ON A QUASILINEAR DEGENERATE SYSTEM ARISING IN SEMICONDUCTORS THEORY. PART I: EXISTENCE AND UNIQUENESS OF SOLUTIONS*

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Abstract. A drift-diffusion model for semiconductors with nonlinear diffusion is considered. The model consists of two quasilinear degenerate parabolic equations for carrier densities and the Poisson equation for electric potential. We assume Lipschitz continuous nonlinearities in the drift and generation-recombination terms.

Existence of weak solutions is proven by using a regularization technique. Uniqueness of solutions is proven when either the diffusion term φ is strictly increasing and solutions have spatial derivatives in $L^1(Q_T)$ or when φ is non-decreasing and a suitable entropy condition is fullfilled by the electric potential.

Key words. Quasilinear degenerate system, semiconductors.

AMS subject classifications. 35K65, 35D05, 35B30, 78A35.

1. Introduction. In solid state physics, drift-diffusion equations are today the most widely used model to describe semiconductor devices. Drift-diffusion models describe the flow of electrons and holes in the conduction and valence band, respectively, of semiconductor materials which are influenced by an electric field. Mathematically, the problem is formulated in terms of a system of parabolic equations for the electron and hole densities u, v, and the Poisson equation for the electric potential w which together with physically motivated auxiliary conditions form the problem

$$(1.1) \begin{cases} u_t - div\,(\nabla \varphi(u) - b(u)\nabla w) = F(u,v) \\ v_t - div\,(\nabla \varphi(v) + b(v)\nabla w) = F(u,v) \\ -\Delta w = v - u + C \\ \nabla\varphi(u)\cdot\nu = 0,\ \nabla\varphi(v)\cdot\nu = 0,\ \nabla w \cdot \nu = 0, \\ \varphi(u) = \varphi(u_D),\ \varphi(v) = \varphi(v_D),\ w = w_D, \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0, \end{cases}$$

with the annotations: $\left.\right\}$ in $Q_T := \Omega \times (0,T),$ on $\Sigma_N := \Gamma_N \times (0,T),$ on $\Sigma_D := \Gamma_D \times (0,T),$ in $\Omega.$

with $\Omega \subset \mathbb{R}^N$, $1 \leq N \leq 3$, the bounded domain occupied by the semiconductor crystal. Here, function C denotes the doping profile (fixed charged background ions) characterizing the semiconductor under consideration, φ the pressure function, $b(s)/s$ the mobility of the particles, and F the recombination-generation rate. The boundary $\partial\Omega$ splits into two disjoint subsets Γ_D and Γ_N. Carrier densities and potential are fixed at the Ohmic contacts, Γ_D, whereas their fluxes are null on Γ_N, the union of insulating boundary segments.

The standard drift-diffusion model corresponds to linear functions φ and b, and $F(u,v) = q(u,v)(u_i^2 - uv)$, with $q(u,v)$ a positive function and $u_i \equiv u_i(x) > 0$ the so-called *intrinsic density*. The standard model can be derived from Boltzmann's

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equation once assumed that the semiconductor device is in the low injection regime, i.e. for small absolute values of the applied voltage. In [24] it is shown that in the high injection regime diffusion terms are no longer linear. A useful choice for φ is then $\varphi(s) = s^\alpha$ for $\alpha = \frac{5}{3}$ and $s \geq 0$. With this pressure function the parabolic equations in (1.1) become of degenerate type and existence of solutions does not follow from standard theory. In this paper we present results including both the low and high injection cases.

Function φ may be interpreted in the language of gas dynamics. Assuming that particles behave, thermodynamically speaking, as an ideal gas satisfying the law $\varphi = u\theta$, with θ the particle temperature, we obtain in the isothermal case a linear pressure term. However, in the isentropic case [10] temperature depends on concentrations as $\theta(u) = u^{2/3}$ and therefore $\varphi(u) = u^{5/3}$.

The standard or low injection model has been mathematically and numerically investigated in many papers, see [33], [34] and references therein. Existence and uniqueness of weak solutions was shown.

The isentropic or high injection model for linear b and monotonic F, including the non-Lipschitz continuous case, has been analyzed in [22]-[25] where existence of weak solutions was proven. However, there is a lacking in results concerning the uniqueness of solutions when the problem actually degenerates. Besides, there are no results for general mobility functions.

As in the question of existence, the main difficulty to prove uniqueness of solutions relies in the simultaneous presence of a transport term and a nonlinear degenerate diffusion term. This kind of difficulty has already received the attention of many authors and has been solved for scalar equations of the type

$$(1.2) \qquad u_t - div\left(\nabla\varphi(u) + b(u)\mathbf{e}\right) = F(u),$$

with $\mathbf{e}$ a prescribed vector field. The most successful technique developed to prove uniqueness of solutions of (1.2) is based on the use of the test function $sign_+(u_1 - u_2)$ in the weak formulation of (1.2), being u_1 and u_2 two, a priori, possible solutions of (1.2) *in some sense*. The core of the problem is to show that solutions have enough regularity to define the sign function as an admissible test function. This justification has been carried out by different means. One of them, introduced by Kruzhkov in [28] to prove an L^1 contraction property of entropy solutions of hyperbolic equations, is based in doubling the time variable and performing a passing to the limit in which these variables collapse. This technique has been applied to parabolic scalar equations, see, e.g., [29], [9], [18], [19], [35], and also to certain systems of parabolic equations coupled through reaction terms, but not through transport terms, see [36]. Notice that when applying succesfully this technique, uniqueness is always obtained as a by-product of a comparison principle. However, systems coupled through transport terms does not exhibit, in general, a comparison property. Therefore, other means have to be applied in order to give criteria to ensure uniqueness of solutions.

