on $R(n,p)$. Since $\Delta V \in L^\infty(Q_T)$, we get, by elliptic theory, the interior regularity $V \in L^\infty(0,T; W^{2,q}(\omega))$ for all $q < \infty$. (Recall that $\omega \subset\subset \Omega$.) Hence, $M = \|\nabla V\|_{L^\infty(0,T;C^0(\omega))}$ is finite. Then

$$\begin{aligned} I_3 &\le \frac{1}{\alpha+1} \int_t^T \int_{\partial B_r(x_0)} (|v_\tau| + |\nabla V| \cdot |v_x|) n^{\alpha+1} \, \mathrm{d}\sigma \, \mathrm{d}\tau \\ &\le (1+M) \int_t^T \int_{\partial B_r(x_0)} n^{\alpha+1} \, \mathrm{d}\sigma \, \mathrm{d}\tau \\ &= (1+M) \int_t^T \|n^\alpha\|_{0,1+1/\alpha,\partial B_r(x_0)}^{1+1/\alpha} \, \mathrm{d}\tau. \end{aligned} \tag{14}$$

Let $\lambda \in (1+\beta/\alpha, 2/\alpha)$; since $\alpha+\beta<2$ by assumption (8), the interval is non-empty. We apply the interpolation-trace Lemma A.2 (see the appendix) with $q = 1+1/\alpha$, $p=2$, $s = 1+\beta/\alpha$ and $r = \lambda$:

$$\|n^\alpha\|_{0,1+1/\alpha,\partial B_r(x_0)} \le c_0 (\|\nabla n^\alpha\|_{0,2,B_r(x_0)} + r^{-\tilde{\delta}} \|n^\alpha\|_{0,1+\beta/\alpha,B_r(x_0)})^{\tilde{\theta}} \|n^\alpha\|_{0,\lambda,B_r(x_0)}^{1-\tilde{\theta}},$$

where

$$\tilde{\theta} = \frac{2\alpha}{\alpha+1} \, \frac{\mathrm{d}(\alpha+1-\lambda\alpha) + \lambda\alpha}{\mathrm{d}\alpha(2-\lambda) + 2\lambda\alpha} \in (0,1), \quad \tilde{\delta} = \frac{2(\alpha+\beta) + \mathrm{d}(\alpha-\beta)}{2(\alpha+\beta)} > 1.$$

We use Hölder's inequality with exponent $Q = (1-\beta)/(1+\alpha-\alpha\lambda) > 1$ for the last norm:

$$\begin{aligned} \|n^\alpha\|_{0,\lambda,B_r(x_0)} &= \left(\int_{B_r(x_0)} n^{(\alpha+\beta)(\alpha+1-\alpha\lambda)/(1-\beta)} n^{(\alpha+1)(\alpha\lambda-\alpha-\beta)/(1-\beta)} \, \mathrm{d}x \right)^{1/\lambda} \\ &\le \left(\int_{B_r(x_0)} n^{\alpha+\beta} \, \mathrm{d}x \right)^{1/\lambda Q} \left(\int_{B_r(x_0)} n^{\alpha+1} \, \mathrm{d}x \right)^{1/\lambda Q'}, \end{aligned}$$

where $Q' = Q/(Q-1)$, and the inequality $(a+b)^2 \le 2(a^2+b^2)$ to get

$$\begin{aligned} \int_{\partial B_r(x_0)} n^{\alpha+1} \, \mathrm{d}x &= \|n^\alpha\|_{0,1+1/\alpha,\partial B_r(x_0)}^{1+1/\alpha} \\ &\le 2c_0^{1+1/\alpha} \left[\int_{B_r(x_0)} |\nabla n^\alpha|^2 \, \mathrm{d}x + r^{-2\tilde{\delta}} \left(\int_{B_r(x_0)} n^{\alpha+\beta} \, \mathrm{d}x \right)^{2\alpha/(\alpha+\beta)} \right]^{\tilde{\theta}(\alpha+1)/2\alpha} \end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{B_r(x_0)} n^{\alpha+\beta}\, \mathrm{d}x \right)^{(1-\tilde\theta)(\alpha+1)/\alpha\lambda Q} \left(\int_{B_r(x_0)} n^{\alpha+1}\, \mathrm{d}x \right)^{(1-\tilde\theta)(\alpha+1)/\alpha\lambda Q'} \\
\le & 2c_0^{1+1/\alpha} \max(1, r(\tau,t)^{-\tilde\delta\tilde\theta(\alpha+1)/\alpha}) \max\left[1, \left(\int_{B_r(x_0)} n^{\alpha+\beta}\, \mathrm{d}x \right)^{2\alpha/(\alpha+\beta)-1}\right]^{\tilde\theta(\alpha+1)/2\alpha} \\
&\times \left(\int_{B_r(x_0)} |\nabla n^\alpha|^2\, \mathrm{d}x + \int_{B_r(x_0)} n^{\alpha+\beta}\, \mathrm{d}x \right)^{\tilde\theta(\alpha+1)/2\alpha} \\
&\times \left(\int_{B_r(x_0)} n^{\alpha+\beta}\, \mathrm{d}x \right)^{(1-\tilde\theta)(\alpha+1)/\alpha\lambda Q} \left(\int_{B_r(x_0)} n^{\alpha+1}\, \mathrm{d}x \right)^{(1-\tilde\theta)(\alpha+1)/\alpha\lambda Q'} \\
\le & K_4(\tau) \left(\int_{B_r(x_0)} |\nabla n^\alpha|^2\, \mathrm{d}x + \int_{B_r(x_0)} n^{\alpha+\beta}\, \mathrm{d}x \right)^{v_1} \left(\int_{B_r(x_0)} n^{\alpha+1}\, \mathrm{d}x \right)^{v_2},
\end{aligned}
$$

where

$$
\begin{aligned}
K_4(\tau) = {} & 2c_0^{1+1/\alpha} \max(1, r(\tau,t)^{-\tilde\delta\tilde\theta(\alpha+1)/\alpha}) \\
& \times \max\left[1, \left(\int_{B_r(x_0)} n(\tau)^{\alpha+\beta}\, \mathrm{d}x \right)^{\tilde\theta(\alpha-\beta)(\alpha+1)/(2\alpha(\alpha+\beta))}\right]
\end{aligned}
$$

and

$$
v_1 = \frac{\tilde\theta(\alpha+1)}{2\alpha} + \frac{(1-\tilde\theta)(\alpha+1)}{\alpha\lambda Q} < 1, \quad v_2 = \frac{(1-\tilde\theta)(\alpha+1)}{\alpha\lambda Q'} > 0. \tag{15}
$$

