

# On the Uniqueness of Solutions of a Nonlinear Elliptic Problem Arising in the Confinement of a Plasma in a Stellarator Device\*

J. I. Díaz,<sup>1</sup> G. Galiano,<sup>1</sup> and J. F. Padial<sup>2</sup>

<sup>1</sup>Departamento de Matemática Aplicada, Fac. de Matemáticas,  
Universidad Complutense de Madrid,  
28040 Madrid, Spain

<sup>2</sup>Departamento de Matemática Aplicada, E.T.S. Arquitectura,  
Universidad Politécnica de Madrid,  
28040 Madrid, Spain

**Abstract.** We study the uniqueness of solutions of a semilinear elliptic problem obtained from an inverse formulation when the nonlinear terms of the equation are prescribed in a general class of real functions. The inverse problem arises in the modeling of the magnetic confinement of a plasma in a Stellarator device. The uniqueness proof relies on an  $L^\infty$ -estimate on the solution of an auxiliary nonlocal problem formulated in terms of the relative rearrangement of a datum with respect to the solution.

**Key Words.** Uniqueness of solution, Nonlinear elliptic equations, Nonlocal equations, Relative rearrangement, Plasma physics.

**AMS Classification.** 35J60, 76W05.

## 1. Introduction

The main goal of this paper is to study the uniqueness of the solution of a two-dimensional free boundary problem modeling the magnetic confinement of a plasma in a Stellarator device. The model consists of a second-order partial differential equation of elliptic type,

---

\* All the authors were partially supported by the DGES (Spain) Project No. PB96-0583. The first author was also supported by Contract 72/93 of the CIEMAT/EURATOM association.

obtained from the three-dimensional ideal MHD system by Hender and Carreras [9] by using toroidal averaging arguments and a suitable system of coordinates: the Boozer vacuum flux coordinates [3]. This problem has recently been studied by Díaz [4] who introduced the following formulation in the form of a *free boundary problem*. Let  $\Omega$  be an open, bounded, regular set of  $\mathbb{R}^2$ , and let

$$\lambda > 0, \quad F_v > 0, \quad a, b \in L^\infty(\Omega), \quad b > 0 \quad \text{a.e. in } \Omega.$$

Given  $\gamma < 0$ , the problem is to find  $u \in H^1(\Omega) \cap L^\infty(\Omega)$  and  $F \in C^0(\mathbb{R}; [0, \infty))$  such that  $F(s) = F_v$  for any  $s \leq 0$ ,  $F^2 \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$ , and  $(u, F)$  satisfies the following *inverse problem*:

$$(\mathcal{P}_I) \quad \begin{cases} -\Delta u = aF(u) + \left(\frac{F^2}{2}\right)'(u) + \lambda b u_+ & \text{in } \Omega, \\ u = \gamma & \text{on } \partial\Omega, \\ 0 = \int_{\{u>t\}} \left(\left(\frac{F^2}{2}\right)'(u) + \lambda u_+ b\right) dx, & \forall t \in (-\infty, \text{ess sup } u], \end{cases}$$

where, for the sake of simplicity in the exposition, we have replaced the second-order symmetric uniformly elliptic operator  $\mathcal{L}$  given in [9] by the Laplace operator and we have taken the pressure term equal to  $(\lambda/2)u_+^2$ . In what follows we refer to the family of integral identities stated in  $(\mathcal{P}_I)$  as the *Stellarator Condition*.

In order to determine the unknown function  $F$ , the above problem was reformulated by Díaz in [5] using the notion of *relative rearrangement*. It was proved there that if  $(u, F)$  is a solution of  $(\mathcal{P}_I)$  such that  $u \in \mathcal{U} \subset C^0(\Omega)$ , where

$$\mathcal{U} = \{u \in W^{2,p}(\Omega), \text{ for any } 1 \leq p < \infty \text{ and } \text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0\},$$

then  $u$  satisfies the following uncoupled *nonlocal problem*:

$$(\mathcal{P}_{\text{NL}}) \quad \begin{cases} -\Delta u = a \left[ F_v^2 - 2\lambda \int_0^{u_+} \sigma b_{*u}(|u > \sigma|) d\sigma \right]_+^{1/2} \\ \quad + \lambda u_+ [b - b_{*u}(|u > u(x)|)] & \text{in } \Omega, \\ u = \gamma & \text{on } \partial\Omega, \end{cases}$$

and necessarily  $F = F_u$  on  $(-\infty, \|u_+\|_{L^\infty(\Omega)})$  with

$$F_u(t) := \left[ F_v^2 - 2\lambda \int_0^{t_+} \sigma b_{*u}(|u > \sigma|) d\sigma \right]_+^{1/2},$$

where  $|u > t|$  denotes  $\text{meas}\{x \in \Omega : u(x) > t\}$ ,  $u_*$  represents the decreasing rearrangement of  $u$ , and  $b_{*u}$  is the relative rearrangement of  $b$  with respect to  $u$  (the definition of both notions are recalled in the next section).