The outline of the paper is as follows. In Section 2 we present the assumptions on the data and the notion of weak solution. Then we prove Theorem 2.1 of existence of weak solutions by means of a regularization technique which involves the consideration of a non-degenerate problem for which existence of solutions is proven by a fixed point argument, see Theorem 2.2. In Section 3 we study the uniqueness of solutions and present three results, see Theorems 3.1, 3.3 and 3.4. The first result covers the situation in which diffusion dominates both transport and reaction. The second result only needs φ non-decreasing, but an entropy type condition must hold for the electric

field. Finally, the third result uses the assumption $\nabla u, \nabla v \in L^1(Q_T)$. We show that this regularity is achieved, at least, for data smooth enough and in space dimension one.

2. Existence of solutions. In this section we prove existence of weak solutions of problem (1.1). The main result is Theorem 2.1 where we prove existence in the most interesting case: when the parabolic equations of (1.1) are of degenerate type. As we already mentioned, the transport terms $div(b(u)\nabla w)$ and $div(b(v)\nabla w)$ are the main difficulty in the proof due to the fact that natural a priori estimates of problem (1.1) are obtained in terms of $\varphi(u)$ (with $\varphi'(0) = 0$) and their spatial derivatives meanwhile transport terms contain $b(u)$ and $b(v)$ which, in general, are not bounded by the former.

This difficulty leads us to consider an auxiliar non-degenerate problem for which we obtain existence of weak solutions, see Theorem 2.2, and which allows us, by means of techniques of regularization and passing to the limit, to prove the result for the general formulation.

Before stating the first result we introduce a set of assumptions on the data as well as the definition of weak solution of (1.1).

ASSUMPTIONS ON THE DATA.

$\mathbf{H}_1$. $\Omega \subset I\!\!R^N$, $N \leq 3$, is an open, bounded and connected set. The boundary of Ω, $\partial\Omega$, is of class $\mathcal{C}^{1,1}$ and it splits in two disjoint components Γ_D, with positive measure, and Γ_N, open in $\partial\Omega$. We assume that for any function ψ satisfying

$$(2.1) \qquad \begin{cases} \Delta\psi \in L^q(\Omega), \\ \psi = 0 \text{ on } \Gamma_D, \\ \nabla\psi \cdot \nu = 0 \text{ on } \Gamma_N, \end{cases}$$

we have the regularity $\psi \in W^{2,q}(\Omega)$, for $q \in [1, \infty)$. Finally, we suppose $T > 0$ is arbitrarily chosen.

DEFINITION. *A function $f : I\!\!R^n \to I\!\!R$ is* sublinear *if there exists a positive constant c such that*

$$|f(s_1, ..., s_n)| \leq c\left(1 + \sum_{i=1}^{n} |s_i|\right), \quad \forall\, (s_1, ..., s_n) \in I\!\!R^n.$$

$\mathbf{H}_2$. We assume

$$\varphi \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^1((0, \infty)), \quad \varphi'(0) = 0, \quad \varphi \text{ non-decreasing,}$$

$$(2.2) \qquad\qquad F \in \mathcal{C}_{loc}^{0,1}([0, \infty)^2; I\!\!R),$$

$$b \in \mathcal{C}^1([0, \infty)) \text{ is sublinear and satisfies}$$

$$(2.3) \qquad\qquad |b'(s)| \leq c\,(1 + \varphi'(s)), \quad \text{for all } s \in [0, \infty),$$

and for some constant $c > 0$.

$\mathbf{H}_3$. The auxiliary data satisfy

$$u_0, v_0 \in L^\infty(\Omega), \quad u_0 \geq 0, \quad v_0 \geq 0 \quad \text{in} \quad \Omega,$$
$$\varphi(u_D), \varphi(v_D) \in L^\infty(0, T; H^1(\Omega)) \cap L^\infty(Q_T) \cap H^1(0, T; L^2(\Omega)),$$
$$w_D \in L^\infty(0, T; W^{2,\infty}(\Omega)).$$

Although physically $C = C(x)$, we shall consider the more general case $C = C(x, t)$ under the assumption $C \in L^\infty(Q_T)$.

H$_4$. If both F and b are nonlinear then we assume $\varphi^{-1} \in \mathcal{C}^{0,\alpha}([0, \infty))$, for some $\alpha \in (0, 1)$.

We remark that the property assumed for problem (2.1) actually represents a condition on the contact angles of the boundary segments Γ_D and Γ_N, see, e.g. [32]. In particular, if both components of the boundary are open and closed, so they do not meet, then the assumption is a well known result, see, e.g. [41]. As stated in (2.2), in this article we shall consider a Lipschitz continuous recombination-generation term F. The case of a monotone F was already treated in [24] obtaining similar results on the existence of weak solutions under somehow stronger conditions on φ and b. As shown in [12], a monotone non-Lipschitz continuous recombination-generation term may imply the formation of *dead cores* (sets where the components u, v of the solution vanish even when the initial data are strictly positive) and play an important role in applications through the phenomenon known as *vacuum solutions*, see [12].

We consider a notion of weak solution similar to that introduced in [1]:

DEFINITION OF WEAK SOLUTION. *Set*

$$\mathcal{V} := \left\{ z \in H^1(\Omega) : z = 0 \quad \text{on} \quad \Gamma_D \right\},$$

and assume **H$_1$-H$_3$**. *Then* (u, v, w) *is a* weak solution *of (1.1) if the following properties hold:*

(i) $u, v \in L^\infty(Q_T)$, $\varphi(u) \in \varphi(u_D) + L^2(0, T; \mathcal{V})$, $\varphi(v) \in \varphi(v_D) + L^2(0, T; \mathcal{V})$ and $w \in w_D + L^2(0, T; \mathcal{V}) \cap L^\infty(Q_T)$.