For future reference we note that, since $\alpha < 2$,

$$
v_1 + v_2 = \frac{\alpha d(2-\lambda) + \alpha\lambda + 2}{\alpha d(2-\lambda) + \alpha\lambda + \alpha} > 1. \tag{16}
$$

Integrating the above estimate for $n^{\alpha+1}$ over (t,T) gives

$$
\begin{aligned}
\int_{\partial_l P} n^{\alpha+1}\, \mathrm{d}\sigma\, \mathrm{d}\tau \le \int_t^T K_4(\tau) & \left(\int_{B_r(x_0)} |\nabla n(\tau)^\alpha|^2\, \mathrm{d}x + \int_{B_r(x_0)} n(\tau)^{\alpha+\beta}\, \mathrm{d}x \right)^{v_1} \\
& \times \left(\int_{B_r(x_0)} n(\tau)^{\alpha+1}\, \mathrm{d}x \right)^{v_2} \mathrm{d}\tau.
\end{aligned}
$$

Since $v_1 < 1$, we can employ Hölder's inequality with exponent $1/v_1$ to get

$$
\int_{\partial_l P} n^{\alpha+1}\, \mathrm{d}\sigma\, \mathrm{d}\tau \le (T-t)^{v_2} b_n(r,t)^{v_2} \left(\int_t^T K_4(\tau)^{1/(1-v_1)}\, \mathrm{d}\tau \right)^{1-v_1} (E_n(t) + C_n(t))^{v_1}. \tag{17}
$$

The integral with $K_4(\tau)$ is well defined if

$$
\int_t^T r(\tau,t)^{-\tilde\delta\tilde\theta(\alpha+1)/(\alpha(1-v_1))}\, \mathrm{d}\tau < \infty.
$$

Choosing $\mu < \alpha(1-v_1)/(\tilde{\delta}\tilde{\theta}(\alpha+1))$

$$-\mu\frac{\tilde{\delta}\tilde{\theta}(\alpha+1)}{\alpha(1-v_1)} > -1,$$

holds and the integral converges. Thus,

$$K_5 \stackrel{\text{def}}{=} \left(\int_t^T K_4(\tau)^{1/(1-v_1)}\,\mathrm{d}\tau\right)^{1-v_1} < \infty,$$

holds and K_5 depends also on the L^∞ norm of n in $\omega\times(0,T)$. Using Young's inequality with exponent $1/v_1$ gives

$$\begin{aligned} b_n^{v_2}(E_n+C_n)^{v_1} &= (E_n+C_n)^{v_1} b_n^{v_1(v_1+v_2-1)} \cdot b_n^{(1-v_1)(v_1+v_2)} \\ &\le v_1(E_n+C_n)b_n^{v_1+v_2-1} + (1-v_1)b_n^{v_1+v_2} \\ &\le (E_n+C_n)b_n^{v_1+v_2-1} + b_n^{v_1+v_2} \\ &= b_n^{v_1+v_2-1}(E_n+C_n+b_n). \end{aligned}$$

Recall that $v_1+v_2-1>0$ (see Eq. (16)). This estimate and the inequality (17) concludes the estimate of I_3 (see Eq. (14)):

$$\begin{aligned} I_3 &\le (1+M)\int_{\partial_l P} n^{\alpha+1}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\le K_5(1+M)(T-t)^{v_2} b_n(r,t)^{v_1+v_2-1}(E_n(t)+C_n(t)+b_n(r,t)). \end{aligned}$$

More generally, we have proven the following result: Let P be given by

$$P = \{(x,\tau)\colon |x-x_0| \le r(\tau),\ \tau\in(\tilde{t},t)\}.$$

Then it holds

$$\int_{\partial_l P} n^{\alpha+1}\,\mathrm{d}\sigma\,\mathrm{d}\tau \le c(t-\tilde{t})^{v_2}\max\left[1,\left(\int_{\tilde{t}}^t r(\tau)^{-v_3}\,\mathrm{d}\tau\right)^{1-v_1}\right] b_n^{v_1+v_2-1}(E_n+C_n+b_n), \tag{18}$$

where $c>0$ only depends on the $L^\infty(0,T;C^0(\omega))$ norm of ∇V and the $L^\infty(\omega\times(0,T))$ norm of n, on Ω, α, β and d. Furthermore, v_1 and v_2 are given by Eq. (15), and

$$v_3 = \frac{\tilde{\delta}\tilde{\theta}(\alpha+1)}{\alpha(1-v_1)}. \tag{19}$$

We need this result in the proof of Theorem 1.2.