The existence of  $u \in \mathcal{U}$ , a solution of  $(\mathcal{P}_{\text{NL}})$ , was proved by Díaz and Rakotonon [7], [8] under some additional assumptions on  $a$  and  $\lambda$ . They also proved that if  $u \in \mathcal{U}$  is a solution of  $(\mathcal{P}_{\text{NL}})$  and we define  $F_u \in C^0((-\infty, \|u_+\|_{L^\infty(\Omega)}] : [0, +\infty))$  by

$$F_u(t) := \left[ F_v^2 - 2\lambda \int_0^{t_+} \sigma b_{*u}(|u > \sigma|) d\sigma \right]_+^{1/2}, \quad \forall t \leq \|u_+\|_{L^\infty(\Omega)}, \quad (1)$$

then  $(u, F_u)$  is a solution of  $(\mathcal{P}_1)$  assuming that  $F_u(t) > 0$  for any  $t \in (-\infty, \|u_+\|_{L^\infty(\Omega)}]$ . Notice that without loss of generality we can prolongate  $F_u$  to  $(\|u_+\|_{L^\infty(\Omega)}, +\infty)$  in such a way that  $F_u \in \mathcal{C}^0(\mathbb{R} : [0, +\infty))$  and  $F_u^2 \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$ .

In this paper we consider the question of the uniqueness of solutions of  $(\mathcal{P}_1)$  under the following special conditions: given  $(u_1, F)$ ,  $(u_2, F)$  solutions of  $(\mathcal{P}_1)$  (i.e., having a common second component) with  $u_i \in \mathcal{U}$ ,  $i = 1, 2$ , find assumptions on  $F$  implying that  $u_1 = u_2$ . The special structure of the equation of  $(\mathcal{P}_1)$  makes it reasonable to assume the function  $(F^2)'$  is locally Lipschitz continuous (i.e.,  $F^2 \in W_{\text{loc}}^{2,\infty}(\mathbb{R})$ ). Notice that, obviously, this assumption is stronger than condition  $F^2 \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$  included in the definition of the solution of  $(\mathcal{P}_1)$  and that neither imply the local Lipschitz continuity of  $F$  since it is not known a priori whether the property

$$F^2(u_i(x)) > 0 \quad \text{a.e. } x \in \Omega, \quad i = 1, 2, \quad (2)$$

holds or not. Notice also that necessarily  $F = F_{u_1} = F_{u_2}$  on  $(-\infty, m]$  (where  $m := \min\{\sup u_i\}$ ) and that

$$(F_{u_i}^2)'(t) = -2\lambda t_+ b_{*u_i}(|u_i > t|) \quad \text{a.e. } t \in (-\infty, \|u_{i+}\|_{L^\infty(\Omega)}]. \quad (3)$$

Moreover, if for instance  $m = \sup u_1$ , then  $F_{u_1}$  can be prolongedated to  $(-\infty, M]$ , with  $M := \max\{\sup u_i\}$  ( $= \sup u_2$  in this case) by means of  $F_{u_2}$  and this prolongation still verifies the requirements  $F_{u_2} \in \mathcal{C}^0((-\infty, M] : [0, \infty))$  and  $F_{u_2}^2 \in W_{\text{loc}}^{1,\infty}(-\infty, M)$ .

The  $\lambda$ -dependence in (3) is the motivation for formulating the condition  $F^2 \in W_{\text{loc}}^{2,\infty}(\mathbb{R})$  in the following quantitative terms:

$$|(F^2(t))' - (F^2(\hat{t}))'| \leq \lambda K |t - \hat{t}|, \quad \forall t, \hat{t} \in (-\infty, M], \quad (4)$$

for some positive constant  $K$  independent of  $\lambda$ . Our uniqueness result can be stated in the following terms:

**Theorem 1.** *Let  $(u_1, F)$ ,  $(u_2, F)$  be solutions of  $(\mathcal{P}_1)$  with  $F \in \mathcal{C}^0(\mathbb{R}; [0, \infty))$  and  $F^2 \in W_{\text{loc}}^{2,\infty}(\mathbb{R})$  satisfying (4). Then there exists a positive constant  $\delta$  such that if  $\lambda < \delta$ , then necessarily  $u_1 \equiv u_2$ .*

We point out that if  $b$  is a positive constant then, for any  $(u, F)$  solution of  $(\mathcal{P}_1)$  with  $u \in \mathcal{U}$ , we have that

$$(F_u^2)'(t) = -2\lambda t_+ b \quad \text{a.e. } t \in (-\infty, \|u_+\|_{L^\infty(\Omega)}]$$

and so assumption (4) holds trivially with  $K = 2b$  (see Remark 2 for other comments on (4)).

One of the main steps of the proof of Theorem 1 is to show that if  $u_i \in \mathcal{U}$  and  $\lambda$  is small enough, then (2) holds and the Lipschitz constant of  $F$  on  $(-\infty, M]$  is also small (Theorem 2). As a consequence, we can apply a general uniqueness criterion for semilinear problems (Lemma 3) implying that necessarily  $u_1 = u_2$ . In order to prove (2) we first obtain some  $L^\infty$ -estimates on  $u_i$  in terms of the parameter  $\lambda$  (Lemma 1). The proof of (2) also uses the characterization  $F = F_u$  given in (1) and some properties of the relative rearrangement. We also give another uniqueness result (Theorem 3) under the condition  $a > 0$  in  $\Omega$  but assuming a sharper bound on  $\lambda$ . Some final remarks are given at the end of the paper.

## 2. Some Properties of the Solutions of $(\mathcal{P}_{\text{NL}})$ : Lipschitz Continuity of $F_u$

We start by recalling the notion of the relative rearrangement of a function with respect to another function introduced by Mossino and Temam in [12]. We need to recall some previous well-known notions.