(ii)

$$(2.4) \quad \begin{aligned} &\int_0^T \langle u_t, \zeta \rangle + \int_0^T \int_\Omega (\nabla\varphi(u) - b(u)\nabla w) \cdot \nabla\zeta = \int_0^T \int_\Omega F(u, v)\zeta, \\ &\int_0^T \langle v_t, \zeta \rangle + \int_0^T \int_\Omega (\nabla\varphi(v) - b(v)\nabla w) \cdot \nabla\zeta = \int_0^T \int_\Omega F(u, v)\zeta, \\ &\int_0^T \int_\Omega \nabla w \cdot \nabla\zeta = \int_0^T \int_\Omega (v - u - C)\zeta \end{aligned}$$

for any test function $\zeta \in L^2(0, T; \mathcal{V})$ *(notice that due to (2.2)* $F(u, v) \in L^2(Q_T)$*).*

(iii) $u_t, v_t \in L^2(0, T; \mathcal{V}')$ *and the initial data are verified in the following sense:*

$$(2.5) \quad \begin{aligned} &\int_0^T \langle u_t, \zeta \rangle + \int_0^T \int_\Omega (u - u_0) \zeta_t = 0, \\ &\int_0^T \langle v_t, \zeta \rangle + \int_0^T \int_\Omega (v - v_0) \zeta_t = 0, \end{aligned}$$

for any test function $\zeta \in L^2(0, T; \mathcal{V}) \cap W^{1,1}(0, T; L^2(\Omega))$ *with* $\zeta(T) = 0$.

Next we state the main result of this section:

THEOREM 2.1. *Assume* **H$_1$-H$_4$** *and suppose that the auxiliary data satisfy*

$$(2.6) \quad \begin{aligned} &k \geq u_0, v_0 \geq m \geq 0 \quad \text{in } \Omega \quad \text{and} \\ &\varphi(ke^{\lambda_0 t}) \geq \varphi(u_D), \varphi(v_D) \geq \varphi(me^{-\lambda_1 t}) \geq 0 \quad \text{in } \Sigma_D, \end{aligned}$$

for some non-negative constants $k, m, \lambda_0, \lambda_1$. *Then there exists* $\lambda \geq 0$, *independent of* φ, *such that problem (1.1) has a weak solution verifying*

$$(2.7) \quad ke^{\lambda t} \geq u(\cdot, t), v(\cdot, t) \geq me^{-\lambda t} \geq 0 \quad \text{in } \Omega \text{ for a.e. } t \in (0, T),$$

$$(2.8) \quad \begin{aligned} &u, v \in \mathcal{C}([0,T]\,;\mathcal{V}'), \\ &w \in L^\infty(0,T;W^{2,s}(\Omega)) \quad \textit{for all } s \in [1,\infty). \end{aligned}$$

Moreover, if $\varphi \in \mathcal{C}^1([0,\infty))$ then

$$\sqrt{\varphi'(u)}\nabla u, \sqrt{\varphi'(v)}\nabla v \in L^2(Q_T).$$

The proof is based on the following previous result for the non-degenerate problem:

THEOREM 2.2. *Assume $\mathbf{H}_1$-$\mathbf{H}_3$ and let φ be sublinear and strictly increasing. Suppose φ^{-1} Lipschitz continuous, F sublinear and auxiliary data satisfaying (2.6) for some non-negative constants $k, m, \lambda_0, \lambda_1$. Then there exists $\lambda \geq 0$, independent of φ, such that problem (1.1) has a weak solution verifying (2.7), (2.8) and*

$$u, v \in \mathcal{C}([0,T]\,;L^1(\Omega)).$$

Proof of Theorem 2.2 is based on a fixed point technique. To define the fixed point operator in L^p spaces we need, due to the lack of regularity of the term $\nabla b(u) \cdot \nabla w$, to uncouple problem (1.1) and to consider two auxiliary problems, see (2.13) and (2.14). First we apply a fixed point argument to obtain the existence of solutions, (u, v), of (2.13) and we show that this solution satisfies (2.7). Then, we solve problem (2.14) and use again a fixed point argument to couple the whole system, obtaining in this way a weak solution of (1.1) with the property (2.7). The additional regularity is obtained by applying general results on L^p spaces, see [39].

Proof of Theorem 2.2

We start introducing some notation and an elementary consequence of Sobolev's theorem: we shall write $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(Q_T)}$, $\|\cdot\|_{L^p(L^q)} := \|\cdot\|_{L^p(0,T;L^q(\Omega))}$ and

$$\||f\|| := \|f\|_{L^\infty(L^2)} + \|f\|_{L^2(0,T;\mathcal{V})}.$$

Let 2^* be the critical Sobolev exponent given by ∞, $s \in [1,\infty)$ or 6 if $N = 1, 2, 3$ respectively. Then we have that for all $f \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathcal{V})$ there exists a positive constant $c \equiv c(\Omega)$ such that

$$(2.9) \quad \|f\|_{L^r} \leq c \||f\||, \quad \text{with} \quad 1 \leq r < 4(1 - \frac{1}{2^*}).$$

Finally, c shall denote a positive constant independent of the relevant quantities of the problem, like $\lim_{s\to 0}\varphi'(s)$ or $\inf_\Omega u_0$, but which shall vary along the proof.

Step **1**. Let $T > 0$, $0 < \rho < c_\rho$, with c_ρ a positive constant to be fixed, and fix r such that

$$(2.10) \quad 3 < r < 4(1 - \frac{1}{2^*}).$$

Let p be an exponent satisfying the following restriction:

$$(2.11) \quad \frac{r}{r-2} < 3 < p < r,$$

with r given by (2.10). Consider the set $K \subset L^p(0,T;W^{2,p}(\Omega)) \cap L^2(0,T;\mathcal{V})$ given by

$$(2.12) \quad h \in K \Leftrightarrow \Delta h \in L^\infty(Q_T) \quad \text{and} \quad \|\Delta h\|_{L^p} + \|\nabla h\|_{L^2} \leq \rho.$$