It remains to estimate the integrals I_4 and I_5:

$$\begin{aligned} I_4+I_5 &\le \|\Delta V\|_{0,\infty,\omega\times(0,T)}\int_P n^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau - \kappa_R\int_P n^{\alpha+\beta}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\le K_6(T-t)b_n(r,t) - \kappa_R C_n(t), \end{aligned}$$

where $K_6 = \|\Delta V\|_{0,\infty,\omega\times(0,T)}$. Therefore, we have shown that

$$\frac{1}{\alpha+1}\int_{B_{r(T,t)}(x_0)} n(T)^{\alpha+1}\,\mathrm{d}x + \int_P |\nabla n^\alpha|^2\,\mathrm{d}x\,\mathrm{d}\tau + \kappa_R\int_P n^{\alpha+\beta}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le \frac{K_3}{\sqrt{\gamma\mu}}\left(-\frac{\mathrm{d}E_n}{\mathrm{d}t}\right)^{1/2}(E_n+b_n)^\mu + K_5(1+M)(T-t)^{\nu_2} b_n^{\nu_1+\nu_2-1}(E_n+C_n+b_n)$$
$$+K_6(T-t)b_n.$$

Since the right-hand side of this inequality is non-decreasing in T, we can replace the left-hand side by

$$\frac{1}{\alpha+1}b_n(t)+E_n(t)+\kappa_R C_n(t).$$

Then, taking $\varepsilon>0$ small enough, setting $t^* = T-\varepsilon$, and using $b_n(r,t)\le K$, where K is the global energy, we get for $t\in(t^*,T)$,

$$\frac{1}{2}\left(\frac{1}{\alpha+1}b_n + E_n + \kappa_R C_n\right) \le \frac{K_3}{\sqrt{\gamma\mu}}\left(-\frac{\mathrm{d}E_n}{\mathrm{d}t}\right)^{1/2}(E_n+C_n+b_n)^\mu. \tag{20}$$

Thus

$$E_n^{2(1-\mu)} \le (E_n+C_n+b_n)^{2(1-\mu)} \le K_7\left(-\frac{\mathrm{d}E_n}{\mathrm{d}t}\right),$$

where the constant

$$K_7^2 \overset{\mathrm{def}}{=} \frac{2K_3}{\sqrt{\gamma\mu}\min(1,\kappa_R,(\alpha+1)^{-1})} > 0$$

is independent of t. Integrating this differential inequality in $(0,t)$ with $t\in(t^*,T)$ gives

$$E_n(t)^{2\mu-1} \le E_n(0)^{2\mu-1} - \frac{t}{K_7} \le K^{2\mu-1} - \frac{t^*}{K_7} \le 0$$

if $K^{2\mu-1}\le t^*/K_7$. Recall that $\mu>\frac{1}{2}$. We conclude

$$E_n(t)^{2\mu-1} = 0 \quad \text{for } t\in(t^*,T),$$

and (see Eq. (20)), for some $T_2\in(t^*,T)$,

$$n(x,\tau)=0 \quad \text{for a.e. } |x-x_0|\le\gamma(\tau-T_2)^\mu,\ \tau\in(T_2,T).$$

This proves the theorem. □

Proof of Theorem 1.2. We consider the cylinder

$$P = P(\rho,t) = B_\rho(x_0)\times(0,t),$$

with $\rho \in (\varepsilon, \rho_1)$, $t \in (0, T)$, and $\varepsilon \in (0, \rho_0)$. Taking into account $v_\tau = 0$ and the hypotheses (8) and (9), we get from Eq. (10):

$$\begin{aligned}
&\frac{1}{\alpha+1}\int_{B_\rho(x_0)} n(t)^{\alpha+1}\,\mathrm{d}x + \int_P |\nabla n^\alpha|^2\,\mathrm{d}x\,\mathrm{d}\tau + \kappa_R \int_P n^{\alpha+\beta}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad \le \frac{\varepsilon_0}{\alpha+1}\int_{B_\rho(x_0)} (\rho-\rho_0)_+^\gamma\,\mathrm{d}x + \int_0^t \int_{\partial B_\rho(x_0)} (\nabla n^\alpha \cdot \nu_x) n^\alpha\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\qquad - \frac{1}{\alpha+1}\int_0^t \int_{\partial B_\rho(x_0)} (\nabla V \cdot \nu_x) n^{\alpha+1}\,\mathrm{d}\sigma\,\mathrm{d}\tau - \frac{\alpha}{\alpha+1}\int_P \Delta V n^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad = I_1 + \cdots + I_4.
\end{aligned} \tag{21}$$

As in the proof of Theorem 1.1 we get the estimate

$$I_2 \le \left(\frac{\partial E_n}{\partial \rho}\right)^{1/2} \|n^\alpha\|_{0,2,\partial_l P} \le K_2 t^{(1-\theta)/2} \left(\frac{\partial E_n}{\partial \rho}\right)^{1/2} (E_n + b_n)^\mu,$$

where

$$K_2^2 = 2c_0^2 \max(1, \varepsilon^{-2\delta}) \max(1, T b_n(\rho_0, T)^{(\alpha-1)/(\alpha+1)})$$

and $\mu \in (\frac{1}{2}, 1)$ (see the proof of Theorem 1.1 for the definition of $c_0 > 0$ and $\delta > 0$). From (18) we conclude

$$\begin{aligned}
I_3 &\le M \int_{\partial_l P} n^{\alpha+1}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\le M c t^{\nu_2} \max(1, (T\varepsilon^{-\nu_3})^{1-\nu_1}) b_n(\rho_0, T)^{\nu_1+\nu_2-1} (E_n + C_n + b_n)(\rho, t),
\end{aligned}$$

where $M = \|\nabla V\|_{L^\infty(0,T;C^0(\overline{B_{\rho_0}(x_0)}))}$, $c > 0$ does not depend on ρ or t, and ν_1, ν_2, $\nu_3 > 0$ are given by Eqs. (15) and (19). Note that $\nu_1 < 1$ and $\nu_1 + \nu_2 > 1$. Thus

$$I_3 \le K_3 t^{\nu_2} (E_n + C_n + b_n),$$

where $K_3 = Mc \max(1, (T\rho_0^{-\nu_3})^{1-\nu_1}) b_n(\rho_0, T)^{\nu_1+\nu_2-1}$.