**Definition 1.** Let  $u: \Omega \rightarrow \mathbb{R}$  be a Lebesgue measurable function. The *distribution function* of  $u$  is defined by

$$m_u(t) := \text{meas}\{x \in \Omega : u(x) > t\} \quad (= |u > t|) \quad \text{for any } t \in \mathbb{R}.$$

The generalized inverse of  $m_u$  is called the *decreasing rearrangement* of  $u$  and is denoted by  $u_*$ , i.e.,  $u_*: (0, |\Omega|) \rightarrow \mathbb{R}$  with  $u_*(s) = \inf\{t \in \mathbb{R} : m_u(t) < s\}$ .

Now we recall the notions of relative rearrangement: Let  $v, u \in L^1(\Omega)$  and define the function  $w: [0, |\Omega|] \rightarrow \mathbb{R}$  by

$$w(s) = \begin{cases} \int_{\{u > u_*(s)\}} v(x) dx & \text{if } |u = u_*(s)| = 0, \\ \int_{\{u > u_*(s)\}} v(x) dx + \int_0^{s - m_u(u_*(s))} (v|_{P_u(u_*(s))})_*(\sigma) d\sigma & \text{if } |u = u_*(s)| \neq 0. \end{cases}$$

Here  $v|_{P_u(u_*(s))}$  denotes the restriction of  $v$  to the set  $P_u(u_*(s))$  where  $P_u(t) := \{x \in \Omega : u(x) = t\}$  and  $(v|_{P_u(u_*(s))})_*$  represents its decreasing rearrangement. It was proved in [12] that if  $u \in L^1(\Omega)$  and  $v \in L^p(\Omega)$  for some  $1 \leq p \leq +\infty$ , then  $w \in W^{1,p}(0, |\Omega|)$  and  $\|dw/ds\|_{L^p(0, |\Omega|)} \leq \|v\|_{L^p(\Omega)}$ .

**Definition 2.** The function  $dw/ds$  is called the *relative rearrangement* of  $v$  with respect to  $u$  and is denoted by  $v_{*u}$ .

The main conclusion of this section is the following:

**Theorem 2.** *There exists a positive constant  $\delta = \delta(a, b, F_v, \gamma, \Omega)$  such that if  $\lambda < \delta$ , then, for any  $(u, F)$  solution of  $(\mathcal{P}_1)$  with  $u \in \mathcal{U}$ , necessarily the function  $F$  is strictly positive and Lipschitz continuous on  $(-\infty, \|u_+\|_{L^\infty(\Omega)})$ . In addition, the Lipschitz constant of  $F$  can be taken as decreasing to zero when  $\lambda \downarrow 0$ .*

The proof of Theorem 2 relies on some lemmata.

**Lemma 1.** *There exists a constant  $C_1 = C_1(\lambda)$ , increasing in  $\lambda$ , also depending on  $F_v, b$ , and  $\Omega$ , and satisfying that  $\lim_{\lambda \downarrow 0} C_1(\lambda) < +\infty$ , such that*

$$\sup_{\Omega} u(x) \leq C_1(\lambda) \|a\|_{L^2(\Omega)} + |\gamma| \quad (5)$$

for any  $(u, F)$  solution of  $(\mathcal{P}_1)$  with  $u \in \mathcal{U}$ .

*Proof.* To obtain (5) we first estimate  $\|u - \gamma\|_{L^\infty(\Omega)}$ . We assume that  $\|u_+\|_{L^\infty(\Omega)} > 0$  (otherwise  $(\mathcal{P}_1)$  becomes a linear problem with  $F \equiv F_v$  and the result is trivial).

Multiplying the equation of  $(\mathcal{P}_1)$  by  $u - \gamma$  we obtain, after integrating by parts in  $\Omega$ , that

$$\begin{aligned} \int_{\Omega} |\nabla(u - \gamma)|^2 dx &= \int_{\Omega} \left( aF(u) + \left( \frac{F^2}{2} \right)'(u) + \lambda b u_+ \right) (u - \gamma) dx \\ &= \int_{\Omega} aF(u)(u - \gamma) dx + \int_{\Omega} z(u - \gamma) dx, \end{aligned}$$

where

$$z = z(x) := \left( \frac{F^2}{2} \right)'(u(x)) + \lambda b(x)u_+(x).$$

By Fubini's theorem

$$\int_{\Omega} z(u - \gamma) dx = \int_{\hat{m}}^{\infty} ds \int_{\{y \in \Omega : u(y) - \gamma > s\}} z(x) dx,$$

where  $\hat{m} = \inf\{u(x) - \gamma : x \in \Omega\}$ . Since  $\{x \in \Omega : u(x) - \gamma > s\} = \{x \in \Omega : u(x) > \gamma + s\}$ , using the Stellarator Condition in  $(\mathcal{P}_1)$  we have

$$\int_{\Omega} z(u - \gamma) dx = 0$$

and hence

$$\int_{\Omega} |\nabla(u - \gamma)|^2 dx = \int_{\Omega} aF(u)(u - \gamma) dx.$$

Moreover, as  $u - \gamma \in H_0^1(\Omega)$ , we can use Poincaré and Hölder inequalities to get

$$\begin{aligned} \int_{\Omega} (u - \gamma)^2 dx &\leq P(\Omega)^2 \int_{\Omega} |\nabla(u - \gamma)|^2 dx \leq P(\Omega)^2 \int_{\Omega} aF(u)(u - \gamma) dx \\ &\leq P(\Omega)^2 \left\{ \int_{\Omega} (aF(u))^2 dx \right\}^{1/2} \left\{ \int_{\Omega} (u - \gamma)^2 dx \right\}^{1/2}, \end{aligned}$$

where  $P(\Omega)$  is the Poincaré constant for  $\Omega$ . From the characterization of  $F = F_u$  mentioned in the Introduction (see also Proposition 1 of [8]) it is easy to see that  $F$  is decreasing and so  $F(u) \leq F_v$ . Thus, using that  $\gamma < 0$  we arrive at

$$\|u_+\|_{L^2(\Omega)} \leq \|(u - \gamma)_+\|_{L^2(\Omega)} \leq F_v P(\Omega)^2 \|a\|_{L^2(\Omega)}. \quad (6)$$