Clearly, K is convex. Moreover, since due to the choice of p we have $2 \leq \frac{Np}{N-p}$, it follows that $\|\Delta w\|_{L^p} + \|\nabla w\|_{L^2}$ is a norm in $L^p(0,T;W^{2,p}(\Omega)) \cap L^2(0,T;\mathcal{V})$ and therefore K is weakly compact in this space. These properties of K will be used later to apply a fixed point argument. Given $h \in K$ we introduce problems

$$(2.13) \quad \begin{cases} u_t - div\,(\nabla\varphi(u) - b(u)\nabla h) = F(u,v) & \text{in } Q_T, \\ v_t - div\,(\nabla\varphi(v) + b(v)\nabla h) = F(u,v) & \text{in } Q_T, \\ \nabla\varphi(u)\cdot\nu = 0, \quad \nabla\varphi(v)\cdot\nu = 0, & \text{on } \Sigma_N, \\ \varphi(u) = \varphi(u_D), \quad \varphi(v) = \varphi(v_D), & \text{on } \Sigma_D, \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, & \text{in } \Omega \end{cases}$$

and

$$(2.14) \quad \begin{cases} -\Delta w = v - u - C & \text{in } Q_T, \\ w = w_D & \text{on } \Sigma_D, \\ \nabla w\cdot\nu = 0 & \text{on } \Sigma_N, \end{cases}$$

with similar notions of weak solutions as for problem (1.1).

Step **2.** *Definition of the fixed point operator for* (2.13). Consider problems

$$(2.15) \quad \begin{cases} u_t - \Delta\varphi(u) = f & \text{in } Q_T, \\ \varphi(u) = \varphi(u_D) & \text{on } \Gamma_D, \\ \nabla\varphi(u)\cdot\nu = 0 & \text{on } \Gamma_{N,} \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

and

$$(2.16) \quad \begin{cases} v_t - \Delta\varphi(v) = g & \text{in } Q_T, \\ \varphi(v) = \varphi(v_D) & \text{on } \Gamma_D, \\ \nabla\varphi(v)\cdot\nu = 0 & \text{on } \Gamma_N, \\ v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases}$$

with $f, g \in L^2(Q_T)$. Since these problems are uniformly parabolic we deduce, see e.g. [30], the existence of a unique weak solution of (2.15) and (2.16) with the regularity $u, v \in L^r(Q_T) \cap \mathcal{C}([0,T];L^1(\Omega))$, $\varphi(u) \in \varphi(u_D) + L^2(0,T;\mathcal{V})$, $\varphi(v) \in \varphi(v_D) + L^2(0,T;\mathcal{V})$, with r given by (2.10).

We introduce the set

$$K^* := \left\{ (f,g) \in L^2(Q_{T^*}) \times L^2(Q_{T^*}) : \|f\|_{L^2}, \|g\|_{L^2} < R \right\}, \quad 0 < T^* \leq T,$$

which is convex and weakly compact in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$, and the mapping $Q : K^* \to L^2(Q_{T^*}) \times L^2(Q_{T^*})$ given by

$$Q(f,g) := \left(F(u,v) - div(b(u)\nabla h), F(u,v) + div(b(v)\nabla h) \right),$$

with u, v solutions of (2.15), (2.16). It can be shown that, as a consequence of (2.3) and the sublinearity of F and b, the operator Q is well defined. Notice also that a fixed point of Q is a weak local solution of (2.13). To prove the existence of such a point we search for R and T^* such that
(i) $Q(K^*) \subset K^*$, and
(ii) Q is weakly-weakly sequentially continuous in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$,

which will allow us to apply the fixed point theorem of [4] to deduce the result. Since problems (2.15) and (2.16) share the same structure we shall only work out the properties satisfied by solutions of (2.15).

Step **3.** *A priori estimates for problems* (2.15) *and* (2.16). *Proof of* $Q(K^*) \subset K^*$. This last condition reads as

$$(2.17) \qquad I_u := \|F(u,v) - \nabla b(u) \cdot \nabla h - b(u) \Delta h\|_{L^2} \le R.$$

Taking $\zeta = \varphi(u) - \varphi(u_D)$ as a test function for problem (2.15) we get

$$\int_\Omega \Phi(u(t)) + \int_{Q_T} |\nabla \varphi(u)|^2 = \int_\Omega \Phi(u_0) - \int_{Q_T} (u - u_0)\, \varphi(u_D)_t + \int_\Omega (u(t) - u_0)\, \varphi(u_D)$$

$$(2.18) \qquad\qquad + \int_{Q_T} \nabla \varphi(u) \cdot \nabla \varphi(u_D) + \int_{Q_T} f\, (\varphi(u) - \varphi(u_D)),$$

with $\Phi(s) := \int_0^s \varphi(\sigma)d\sigma$. Using φ^{-1} Lipschitz continuous and standard inequalities we get from (2.18)

$$(2.19)\ c_\varphi \left(\|u\|^2_{L^\infty(L^2)} + \|\nabla u\|^2_{L^2} \right) \le \|\Phi(u)\|_{L^\infty(L^1)} + \|\nabla \varphi(u)\|^2_{L^2} \le \Lambda + \|f\|^2_{L^2},$$

with c_φ, Λ constants depending on the Lipschitz continuity constant of φ^{-1} and on the auxiliary data, respectively. Using (2.9) and $f \in K^*$ we deduce

$$(2.20) \qquad \|u\|_{L^r} \le c\, \|\|u\|\| \le c\, (\Lambda + R).$$

Since $r > 2$ we have

$$(2.21) \qquad \|u\|_{L^2} \le A_0(T^*)\, \|u\|_{L^r} \le c A_0(T^*)\, (\Lambda + R),$$

with $A_0(T^*) := |Q_{T^*}|^{\frac{r-2}{2r}}$. Finally, since φ is sublinear we deduce the existence of a continuous non-decreasing function $\eta : (0, \infty) \to (0, \infty)$ such that