Finally, the integral I_4 is estimated by

$$I_4 \le K_4 t b_n,$$

with $K_4 = \|\Delta V\|_{0,\infty,B_{\rho_0}(x_0)\times(0,T)}$. Therefore, we obtain from Eq. (21)

$$\begin{aligned}
&\frac{1}{\alpha+1}\int_{B_\rho(x_0)} n(t)^{\alpha+1}\,\mathrm{d}x + E_n(\rho, t) + \kappa_R C_n(\rho, t) \\
&\quad \le \varepsilon_0 K_5 (\rho-\rho_0)_+^\gamma + K_2 t^{(1-\theta)/2} \left(\frac{\partial E_n}{\partial \rho}\right)^{1/2} (E_n + b_n)(\rho, t)^\mu \\
&\qquad + K_3 t^{\nu_2} (E_n + C_n + b_n)(\rho, t) + K_4 t b_n(\rho, t),
\end{aligned}$$

where $K_5 = \text{meas}(B_{\rho_0}(x_0))$. Since the right-hand side is non-decreasing in t, we can replace the left-hand side by

$$\frac{1}{\alpha+1} b_n(\rho, t) + E_n(\rho, t) + \kappa_R C_n(\rho, t).$$

Choosing $t > 0$ small enough, we get

$$(b_n + E_n + C_n) \le 2K_6 K_5 \varepsilon_0 (\rho - \rho_0)_+^{\gamma} + 2K_6 K_2 t^{(1-\theta)/2} \left(\frac{\partial E_n}{\partial \rho} \right)^{1/2} (E_n + b_n)^{\mu},$$

where $K_6^{-1} = \min(1, \kappa_R, (\alpha+1)^{-1})$. By Young's inequality with exponent $1/\mu > 1$ we get

$$(1-\mu)(b_n + E_n + C_n) \le 2K_6 K_5 \varepsilon_0 (\rho - \rho_0)_+^{\gamma} + (1-\mu) K_7 \left(\frac{\partial E_n}{\partial \rho} \right)^{1/2(1-\mu)},$$

where $K_7 = (2K_6 K_2 T^{(1-\theta)/2})^{1/(1-\mu)}$. Therefore, setting $K_8 = (2K_5 K_6/(1-\mu))^{2(1-\mu)}$,

$$E_n(\rho, t)^{2(1-\mu)} \le K_8 \varepsilon_0^{2(1-\mu)} (\rho - \rho_0)_+^{2\gamma(1-\mu)} + K_7^{2(1-\mu)} \frac{\partial E_n}{\partial \rho}(\rho, t),$$

where $\rho \in (\varepsilon, \rho_1)$. Now we can apply the following lemma (cf. [3, 4, 6, 13] for a proof):

Lemma 2.1. *Let* $\eta \in (0,1)$, $\varepsilon_0, \rho_0, \delta > 0$, $0 \le \varepsilon < \rho_0$, *and let* $\phi \in C^0([\varepsilon, \rho_0 + \delta] \times [0, T])$ *be a non-negative function, non-decreasing in both variables and satisfying* $\phi(\rho_0 + \delta, 0) = 0$ *and*

$$\phi(\rho, t)^{\eta} \le K \frac{\partial \phi}{\partial \rho}(\rho, t) + \varepsilon_0 (\rho - \rho_0)_+^{\eta/(1-\eta)}$$

for $\rho \in [\varepsilon, \rho_0 + \delta]$, $t \in [0, T]$. *Then there exist* $\varepsilon_1 > 0$ *and* $t^* \in (0, T)$ *such that if* $\varepsilon_0 < \varepsilon_1$ *then*

$$\phi(\rho_0, t) = 0 \quad \text{for } t \in (0, t^*).$$

We finish the proof of the theorem before proving the above lemma. Since $\eta \stackrel{\text{def}}{=} 2(1-\mu) < 1$ and $2\gamma(1-\mu) = \eta/(1-\eta)$, the assumptions of the lemma are satisfied and we conclude the existence of $\varepsilon_1 > 0$ and $T_2 \in (0, T)$ such that for all $\varepsilon_0 \in (0, \varepsilon_1)$

$$E_n(\rho, t) = 0 \quad \text{for } \rho \in [0, \rho_0], \ t \in [0, T_2],$$

that means, $n(x, t) = 0$ for $x \in B_{\rho_0}(x_0)$, $t \in (0, T_2)$. This proves the theorem. $\square$

3. Numerical examples

We present numerical examples which illustrate the properties of the vacuum sets $\{n = 0\}$ and $\{p = 0\}$ proved in the previous section, namely (i) finite speed of propagation, (ii) waiting time, and (iii) formation of vacuum.

We use the following formulation in one space dimension:

$$\partial_t n - \partial_x(\partial_x(n^{5/3}) - n\partial_x V) = -R(n, p),$$

$$\partial_t p - \partial_x(\partial_x(p^{5/3}) + p\partial_x V) = -R(n, p),$$

$$\lambda^2 \partial_{xx} V = n - p - C \quad \text{in } (0,1) \times (0,T)$$

with initial and boundary conditions

$$n(0,t) = n_0, \quad p(0,t) = p_0, \quad V(0,t) = V_0,$$

$$n(1,t) = n_1, \quad p(1,t) = p_1, \quad V(1,t) = V_1,$$

$$n(x,0) = n_I(x), \qquad p(x,0) = p_I(x), \quad x \in (0,1).$$

The equations are in dimensionless form (see the papers [17, 19] for details of the scaling). The constant $\lambda > 0$ is called the (scaled) Debye length. We take the numerical value $\lambda^2 = 1.6 \times 10^{-3}$. The doping profile is given by

$$C(x) = \begin{cases} -1 & \text{if } 0 < x < 0.7, \\ +1 & \text{if } 0.7 < x < 1. \end{cases}$$

This choice of parameter and functions corresponds to a silicon pn-junction diode of length $L = 10^{-3}$ cm with the moderate doping concentration $|C| = 10^{15}\,\text{cm}^{-3}$ (see [19]).