The next step is to obtain an  $L^2$ -estimate for  $\Delta u$ . As  $u$  is also a solution of  $(\mathcal{P}_{\text{NL}})$ , we get that

$$\begin{aligned} \int_{\Omega} |\Delta u|^2 dx &= \int_{\Omega} (aF(u))^2 dx + \lambda^2 \int_{\Omega} u_+^2 (b - b_{*u}(|u > u(x)|))^2 dx \\ &\quad + 2\lambda \int_{\Omega} aF(u)u_+ [b - b_{*u}(|u > u(x)|)] dx. \end{aligned}$$

Using the well-known estimate  $\|b_{*u}\|_{L^\infty(\Omega_*)} \leq \|b\|_{L^\infty(\Omega)}$  we obtain

$$\begin{aligned} \int_{\Omega} |\Delta u|^2 dx &\leq (F_v \|a\|_{L^2(\Omega)})^2 + (2\lambda \|b\|_{L^\infty(\Omega)} \|u_+\|_{L^2(\Omega)})^2 \\ &\quad + 4\lambda F_v \|a\|_{L^2(\Omega)} \|b\|_{L^\infty(\Omega)} \|u_+\|_{L^2(\Omega)}. \end{aligned}$$

In a last step, from (6) and the above estimate, we deduce that

$$\int_{\Omega} |\Delta(u - \gamma)|^2 dx = \int_{\Omega} |\Delta u|^2 dx \leq C \|a\|_{L^2(\Omega)},$$

where

$$C = F_v^2 [2\lambda \|b\|_{L^\infty(\Omega)} P^2(\Omega) + 1]^2.$$

Therefore

$$\|\Delta(u - \gamma)\|_{L^2(\Omega)} \leq \sqrt{C} \|a\|_{L^2(\Omega)}.$$

Now, by the Agmon–Douglis–Nirenberg regularity result, there exists another constant  $T(\Omega)$  such that  $\|\psi\|_{H^2(\Omega)} \leq T(\Omega) \|\Delta\psi\|_{L^2(\Omega)}$  for any  $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ . Moreover, by the Sobolev Embedding Theorem

$$\begin{aligned} \sup_{\Omega} u &\leq \|u\|_{L^\infty(\Omega)} \leq \|(u - \gamma)\|_{L^\infty(\Omega)} + |\gamma| \\ &\leq S(\Omega) \|(u - \gamma)\|_{H^2(\Omega)} + |\gamma| \end{aligned}$$

for some positive constant  $S(\Omega)$ . Then if we define  $\hat{S}(\Omega) := S(\Omega)T(\Omega)$  we have

$$\begin{aligned} \sup_{\Omega} u &\leq \hat{S}(\Omega) \|\Delta(u - \gamma)\|_{L^2(\Omega)} + |\gamma| \\ &\leq \hat{S}(\Omega) \sqrt{C} \|a\|_{L^2(\Omega)} + |\gamma|. \end{aligned}$$

Therefore (5) holds and the constant  $C_1(\lambda)$  can be taken as

$$C_1(\lambda) = \hat{S}(\Omega) F_v [2\lambda P^2(\Omega) \|b\|_{L^\infty(\Omega)} + 1]. \quad \square$$

The second ingredient of the proof of Theorem 2 is the following:

**Lemma 2.** *Let  $u \in \mathcal{U}$  be any solution of  $(\mathcal{P}_{\text{NL}})$ . Assume that*

$$\sup_{\Omega} u < \frac{F_v}{\|b\|_{L^\infty(\Omega)}^{1/2}} \frac{1}{\lambda^{1/2}}. \quad (7)$$

*Then the function  $F_u$  defined by (1) is strictly decreasing on  $(0, \|u_+\|_{L^\infty(\Omega)}]$ . Moreover,  $F_u$  is strictly positive and Lipschitz continuous on  $(-\infty, \|u_+\|_{L^\infty(\Omega)}]$ . More precisely*

$$|F_u(t) - F_u(\hat{t})| \leq \frac{\|u_+\|_{L^\infty(\Omega)}}{F_u(\|u_+\|_{L^\infty(\Omega)})} \lambda \|b\|_{L^\infty(\Omega)} |t - \hat{t}|, \quad \forall t, \hat{t} \in (-\infty, \|u_+\|_{L^\infty(\Omega)}].$$

*Proof.* We recall that if  $w \in \mathcal{U}$  and  $v \in L^p(\Omega)$  (for some  $p \in [1, \infty]$ ), then

$$v_{*w}(s) = \frac{\int_{\{w=w_*(s)\}} (v/|\nabla w|) d\Gamma}{\int_{\{w=w_*(s)\}} (1/|\nabla w|) d\Gamma} \quad \text{a.e. } s \in (0, |\Omega|)$$

(see, e.g., [12] and [14]). This implies that if  $v > 0$  a.e.  $s \in \Omega$ , then  $v_{*w} > 0$  a.e. in  $(0, |\Omega|)$ . Now define the function  $G_u: \mathbb{R} \rightarrow [0, \infty)$  by

$$G_u(t) := F_v^2 - 2\lambda \int_0^t s b_{*u}(|u| > s) ds \quad \text{for any } t \in (0, \|u_+\|_{L^\infty(\Omega)}].$$