$$(2.22) \qquad \|\nabla \varphi(u)\|^2_{L^2} \le \Lambda + \eta(T^*),$$

with $\eta(T^*) \to 0$ as $T^* \to 0$, see [16]. We are now ready to estimate the terms in (2.17): F sublinear and (2.21) imply

$$(2.23) \qquad \|F(u,v)\|_{L^2} \le c A_1(T^*),$$

with $A_1(T^*) := A_0(T^*)(\Lambda + 2R) + |Q_{T^*}|^{1/2}$. From (2.3), (2.22) and the regularity of h we get

$$(2.24) \quad \|\nabla b(u) \cdot \nabla h\|_{L^2} \le c\, \|\nabla \varphi(u)\|_{L^2} \|\nabla h\|_{L^\infty} \le c\, (\Lambda + \eta(T^*))\, \|\nabla h\|_{L^\infty},$$

and since b is sublinear

$$(2.25) \qquad \|b(u)\Delta h\|_{L^2} \le \|b(u)\|_{L^2} \|\Delta h\|_{L^\infty} \le c A_2(T^*)\, \|\Delta h\|_{L^\infty},$$

with $A_2(T^*) := |Q_{T^*}|^{\frac{1}{2}} + |Q_{T^*}|^{\frac{r-2}{2r}} (\Lambda + R)$. Gathering (2.23), (2.24) and (2.25) we obtain

$$I_u \le c A_1(T^*) + c\, (\Lambda + \eta(T^*))\, \|\nabla h\|_{L^\infty} + c A_2(T^*)\, \|\Delta h\|_{L^\infty},$$

and since we want $I_u \leq R$, it is sufficient to make

$$(2.26) \qquad cA_1(T^*) + c\left(\Lambda + \eta(T^*)\right)\|\nabla h\|_{L^\infty} + cA_2(T^*)\|\Delta h\|_{L^\infty} \leq R.$$

Since A_1, A_2, η are non-decreasing continuous functions in $\mathbb{R}_+$, we have that, fixing R such that

$$R > cA_1(T) + c\left(\Lambda(u_0, \varphi(u_D)) + \eta(T)\right)\|\nabla h\|_{L^\infty(Q_T)} + cA_2(T)\|\Delta h\|_{L^\infty(Q_T)},$$

inequality (2.26) is satisfied for all $T^* \in [0, T]$. An identical argument allows us to get $I_v \leq R$. Therefore, we have proven the existence of R and T^*, which can be taken as $T^* = T$, such that $Q(K^*) \subset K^*$.

Step 4. *Continuity of Q.* Consider any sequence $(f_j, g_j) \subset K^* \to (f, g)$ weakly in $L^2(Q_T) \times L^2(Q_T)$ and let us show

$$\begin{aligned} div(b(u_j)\nabla h) &\to div(b(u)\nabla h) && \text{weakly in } L^2(Q_T), \\ F(u_j, v_j) &\to F(u, v) && \text{weakly in } L^2(Q_T), \end{aligned}$$

with u_j, v_j, u, v solutions of (2.15), (2.16) corresponding to f_j, g_j, f, g, respectively. By (2.20) we have $\|u_j\|_{L^r} \leq c\,\|u_j\| \leq$ const., and from (2.15) we also deduce $\|u_{jt}\|_{L^2(0,T;\mathcal{V}')} \leq$ const. Passing to a subsequence, if necessary, we obtain

$$\begin{aligned} u_j &\to u && \text{weakly in } L^r(Q_T), \\ u_j &\to u && \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ \nabla u_j &\to \nabla u && \text{weakly in } L^2(Q_T), \\ u_{jt} &\to u_t && \text{weakly in } L^2(0, T; \mathcal{V}'). \end{aligned}$$

Since F is sublinear and u_j, v_j are bounded in $L^r(Q_T)$ it follows $\|F(u_j, v_j)\|_{L^r} \leq$ const. and then $F(u_j, v_j) \to \tilde{F}$ in $L^r(Q_T)$ for some $\tilde{F} \in L^r(Q_T)$. Continuity of F together with pointwise convergence of u_j, v_j imply that $F(u_j, v_j) \to F(u, v)$ a.e. in Q_{T^*}, and therefore $\tilde{F} \equiv F(u, v)$. Hence, $F(u_j, v_j) \to F(u, v)$ strongly in $L^r(Q_T)$. A similar argument shows that $b(u_j) \to b(u)$ strongly in $L^r(Q_T)$. Finally, since $r > 2$ and $\nabla h \in L^\infty(Q_T)$ we deduce $div(b(u_j)\nabla h) \to div(b(u)\nabla h)$ weakly in $L^2(Q_T)$. Hence, Q is weakly-weakly sequentially continuous. By the fixed point theorem [4] we deduce the existence of a weak solution (u, v) of (2.13) with the same regularity obtained for the solutions of (2.15) and (2.16) when $f, g \in L^2(Q_T)$ is assumed. Notice that the solution found is global in time.

Step 5. *Lower bound and L^∞ regularity of u, v.* We introduce in problem (2.13) the change of unknowns $U := ue^{-\beta t}$ and $V := ve^{-\beta t}$ with $\beta > 0$. Then (U, V) satisfies

$$\begin{cases} U_t + \beta U - e^{-\beta t} div\left(\nabla \varphi(e^{\beta t}U) - b(e^{\beta t}U)\nabla h\right) = \hat{F} & \text{in } Q_T, \\ V_t + \beta V - e^{-\beta t} div\left(\nabla \varphi(e^{\beta t}V) + b(e^{\beta t}V)\nabla h\right) = \hat{F} & \text{in } Q_T, \\ \nabla \varphi(e^{\beta t}U) \cdot \nu = 0, \quad \nabla \varphi(e^{\beta t}V) \cdot \nu = 0, & \text{on } \Sigma_N, \\ \varphi(e^{\beta t}U) = \varphi(u_D), \quad \varphi(e^{\beta t}V) = \varphi(v_D), & \text{on } \Sigma_D, \\ U(\cdot, 0) = u_0, \quad V(\cdot, 0) = v_0, & \text{in } \Omega, \end{cases}$$