The above system is numerically solved by using an exponentially fitted mixed finite element method for the discretization with respect to the space variable and an explicit Euler method for the discretization with respect to the time variable (see [17]).

Example 1. We take the boundary values

$$n_0 = 0, \quad p_0 = 1, \quad V_0 = -2.5,$$

$$n_1 = 1, \quad p_1 = 0, \quad V_1 = -3.75.$$

In semiconductor simulation, the boundary values are usually chosen such that (i) the total space charge $-n + p + C$ vanishes at the Ohmic contacts $x = 0$ and $x = 1$, (ii) the boundary densities are in thermal equilibrium and the boundary potential is the superposition of the thermal equilibrium value and the applied potential [23]. The above values satisfy these conditions with an applied voltage of $U = 1.0V$. Thus, we are modeling a forward biased pn diode. We neglect in this example recombination–generation effects: $R(n, p) = 0$ (see Examples 2 and 3 for non-vanishing $R(n, p)$).

The initial densities are shown in Figs 2 and 3. In Fig. 2 the temporal evolution of the hole density p is depicted. Initially, there is a vacuum region for p consisting of the interval $[0.2, 1.0]$. For increasing $t > 0$, the vacuum set becomes smaller and finally, it becomes trivial (i.e. only $p(1,t) = 0$) after some time. A similar behavior can be observed for the electron density in Fig. 3. This shows the finite speed of propagation

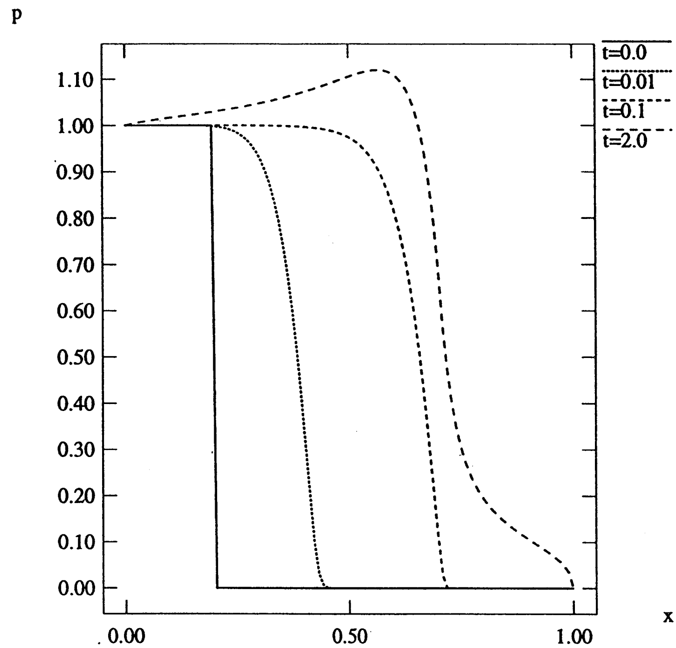

Fig. 2. Example 1: hole density, $R(n, p) = 0$.

of the support of p and n (see Theorem 1.1). In Fig. 4 the electron density is shown for small values of time. Here, the vacuum set for n becomes larger for small time ($t = 0.01$), and later ($t = 0.05$), the size of the vacuum set is decreasing. This illustrates the waiting time property even in the absence of recombination–generation effects (see Theorem 1.2).

Example 2. We use the same boundary values as in the first example but different initial functions (see Fig. 5). In this example, we study the effects of the recombination–generation term. We choose

$$R(n, p) = c_R(np)^\beta.$$

From Fig. 5, where $c_R = 0$, we see that the vacuum set for the electrons is trivial (i.e. only $n(0, t) = 0$) for sufficiently large time. At $t = 4.0$ the electron density has almost reached the stationary state. If $c_R = 1$ and $\beta = 0.2$ there are (non-trivial) vacuum sets for n for sufficiently large time, e.g. $t \geq 0.7$ (see Fig. 6). This example shows the property of formation of vacuum due to the presence of a strong recombination term (see Theorem 1.3). Note that the assumption $\alpha + \beta < 2$ is satisfied. (The hole density is positive in the region where $n = 0$.) Choosing $c_R = 1$ and $\beta = 0.5$, there are no vacuum solutions for n for $t \geq 0.7$ (Fig. 7). In this situation the condition $\alpha + \beta < 2$ is not satisfied.

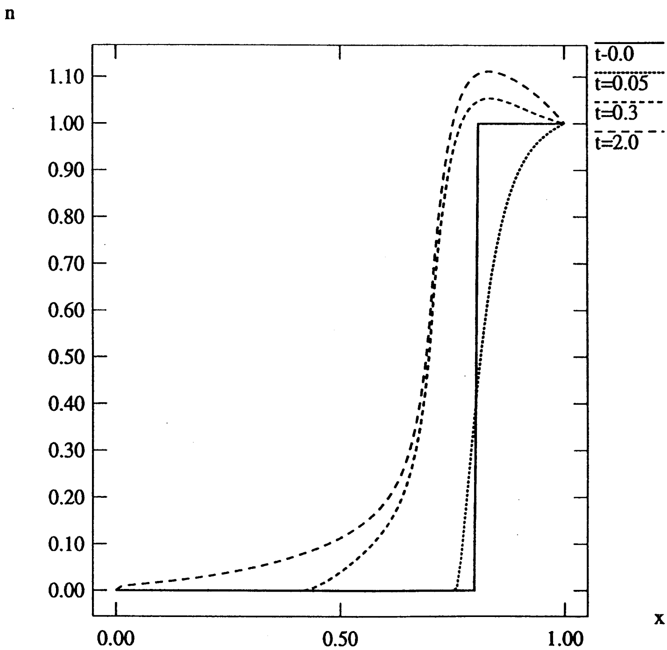

Fig. 3. Example 1: electron density, $R(n, p) = 0$.