We deduce that  $G_u(t)$  is a strictly decreasing function as a consequence of the positivity of the integral terms. Therefore, if  $t \in (0, \|u_+\|_{L^\infty(\Omega)})$  then, taking into account assumption (7) and that  $\|b_{*u}\|_{L^\infty(\Omega_*)} \leq \|b\|_{L^\infty(\Omega)}$ , we conclude that

$$\begin{aligned} G_u(t) &\geq F_v^2 - 2\lambda \|b\|_{L^\infty(\Omega)} \int_0^t s ds \\ &\geq F_v^2 - \lambda \|b\|_{L^\infty(\Omega)} \|u_+\|_{L^\infty(\Omega)}^2 \\ &\geq F_v^2 - \lambda \|b\|_{L^\infty(\Omega)} \left( \frac{F_v}{(\lambda \|b\|_{L^\infty(\Omega)})^{1/2}} \right)^2 = 0. \end{aligned}$$

This proves that  $F_u(t) > 0$  if  $t \in (0, \|u_+\|_{L^\infty(\Omega)})$ . Furthermore, if  $t, \hat{t} \in (0, \|u_+\|_{L^\infty(\Omega)})$  and, for instance  $\hat{t} > t$ , then

$$\begin{aligned} |F_u(t) - F_u(\hat{t})| &= \frac{F_u^2(t) - F_u^2(\hat{t})}{F_u(t) + F_u(\hat{t})} = \frac{2\lambda \int_t^{\hat{t}} \sigma b_{*u}(|u| > \sigma) d\sigma}{F_u(t) + F_u(\hat{t})} \\ &\leq \frac{\lambda \|b\|_{L^\infty(\Omega)} (\hat{t}^2 - t^2)}{2F_u(\|u_+\|_{L^\infty(\Omega)})} \leq \frac{\|u_+\|_{L^\infty(\Omega)}}{F_u(\|u_+\|_{L^\infty(\Omega)})} \lambda \|b\|_{L^\infty(\Omega)} (\hat{t} - t). \quad \square \end{aligned}$$

*Proof of Theorem 2.* Let  $\lambda > 0$  be such that

$$\lambda^{1/2} [C_1(\lambda) \|a\|_{L^2(\Omega)} + |\gamma|] < \frac{F_v}{\|b\|_{L^\infty(\Omega)}^{1/2}}. \quad (8)$$

Then if  $(u, F)$  is any solution of  $(\mathcal{P}_1)$  with  $u \in \mathcal{U}$  we conclude from Lemma 1 that  $u$  satisfies (7). By the results of [8] we know that  $u$  is also a solution of  $(\mathcal{P}_{\text{NL}})$  and that  $F = F_u$  on  $(-\infty, \|u_+\|_{L^\infty(\Omega)})$ . Thus, if (8) holds we can apply Lemma 2 and we deduce the first part of Theorem 2 since the left-hand side in inequality (8) is a continuous function of  $\lambda$  vanishing for  $\lambda = 0$  and so (8) holds for  $\lambda$  small enough. Finally, if we use the decreasing function  $G_u$  introduced in the proof of Lemma 2, applying Lemma 1, we have that

$$\begin{aligned} F_u(\|u_+\|_{L^\infty(\Omega)}) &= [G_u(\|u_+\|_{L^\infty(\Omega)})]^{1/2} \\ &\geq [F_v^2 - \lambda \|b\|_{L^\infty(\Omega)} (C_1(\lambda) \|a\|_{L^2(\Omega)} + |\gamma|)^2]^{1/2}. \end{aligned}$$

Again using Lemma 1 we conclude that

$$|F_u(t) - F_u(\hat{t})| \leq \kappa(\lambda) |t - \hat{t}| \quad \text{for any } t, \hat{t} \in (-\infty, \|u_+\|_{L^\infty(\Omega)}], \quad (9)$$

where

$$\kappa(\lambda) := \frac{\lambda(C_1(\lambda)\|a\|_{L^2(\Omega)} + |\gamma|)\|b\|_{L^\infty(\Omega)}}{[F_v^2 - \lambda\|b\|_{L^\infty(\Omega)}(C_1(\lambda)\|a\|_{L^2(\Omega)} + |\gamma|)^2]^{1/2}} \quad (10)$$

and so

$$\lim_{\lambda \downarrow 0} \kappa(\lambda) = 0,$$

which ends the proof of Theorem 2.  $\square$

**Remark 1.** The strict positivity of function  $F_u$  was already proved in Lemma 25 of [8], under a different hypothesis on the data and for some special solutions of  $(\mathcal{P}_{\text{NL}})$  (the ones constructed in the proof of the existence of solutions of  $(\mathcal{P}_{\text{NL}})$ ). An  $L^\infty$ -estimate for such a special class of solutions was also given in [8] (see Lemma 24) under an additional assumption on  $b$ .