with $\hat{F} := e^{-\beta t}F(e^{\beta t}U, e^{\beta t}V)$. To obtain the lower bound we compare U and V with $z(t) := me^{-(\lambda+\beta)t}$, for a suitable $\lambda > \lambda_1$. By assumption, $u_D \geq me^{-\lambda_1 t} \geq me^{-\lambda t}$ and

then we can take $Z_u := \min\{U - z, 0\}$ as test function obtaining

$$
\begin{aligned}
(2.27) \qquad \int_\Omega Z_u \left(U - z\right)_t & - \lambda \int_\Omega z Z_u + \beta \int_\Omega Z_u^2 + e^{-\beta t} \int_\Omega \nabla\varphi(e^{\beta t}U) \cdot \nabla Z_u \\
& = -e^{-\beta t} \int_\Omega Z_u \left[\nabla b(e^{\beta t}U) \cdot \nabla h + b(e^{\beta t}U)\Delta h\right] \\
& \quad + e^{-\beta t} \int_\Omega Z_u F(e^{\beta t}U, e^{\beta t}V).
\end{aligned}
$$

Since b is Lipschitz continuous (with constant M_b) by estimating

$$
\int_\Omega Z_u b'(e^{\beta t}U)\nabla Z_u \cdot \nabla h \leq M_b \int_\Omega |\nabla Z_u|^2 + M_b \|\nabla h\|_{L^\infty}^2 \int_\Omega Z_u^2
$$

and

$$
\begin{aligned}
\int_\Omega Z_u b(e^{\beta t}U)\Delta h & = \int_\Omega Z_u \left(b(e^{\beta t}U) - b(e^{\beta t}z) + b(e^{\beta t}z)\right)\Delta h \\
& \leq c e^{\beta t} M_b \|\Delta h\|_{L^\infty} \left(\int_\Omega Z_u^2 + \int_\Omega z |Z_u|\right)
\end{aligned}
$$

we obtain from (2.27)

$$
(2.28) \qquad \frac{d}{dt}\int_\Omega Z_u^2 + \lambda' \int_\Omega z |Z_u| + \beta' \int_\Omega Z_u^2 \leq e^{-\beta t} \int_\Omega Z_u F(e^{\beta t}U, e^{\beta t}V),
$$

with $\lambda' := \lambda - cM_b \|\Delta h\|_{L^\infty}$, $\beta' := \beta - M_b \|\nabla h\|_{L^\infty}^2 - cM_b \|\Delta h\|_{L^\infty}$, where we used $-\lambda Z_u = \lambda |Z_u|$. Since F is Lipschitz continuous we can use a similar argument to show that

$$
(2.29) \qquad Z_u F(e^{\beta t}U, e^{\beta t}V) \leq cZ_u \left(Z_u + Z_v + F(e^{\beta t}z, e^{\beta t}z)\right),
$$

with $Z_v := \min\{V - z, 0\}$. Adding to (2.28) the similar estimate for V we get in the right hand side of the resulting inequality the term

$$
(2.30) \qquad e^{-\beta t} \int_\Omega F(e^{\beta t}U, e^{\beta t}V)\left(Z_u + Z_v\right).
$$

Using (2.29) and the similar estimate for $Z_v F(e^{\beta t}U, e^{\beta t}V)$ (2.30) may be estimated as

$$
e^{-\beta t} \int_\Omega F(e^{\beta t}U, e^{\beta t}V)\left(Z_u + Z_v\right) \leq c \int_\Omega \left(Z_u^2 + Z_v^2 + z |Z_u| + z |Z_v|\right).
$$

Then, for $\beta \geq M_b \|\nabla h\|_{L^\infty}^2 + cM_b \|\Delta h\|_{L^\infty} + c$ and $\lambda \geq cM_b \|\Delta h\|_{L^\infty} + c$ we obtain

$$
\frac{d}{dt}\int_\Omega \left(Z_u^2 + Z_v^2\right) \leq 0,
$$

from where the result follows. Notice that neither β nor λ depend on φ. Finally, the estimate $u, v \in L^\infty(Q_T)$ is obtained using a similar technique and we omit therefore the proof, see [20] for details.

Step **6**. *End of proof of existence of local solutions of* (1.1). Let $\tilde{T} \in (0, T]$ to be fixed, $\tilde{K}$ defined by (2.12) with T replaced by $\tilde{T}$, $h \in \tilde{K}$ and u, v be solutions of

(2.15), (2.16) corresponding to h. Consider problem (2.14) in $Q_{\tilde{T}}$. Since $u, v, C \in L^\infty(Q_{\tilde{T}})$ (2.14) has a unique solution $w \in L^\infty(0, \tilde{T}, W^{2,s}(\Omega)) \cap L^2(0, T; \mathcal{V})$ with $\Delta w \in L^\infty(Q_{\tilde{T}})$. In particular, $w \in L^p(0, \tilde{T}, W^{2,p}(\Omega)) \cap L^2(0, T; \mathcal{V})$. We define $P : \tilde{K} \to L^p(0, \tilde{T}, W^{2,p}(\Omega)) \cap L^2(0, T; \mathcal{V})$ by $P(h) = w$, being w such solution. Notice that if w is a fixed point for P then (u, v, w) is a local solution of (1.1). To prove the existence of a fixed point we use the same technique than before, which consists on showing
(i) $P(\tilde{K}) \subset \tilde{K}$, i.e., $\Delta w \in L^\infty(Q_T)$, and $\|\Delta w\|_{L^p} + \|\nabla w\|_{L^2} < \rho$ and
(ii) P is weakly-weakly sequentially continuous in $L^p(0, \tilde{T}, W^{2,p}(\Omega)) \cap L^2(0, T; \mathcal{V})$.
From (2.14):

$$(2.31) \qquad \|\Delta w\|_{L^s} \le \|u\|_{L^s} + \|v\|_{L^s} + \|C\|_{L^s}, \quad \text{for all } s \in [1, \infty].$$