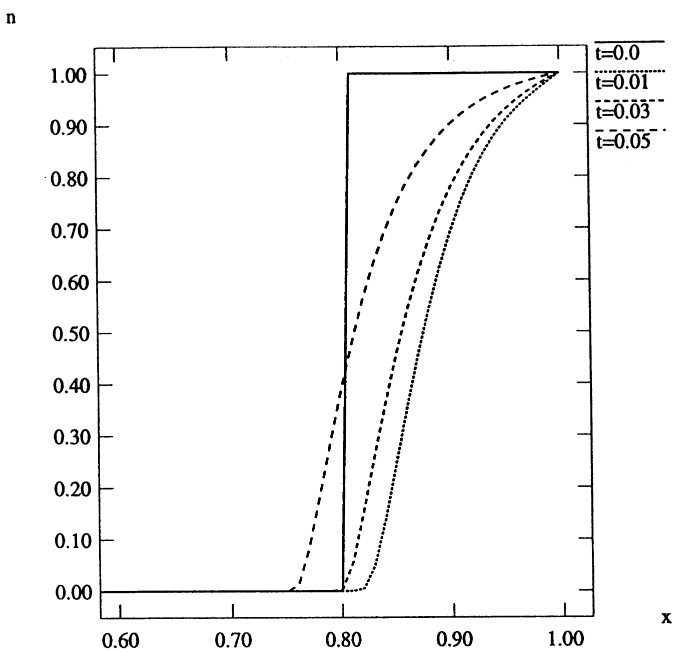

Fig. 4. Example 1: electron density, $R(n, p) = 0$.

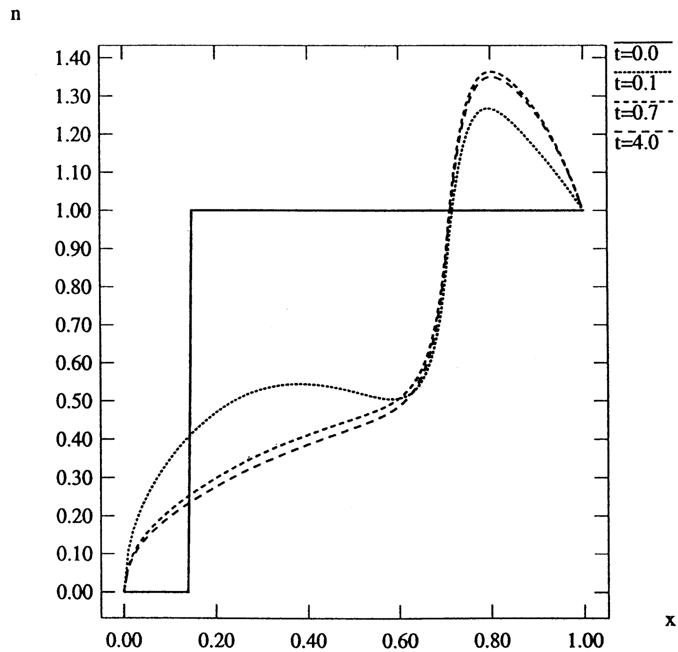

Fig. 5. Example 2: electron density, $R(n, p) = 0$.

Example 3. For the last example we choose initial conditions such that $n(x,0)$ and $p(x,0)$ are strictly positive. Therefore, we use different boundary values than in Examples 1 and 2:

$$n_0 = 1, \quad p_0 = 2, \quad V_0 = -\tfrac{5}{2}(\sqrt[3]{4} - 1),$$

$$n_1 = 2, \quad p_1 = 1, \quad V_0 = -\tfrac{5}{2}(\sqrt[3]{4} - 1) - \tfrac{25}{4}.$$

Here the total space charge vanishes, but the boundary functions are not in thermal equilibrium. Again, for vanishing recombination–generation, the vacuum sets $\{n(t) = 0\}$ are empty for all $t \geq 0$ (Fig. 8). If $c_R = 5$ and $\beta = 0.1$, there exists $t_0 > 0$ such that the vacuum sets for $n(t)$ have positive measure for all t larger than t_0 (Fig. 9). In this situation, it holds $\alpha + \beta < 2$. In Fig. 10 the electron density in the case $c_R = 5$ and $\beta = 0.6$ is presented. The condition $\alpha + \beta < 2$ is not satisfied, the vacuum sets are empty for all time. For $t = 2$, the solution is close to the steady state. However, if we choose $c_R = 10$ and $\beta = 0.6$, there are vacuum solutions for $t \geq 2$ (Fig. 11), although it holds $\alpha + \beta > 2$. The recombination effects are so strong that vacuum occurs. This does not contradict the non-vacuum result mentioned in Section 1 since this result is only valid for $\beta \geq 1$.