### 3. On the Uniqueness of Solutions of $(\mathcal{P}_1)$ When $F$ Is Prescribed

The uniqueness of solutions of  $(\mathcal{P}_1)$  when  $F$  is prescribed can be proved in different ways under the assumption (4) for  $\lambda$  small enough. Indeed, thanks to Theorem 2 we know that the whole right-hand side of the equation of  $(\mathcal{P}_1)$  is a Lipschitz continuous function on  $u$  such that its Lipschitz constant decreases to zero as  $\lambda \downarrow 0$ . Then the conclusion can be obtained with the help of an easy technical result:

**Lemma 3.** *Consider the problem*

$$\begin{cases} -\Delta u = f(x, u; \lambda) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (11)$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$ ,  $\varphi \in H^1(\Omega)$ ,  $\lambda$  is a positive constant, and the real function  $f$  satisfies

$$|f(x, t_1; \lambda) - f(x, t_2; \lambda)| \leq J(\lambda)H(x) |t_1 - t_2| \quad (12)$$

for all  $t_1, t_2 \in \mathbb{R}$ , a.e.  $x \in \Omega$ , and for all  $\lambda > 0$ . Assume that  $J$  is continuous,  $J(0) = 0$ , and  $J(\lambda) > 0$  if  $\lambda > 0$ , and that  $H$  is an a.e. positive function of  $L^r(\Omega)$ , with  $r > N/2$ . Then, if

$$J(\lambda) < \mu_1, \quad (13)$$

with  $\mu_1$  the first eigenvalue of

$$\begin{cases} -\Delta w = \mu H(x)w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (14)$$

problem (11) has, as such, a unique nontrivial solution in  $H^1(\Omega)$ .



*Proof.* Although several uniqueness results of this type are consequences of fixed point theorems (see [10] and the references therein) a direct proof can be obtained as follows: Suppose that there exist two solutions of (11)  $u, v \in H^1(\Omega)$ . Set  $U = u - v$ . Then  $U$  satisfies

$$\begin{cases} -\Delta U = \Phi(x; \lambda)U & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \quad (15)$$

where

$$\Phi(x; \lambda) = \begin{cases} \frac{f(x, u; \lambda) - f(x, v; \lambda)}{u(x) - v(x)} & \text{if } U(x) \neq 0, \\ 0 & \text{if } U(x) = 0. \end{cases}$$

From the assumptions it is straightforward to see that  $|\Phi(x; \lambda)| \leq J(\lambda)H(x)$ . Then, multiplying (15) by  $U$ , using the assumptions on  $f$  and  $J$  and the characterization of the first eigenvalue of the  $\Delta$  operator we obtain that necessarily  $U \equiv 0$ .  $\square$

*Proof of Theorem 1.* It suffices to check that the assumptions of Lemma 3 are fulfilled. We set  $f(x, t; \lambda) = a(x)F(t) + (F^2/2)'(t) + \lambda b(x)t_+$ . In order to check the assumptions on  $f$  we define

$$g_1(x) := \begin{cases} \frac{F(u_1(x)) - F(u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x), \\ 0 & \text{if } u_1(x) = u_2(x), \end{cases}$$

$$g_2(x) := \begin{cases} \frac{(F^2/2)'(u_1(x)) - (F^2/2)'(u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x), \\ 0 & \text{if } u_1(x) = u_2(x), \end{cases}$$

$$h(x) := \begin{cases} \frac{u_{1+}(x) - u_{2+}(x)}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x), \\ 0 & \text{if } u_1(x) = u_2(x), \end{cases}$$

and finally

$$\Phi(x; \lambda) = a(x)g_1(x) + g_2(x) + \lambda b(x)h(x).$$

We prove that

$$|\Phi(x; \lambda)| \leq J(\lambda)H(x), \quad (16)$$

with  $J$  and  $H$  satisfying the conditions of Lemma 3. We have that

$$|\Phi(x; \lambda)| \leq |a(x)| |g_1(x)| + |g_2(x)| + \lambda b(x) \quad \text{a.e. } x \in \Omega.$$

Since  $|h(x)| \leq 1$ , from hypothesis (4), we get that if  $\lambda$  satisfies (8), then  $F$  is decreasing and so

$$|\Phi(x; \lambda)| \leq |a(x)| |g_1(x)| + \lambda \left( \frac{K}{2} + b(x) \right) \quad \text{a.e. } x \in \Omega.$$

If we suppose, for instance, that  $\|u_{1+}\|_{L^\infty(\Omega)} \leq \|u_{2+}\|_{L^\infty(\Omega)}$ , then  $F(u_1(x)) - F(u_2(x)) = F_{u_2}(u_1(x)) - F_{u_2}(u_2(x))$ . From (8) we know that inequalities (9) and (10) hold (see the proof of Theorem 2). Then we deduce that

$$|\Phi(x; \lambda)| \leq |a(x)| \kappa(\lambda) + \lambda \left( \frac{K}{2} + b(x) \right) \quad \text{a.e. } x \in \Omega,$$

and inequality (16) is verified with

$$J(\lambda) := \max\{\lambda, \kappa(\lambda)\} \quad (17)$$

and

$$H(x) = \max \left\{ |a(x)|, \frac{K}{2} + b(x) \right\}. \quad (18)$$

The conclusion of Theorem 1 follows by taking  $\lambda$  small enough so that (8) and (13) hold.  $\square$

Another uniqueness result, under a sharper bound on  $\lambda$ , can be obtained by a different technique assuming a sign condition on the coefficient  $a$ .