Multiplying the equation in (2.14) by $w - w_D$ and using Hölder and Poincaré's inequalities we obtain

$$(2.32) \qquad \|\nabla w\|_{L^2} \le c\left(\|u\|_{L^2} + \|v\|_{L^2} + \|C\|_{L^2} + \|w_D\|_{L^2} + \|\nabla w_D\|_{L^2}\right).$$

From (2.31), (2.32) and $p > 2$ we get

$$(2.33) \|\Delta w\|_{L^p} + \|\nabla w\|_{L^2} \le c\left(\|u\|_{L^p} + \|v\|_{L^p} + \|C\|_{L^p} + \|w_D\|_{L^2} + \|\nabla w_D\|_{L^2}\right).$$

By (2.11) we have $p < r$ and therefore

$$(2.34) \qquad \|u\|_{L^p} \le A(\tilde{T}) \|u\|_{L^r} \le cA(\tilde{T}) \|u\|,$$

with $A(\tilde{T}) := |Q_{\tilde{T}}|^{\frac{rp}{r-p}}$. Assume that the estimate

$$(2.35) \qquad \|u\| \le G(\rho, \tilde{T})$$

holds, with G continuous, bounded as a function of $\tilde{T}$ and increasing with respect to ρ in an interval $(0, c_\rho)$, with $c_\rho > 0$. We shall prove this estimate later on, see Step **7**. Then, from (2.34) $\|u\|_{L^p} \le cA(\tilde{T})G(\rho, \tilde{T})$. A similar estimate holds for v. Since $C \in L^\infty(Q_{\tilde{T}})$ and $w_D \in L^\infty(0, T; H^1(\Omega))$ we have $\|C\|_{L^p} + \|w_D\|_{L^2} + \|\nabla w_D\|_{L^2} = B_0(\tilde{T})$ for a non-decreasing continuous function B_0 satisfying $B_0(0) = 0$. From (2.33) we deduce

$$\|\Delta w\|_{L^p} + \|\nabla w\|_{L^2} \le A(\tilde{T})G(\rho, \tilde{T}) + B_0(\tilde{T}),$$

and since we want to make $\|\Delta w\|_{L^p} + \|\nabla w\|_{L^2} \le \rho$, it suffices to find $\tilde{T} > 0$ such that

$$A(\tilde{T})G(\rho, \tilde{T}) + B_0(\tilde{T}) = \rho.$$

Since G is bounded as a function of ρ and $A(\tilde{T}), B_0(\tilde{T}) \downarrow 0$ as $\tilde{T} \downarrow 0$ it is straightforward to see that such $\tilde{T}$ exists, so (i) is satisfied.

To prove the continuity we consider a sequence $h_n \in \tilde{K}$ such that $h_n \to h$ weakly in $L^p(0, \tilde{T}; W^{2,p}(\Omega)) \cap L^2(0, T; \mathcal{V})$ and we show that $w_n \to w$ weakly in $L^p(0, \tilde{T}; W^{2,p}(\Omega)) \cap L^2(0, T; \mathcal{V})$, with w_n, w solutions of (2.14) corresponding to h_n, h. Since $h_n \in \tilde{K}$, $\|\Delta h_n\|_{L^p} + \|\nabla w\|_{L^2} \le \rho < c_\rho$, and then from (2.35) we get $\|u_n\|, \|v_n\| \le G(c_\rho, \tilde{T}) \le \text{const.}$ Using (2.9) we also obtain $\|u_n\|_{L^r}, \|v_n\|_{L^r} \le \text{const.}$ Then $u_n \to u$ weakly in $L^r(Q_{\tilde{T}})$, and similarly for v_n. Since $p < r$ we find $\Delta w_n \to \Delta w$ weakly in $L^p(Q_{\tilde{T}})$. From $r > 2$ we also have $w_n \to w$ weakly in $L^2(0, T; \mathcal{V})$. We deduce from

the fixed point theorem [4] that P has a fixed point, (u, v, w), which is a weak local solution of (1.1) in $Q_{\tilde{T}}$ with the regularity inherited from problems (2.13) and (2.14). Moreover, since the estimates do not blow up when $\tilde{T}$ increases, see Step **7**, we may use a standard continuation argument to deduce that the solution is global in time.

Step **7**. *Estimating* $\|\|u\|\|^2 + \|\|v\|\|^2$ *of problem* (2.13). Taking $\varphi(u) - \varphi(u_D)$ as test function for (2.13) and reasoning as in (2.18) with $f := F(u, v) - div(b(u)\nabla h)$ we get

$$(2.36) \qquad \|\Phi(u)\|_{L^\infty(L^1)} + \|\nabla\varphi(u)\|_{L^2}^2 \leq \Lambda + \int_{Q_{\tilde{T}}} F(u, v)\left(\varphi(u) - \varphi(u_D)\right)$$
$$+ \int_{Q_{\tilde{T}}} b(u)\nabla h \cdot \nabla\left(\varphi(u) - \varphi(u_D)\right),$$

with Λ depending only on the auxiliary data. Since F is sublinear we again obtain (2.23). Defining $B(s) := b(s)\varphi(s) - \int_0^s b'(\sigma)\varphi(\sigma)d\sigma$ and using the sublinearity of φ and b and (2.3) we get $|B(s)| \leq c(1 + |s| + s^2)$. Then

$$\int_{Q_{\tilde{T}}} b(u)\nabla h \cdot \left(\nabla\varphi(u) - \nabla\varphi(u_D)\right) = -\int_{Q_{\tilde{T}}} \left(B(u) - B(u_D)\right)\Delta h$$
$$- \int_{Q_{\tilde{T}}} \left(b(u) - b(u_D)\right)\nabla h \cdot \nabla\varphi(u_D).$$