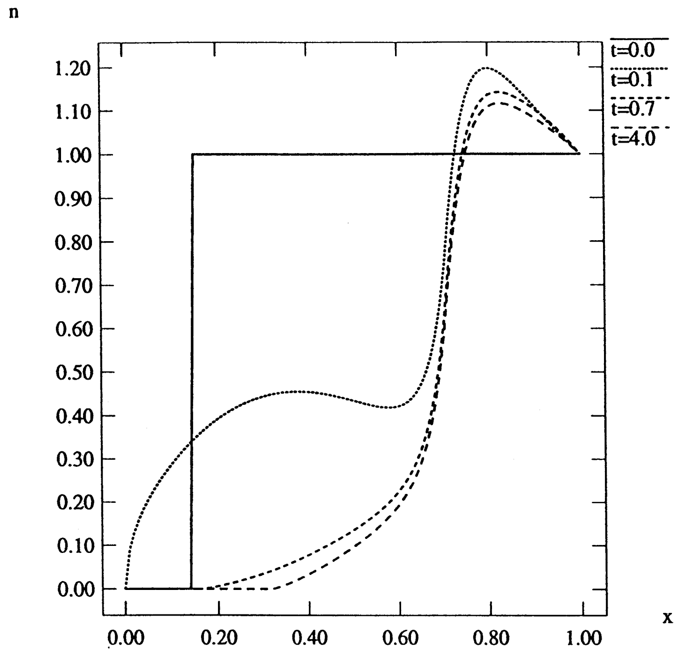

Fig. 6. Example 2: electron density, $R(n, p) = (np)^{0.2}$.

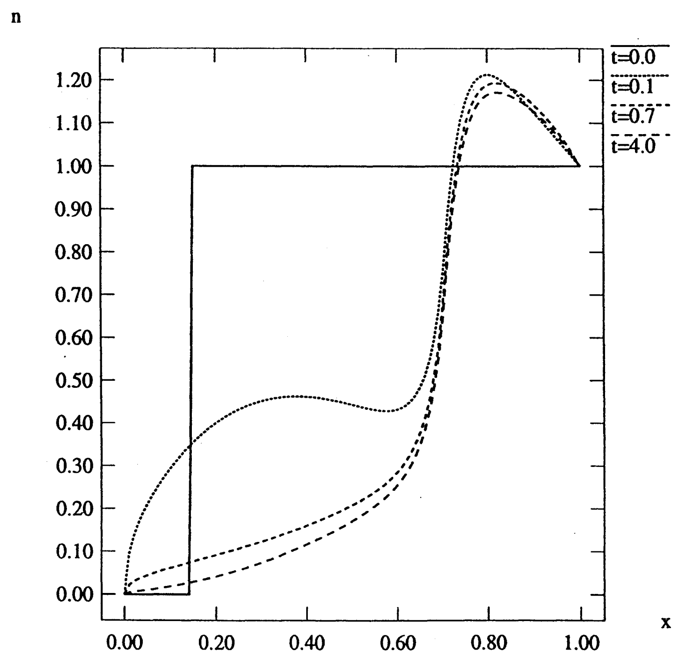

Fig. 7. Example 2: electron density, $R(n, p) = (np)^{0.5}$.

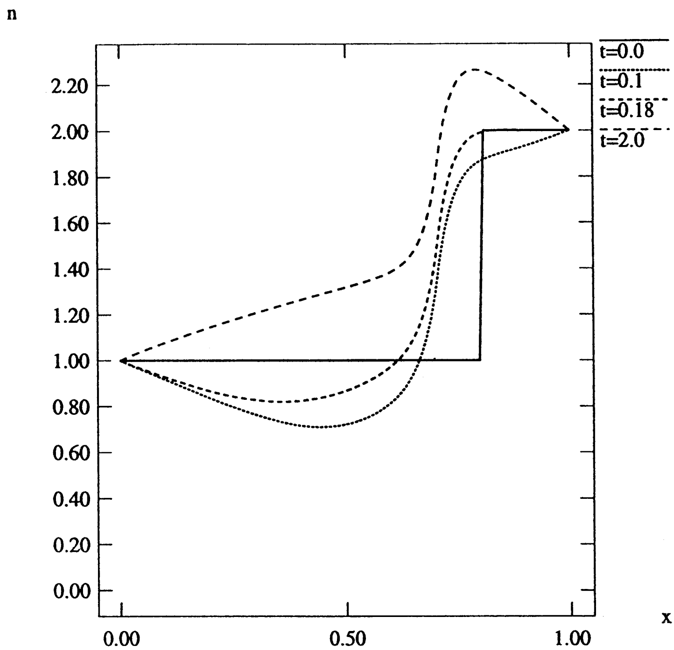

Fig. 8. Example 3: electron density, $R(n, p) = 0$.

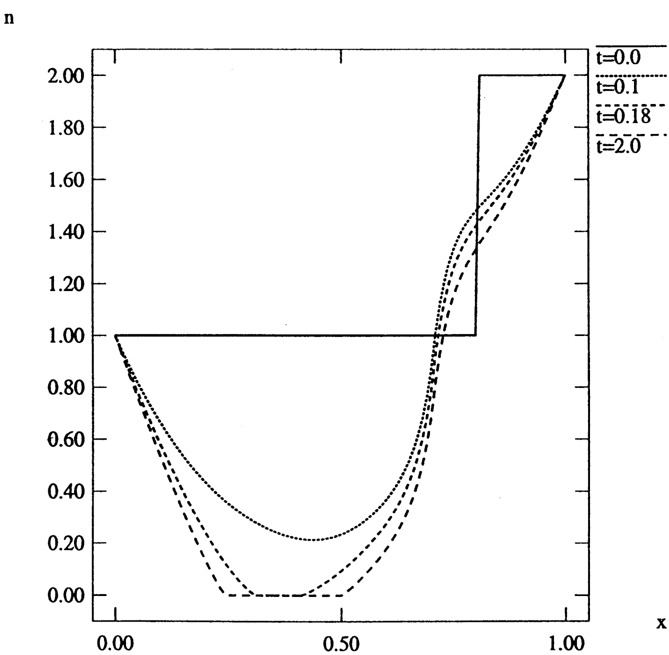

Fig. 9. Example 3: electron density, $R(n, p) = 5(np)^{0.1}$.