**Theorem 3.** *Assume  $a > 0$  on  $\Omega$  and let  $(u_1, F)$ ,  $(u_2, F)$  be as in Theorem 1 with  $F$  satisfying (4). Then there exists  $\hat{\delta} > 0$  (with  $\hat{\delta}$  depending on the second eigenvalue of problem (14) and with  $H$  given by (18)) such that if  $\lambda < \hat{\delta}$ , then necessarily  $u_1 \equiv u_2$ .*

*Proof.* We follow a technique similar to the one of Puel [13]. We set  $U := u_1 - u_2$ . We proceed in two steps. First we show that  $U$  has a definite sign in  $\Omega$  and later we obtain a contradiction to this fact. So we claim that  $U$  does not change its sign in  $\Omega$ . If it does, we define the subsets

$$\Omega_+ := \{x \in \Omega : U(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega : U(x) < 0\}$$

and choose  $\alpha > 0$  such that the function defined by

$$\tilde{U}(x) := \begin{cases} U(x) & \text{if } x \in \Omega_+, \\ \alpha U(x) & \text{if } x \in \Omega_-, \end{cases}$$

is orthogonal in the weighted space  $L_H^2(\Omega)$  to the first eigenvector  $w_1$  of (14), i.e.,

$$\alpha = \frac{-\int_{\Omega_+} U w_1 H dx}{\int_{\Omega_-} U w_1 H dx}.$$

Notice that  $\tilde{U} \in H_0^1(\Omega)$ . Then, if  $\mu_2$  is the second eigenvalue of (14), by definition

$$\mu_2 = \min \left\{ \frac{\int_{\Omega} |\nabla \psi|^2 dx}{\int_{\Omega} \psi^2 H dx} : \psi \in H_0^1(\Omega), \int_{\Omega} \psi w_1 H dx = 0 \right\}.$$

Therefore

$$\mu_2 \leq \frac{\int_{\Omega} |\nabla \tilde{U}|^2 dx}{\int_{\Omega} \tilde{U}^2 H dx}.$$

Multiplying by  $U_+$  in (15) and integrating by parts we obtain that

$$\int_{\Omega_+} |\nabla U|^2 dx = \int_{\Omega_+} [ag_1 + g_2 + \lambda bh] U^2 dx,$$

and an analogous equality is obtained in  $\Omega_-$ . Using that

$$\int_{\Omega} |\nabla \tilde{U}|^2 dx = \int_{\Omega_+} |\nabla U|^2 dx + \alpha^2 \int_{\Omega_-} |\nabla U|^2 dx$$

and assuming (8) we obtain

$$0 \geq \int_{\Omega} (\mu_2 - J(\lambda)) H \tilde{U}^2 dx, \quad (19)$$

where  $J(\lambda)$  is given by (17). Therefore inequality (19) leads to a contradiction if we assume

$$J(\lambda) < \mu_2. \quad (20)$$

Now, as  $U$  does not change its sign, we can assume without loss of generality that  $U \geq 0$  in  $\Omega$ , i.e., that  $u_1 \geq u_2$  in  $\Omega$ . If we integrate the equation of  $(\mathcal{P}_1)$  satisfied by  $u_1, u_2$ , apply the Divergence Theorem, and assume (8) then, by Theorem 2,  $F$  is strictly decreasing on  $(0, \|u_{1+}\|_{L^\infty(\Omega)})$  and by the Stellarator Condition stated in  $(\mathcal{P}_1)$  we obtain

$$\begin{aligned} - \int_{\partial\Omega} \nabla u_1 \cdot \mathbf{n} d\Gamma &= - \int_{\Omega} \Delta u_1 dx = \int_{\Omega} a F_{u_1}(u_1) dx \\ &< \int_{\Omega} a F_{u_1}(u_2) dx = - \int_{\Omega} \Delta u_2 dx \\ &= - \int_{\partial\Omega} \nabla u_2 \cdot \mathbf{n} d\Gamma, \end{aligned}$$

where  $\mathbf{n}$  is the outward normal to  $\Omega$ . Finally, we have three cases to consider: If none of the solutions has a null positive part then, as  $u_1 = v_2$  on  $\partial\Omega$  and  $u_1 \geq u_2$  in  $\Omega$ , this implies that  $\nabla u_1 \cdot \mathbf{n} \leq \nabla u_2 \cdot \mathbf{n}$  on  $\partial\Omega$  leading to a contradiction. If both solutions have null positive part, then the conclusion is obvious because the problem  $(\mathcal{P}_1)$  becomes

linear. Finally, in the case in which one of the solutions has a null positive part but the other does not (and as  $u_1 \geq u_2$  it has to be  $u_{2+} \equiv 0$ ), then

$$\begin{aligned} \int_{\Omega} F_{u_1}(u_1) dx &= \int_{\{u_1 < 0\}} F_{u_1}(u_1) dx + \int_{\{u_1 > 0\}} F_{u_1}(u_1) dx < F_v |\Omega| \\ &= \int_{\Omega} F_{u_1}(u_2) dx, \end{aligned}$$

arriving again at a contradiction. Hence, the conclusion follows taking  $\lambda$  small enough such that (8) and (20) hold.  $\square$

**Remark 2.** As  $u \in \mathcal{U}$  then

$$(F_u^2)' = -2\lambda t_+ b_{*u}(|u > t|) = -2\lambda t_+ \frac{\int_{\omega(t)} (b/|\nabla u|) d\Gamma}{\int_{\omega(t)} (1/|\nabla u|) d\Gamma},$$

where

$$\omega(t) := \{x \in \Omega : b(x) = b_{*u}(|u > t|)\}.$$

Then, as mentioned in the Introduction, if  $b$  is a positive constant ( $F_u^2)' = -2\lambda t_+ b$  and so assumption (4) holds with  $K = 2b$ . Moreover, in general, the Lipschitz continuity of  $(F_u^2)'$  is related to the Lipschitz continuity of the functions  $s \rightarrow b_*(s)$  and  $t \rightarrow |u > t|$ . In a recent paper, Díaz et al. [6] (see their Lemma 4) have shown that if  $b \in H^1(\Omega)$ ,  $\Delta b \in L^\infty(\Omega)$ ,  $b = B$  on  $\partial\Omega$ , and  $b(x) \geq B$  a.e.  $x \in \Omega$ , for some constant  $B$ , then the function  $s \rightarrow b_*(s)$  is Lipschitz continuous (recall that  $\Omega \subset \mathbb{R}^2$ ). The Lipschitz regularity of  $t \rightarrow |u > t|$  is more delicate and remains open as far as we know.

**Remark 3.** Although in [5] and [8] the problem is formulated in a more complicated framework, we have preferred to simplify it for the sake of clarity in the exposition. Nevertheless, all the calculus involved in this article can be extended to that case. To do this it suffices to replace  $\Delta$  by the uniformly elliptic operator  $\mathcal{L}$  given in [9],  $\Omega$  by the ball centered in the origin of radius  $R$ , the Cartesian coordinates  $x = (x_1, x_2)$  by the polar Boozer coordinates  $(\rho, \theta)$ , the functional spaces defined in this article (in terms of the Lebesgue measure  $dx$ ) by their analogous weighted functional spaces (using the measure  $\rho d\rho d\theta$ ), the usual relative rearrangement by the weighted relative rearrangement, the estimates in terms of  $\int_{\Omega} |\nabla(u - \gamma)|^2 dx$  by the coerciveness property of  $\mathcal{L}$ , the Poincaré and Hölder inequalities by its weighted versions and, finally, the Agmon–Douglas–Nirenberg theorem by its corresponding version on weighted functional spaces.

We also remark that the pressure term can be generalized to any  $C^1$  function  $p(u)$  such that  $0 \leq p'(u) \leq \lambda u_+$  on  $\mathbb{R}_+$ .

**Remark 4.** The requirement on the smallness of  $\lambda$  is physically expected. As the parameter  $\lambda$  represents the ratio between the particle pressure and the magnetic pressure, a large value of  $\lambda$  would give rise to the so-called magnetic islands (the plasma splits in several disconnected toroidal volumes) which mathematically corresponds to the

bifurcation of the branch of solutions. This was proved mathematically in the case of confinement of a plasma in a Tokamak device by Schaeffer [16]. For general expositions on the Tokamak case see, e.g., [1], [2], [11], [14], [15], [17], [18], and their references.

## Acknowledgments

The authors thank the anonymous referee for indicating the present proof of Lemma 3, which is shorter than the one given in a previous manuscript.

## References

1. Berestycki H, Brezis H (1980) On a free boundary problem arising in plasma physics. *Nonlinear Anal* 4:415–436
2. Blum J (1989) *Numerical Simulation and Optimal Control in Plasma Physics*. Gauthier Vilars, Paris
3. Boozer AH (1982) Establishment of magnetic coordinates for a given magnetic field. *Phys Fluids* 25:520
4. Díaz JI (1992) Modelos bidimensionales de equilibrio magnetohidrodinámico para Stellarators: Informe # 1. Association EURATOM-CIEMAT, Madrid
5. Díaz JI (1993) Modelos bidimensionales de equilibrio magnetohidrodinámico para Stellarators: Informe # 2. Association EURATOM-CIEMAT, Madrid
6. Díaz JI, Padial JF, Rakotoson JM (1998) Mathematical treatment of the magnetic confinement in a current carrying Stellarator. *Nonlinear Anal TMA* 34:857–887
7. Díaz JI, Rakotoson JM (1993) On a two-dimensional stationary free boundary problem arising in the confinement of a plasma in a Stellarator. *CR Acad Sci Paris Ser I* 317:353–358
8. Díaz JI, Rakotoson JM (1996) On a nonlocal stationary free boundary problem arising in the confinement of a plasma in a Stellarator geometry. *Arch Rational Mech Anal* 134:53–95
9. Hender TC, Carreras BA (1984) Equilibrium calculations for helical axis Stellarators. *Phys Fluids* 27:2101–2109
10. Hernández J (1986) Qualitative methods for nonlinear diffusion equations. In *Nonlinear Diffusion Problems* (A. Fasano and M. Primicerio eds). Lecture Notes in Mathematics 1224. Springer-Verlag, Berlin, pp 47–118
11. Mossino J (1995) On some nonlocal models for the equilibrium of a confined plasma. In *Modelos Matemáticos en Física de Plasmas* (J.I. Díaz and A. Galindo eds). Memorias de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie de Ciencias Exactas. Tomo XXX. Madrid, pp 145–156
12. Mossino J, Temam R (1981) Directional derivative of the increasing rearrangement mapping and application to a queer differential equation in plasma physics. *Duke Math J* 48:475–495
13. Puel P (1977) Sur un problème de valeur propre non lineaire et de frontière libre. *CR Acad Sci Paris Ser I* 284:861–863
14. Rakotoson JM (1988) Some properties of the relative rearrangement. *J Math Anal Appl* 135:475–495
15. Rakotoson JM (1995) Relative rearrangement for highly nonlinear equations. *Nonlinear Anal TMA* 24(4):493–507
16. Schaeffer DG (1977) Nonuniqueness in the equilibrium shape of a confined plasma. *Comm Partial Differential Equations* 2(6):587–600
17. Temam R (1975) A nonlinear eigenvalue problem: equilibrium shape of a confined plasma. *Arch Rational Mech Anal* 60:51–73
18. Temam R (1977) Remarks on a free boundary value problem arising in plasma physics. *Comm Partial Differential Equations* 2:563–585