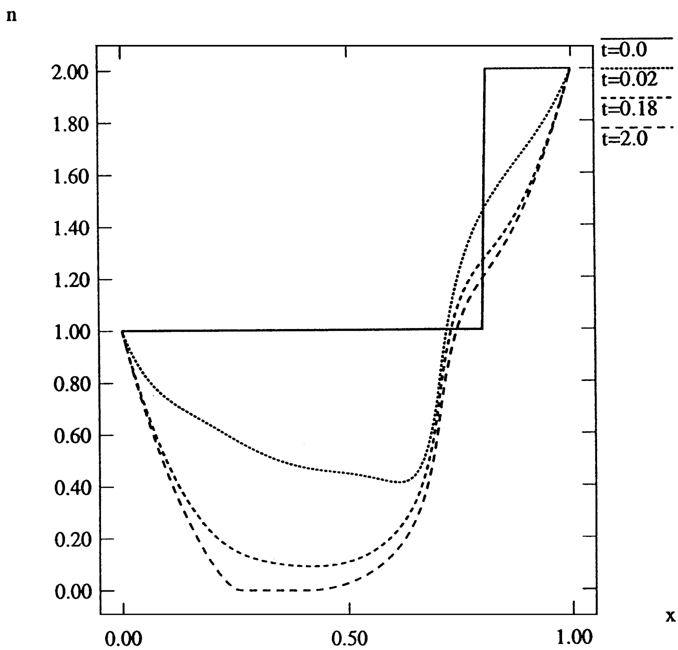

Fig. 10. Example 3: electron density, $R(n, p) = 5(np)^{0.6}$.

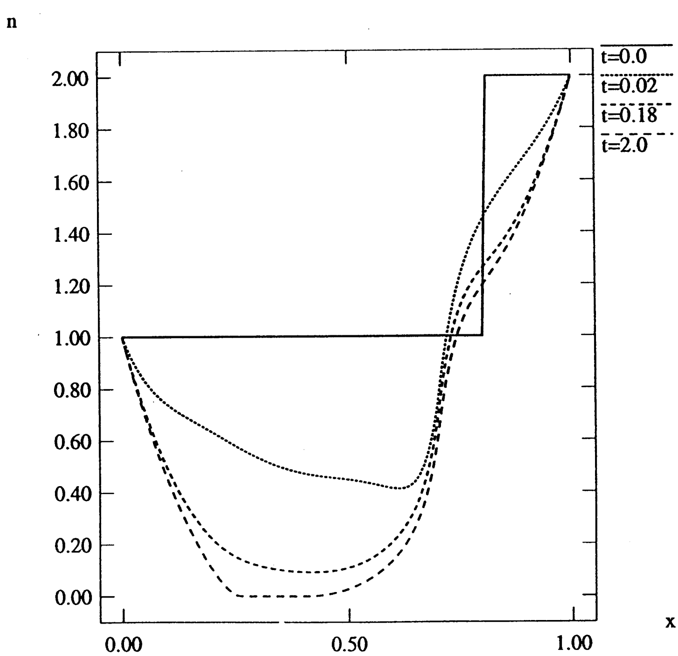

Fig. 11. Example 3: electron density, $R(n, p) = 10(np)^{0.6}$.

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Appendix

In this appendix we present two technical lemmas which are used in the proofs of Theorems 1.1–1.3. The first lemma is a local integration by parts:

Lemma A.1. *Assume that* $P \subset Q_T$. *Then for almost all* $\tilde{t}, t \in [0, T]$, $\tilde{t} < t$,

$$\int_P (\nabla n^\alpha - n\nabla V) \cdot \nabla n^\alpha \,\mathrm{d}x\,\mathrm{d}\tau - \int_{\partial_l P} (\nabla n^\alpha - n\nabla V) \cdot v_x n^\alpha \,\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$\leq \frac{1}{\alpha+1}\int_{P\cap\{\tau=\tilde{t}\}} n(\tilde{t})^{\alpha+1}\,\mathrm{d}x - \frac{1}{\alpha+1}\int_{P\cap\{\tau=t\}} n(t)^{\alpha+1}\,\mathrm{d}x - \frac{1}{\alpha+1}\int_{\partial_l P} n^{\alpha+1} v_\tau \,\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$- \int_P R(n, p) n^\alpha \,\mathrm{d}x\,\mathrm{d}\tau \tag{22}$$

holds.

By using spherical coordinates and Fubini's theorem, it can be seen that the integrals over $\partial_l P$ are well defined for almost all $r(\tau)$. A similar inequality holds for the hole density p. The proof is a straightforward extension of the proof of the local integration by parts formula in [12].

The second technical tool is an interpolation-trace lemma.

Lemma A.2. *Let* $B = B_R(x_0) \subset \mathbb{R}^d$ *be a ball of radius* $R > 0$ *and center* x_0 *and let* $u \in W^{1,p}(B)$ *with* $1 < p < \infty$. *Then*

$$\|u\|_{0,q,\partial B} \leq c_0 (\|\nabla u\|_{0,p,B} + R^{-\delta}\|u\|_{0,s,B})^\theta \|u\|_{0,r,B}^{1-\theta}, \tag{23}$$

where $c_0 > 0$ *is independent of* u *and* R, $1 \leq s < \infty$,

$$1 \leq q < \frac{p(d-1)}{d-p}, \quad 1 \leq r < \frac{dp}{d-p} \quad \text{if } p < d,$$
$$1 \leq q, r < \infty \quad \text{if } p = d,$$
$$1 \leq q, p \leq \infty \quad \text{if } p > d,$$

and the exponents are given by

$$\theta = \frac{p}{q}\,\frac{qd - r(d-1)}{p(d+r) - dr} \in (0,1), \qquad \delta = \frac{ps + d(p-s)}{ps} > 1.$$

The proof can be found in [5]. In the case $q = p$ and $s = r$ the lemma is proved in [12].

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