The first term is estimated as

$$\int_{Q_{\tilde{T}}} \left(B(u) - B(u_D)\right)\Delta h \leq \|B(u) - B(u_D)\|_{L^{p'}} \|\Delta h\|_{L^p}$$
$$\leq c\left(|Q_{\tilde{T}}|^{1/p'} + \|u\|_{L^{2p'}}^2 + \|u_D\|_{L^{2p'}}^2\right)\|\Delta h\|_{L^p}.$$

Since $h \in \tilde{K}$, and $2p' \leq r$ due to the choice of p, see (2.11), we deduce $\|u\|_{L^{2p'}} \leq c\|u\|_{L^r} \leq c\|\|u\|\|$ and therefore

$$(2.37) \qquad \int_{Q_{\tilde{T}}} \left(B(u) - B(u_D)\right)\Delta h \leq c\left(|Q_{\tilde{T}}|^{1/p'} + \|u_D\|_{L^{2p'}}^2 + \|\|u\|\|^2\right)\rho.$$

The second term is estimated as follows:

$$\int_{Q_{\tilde{T}}} \left(b(u) - b(u_D)\right)\nabla h \cdot \nabla\varphi(u_D) \leq \|b(u) - b(u_D)\|_{L^r(L^2)} \|\nabla h\|_{L^p(L^\infty)} \|\nabla\varphi(u_D)\|_{L^2}.$$

Since $p > N$ we have $\|\nabla h\|_{L^p(L^\infty)} \leq c\|\Delta h\|_{L^\infty}$. Using b sublinear, $r > 2$ and (2.10)-(2.11) we obtain

$$(2.38) \qquad \int_{Q_{\tilde{T}}} \left(b(u) - b(u_D)\right)\nabla h \cdot \nabla\varphi(u_D) \leq c\|\nabla\varphi(u_D)\|_{L^2}$$
$$\times \left(|Q_{\tilde{T}}|^{1/p'} + \|u_D\|_{L^r} + \|\|u\|\|\right)\rho.$$

We may obtain similar estimates from v−equation. Using φ^{-1} Lipschitz continuous as in (2.19) we get from (2.36)-(2.38)

$$\|\|u\|\|^2 + \|\|v\|\|^2 \leq \Lambda_1(\rho) + c_3\left(\|\|u\|\|^2 + \|\|v\|\|^2\right)\left(\rho + \rho^2\right),$$

with $\Lambda_1(\rho) := c_1 + c_2\rho$, and c_1, c_2, c_3 depending on the norms of the auxiliary conditions and on $\tilde{T}^\beta$, for some $\beta > 0$. Hence, defining $G(\rho, \tilde{T}) := \frac{\Lambda_1(\rho)}{\sqrt{1 - c_3(\rho + \rho^2)}}$ with $\rho \in (0, c_\rho)$ and $c_\rho := \min\left\{1, \frac{1}{2c}\right\}$ we finish. $\square$

Now we can afford the

Proof of Theorem 2.1. The proof uses a regularization technique and Theorem 2.2. In view of the constructive method that we shall use in one of the uniqueness results, we consider two different regularizations of problem (1.1) depending on whether φ is strictly increasing or only non-decreasing. In the first case we consider the following perturbation of the auxiliary data

$$(2.39) \qquad \begin{cases} \varphi(u_{D\varepsilon}) = \varphi(u_D) + \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_D, \\ \varphi(v_{D\varepsilon}) = \varphi(v_D) + \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_D, \\ u_{0\varepsilon} = u_0 + \varepsilon, \quad v_{0\varepsilon} = v_0 + \varepsilon & \text{in } \Omega, \end{cases}$$

for some $\lambda_1 > 0$, remaining the other auxiliary conditions the same, and we consider

$$(2.40) \qquad \varphi_\varepsilon(s) := \begin{cases} \varphi(\varepsilon e^{-\lambda T}) \exp\left\{\mu(s - \varepsilon e^{-\lambda T})\right\} & \text{si } s < \varepsilon e^{-\lambda T}, \\ \varphi(s) & \text{si } s \in [\varepsilon e^{-\lambda T}, k], \\ \varphi'(k)s + \varphi(k) - k\varphi'(k) & \text{si } s \geq k, \end{cases}$$

with k an L^∞ constant bound of the auxiliary data and $\mu := \frac{\varphi'(\varepsilon e^{-\lambda T})}{\varphi(\varepsilon e^{-\lambda T})}$. We have $\varphi_\varepsilon \in \mathcal{C}^1([0, \infty))$, $\varphi_\varepsilon(0) > 0$ and $\varphi_\varepsilon' > 0$. It is straightforward to check that the sequence of problems $(1.1)_\varepsilon$ corresponding to (2.39) and (2.40) satisfy the conditions of Theorem 2.2. Finally, notice that φ and φ_ε coincides in the range of $u_\varepsilon, v_\varepsilon$.

In the case in which φ is non-decreasing we consider, for each $\varepsilon > 0$, the regularization given by $\varphi_\varepsilon(s) := \varphi(s) + \varepsilon s$ and leave the auxiliary conditions unchanged. It is easy to see that the requirements of Theorem 2.2 are satisfied, obtaining therefore the existence of a sequence of solutions of $(1.1)_\varepsilon$ with the regularity and properties stated in that theorem.

A priori estimates. In both cases we proceed in a similar way: we use $\varphi_\varepsilon(u_\varepsilon) - \varphi(u_{D\varepsilon})$ as a test function for the first equation in (1.1) and as in step **7** of the proof of Theorem 2.2, we obtain

$$(2.41) \qquad \sup_{0 \leq t \leq T} \int_\Omega \Phi(u_\varepsilon(t)) + \int_{Q_T} |\nabla\varphi(u_\varepsilon)|^2 + \varepsilon \int_{Q_T} |\nabla u_\varepsilon|^2 \leq C,$$

with C independent of ε (because the L^∞ bounds of $u_\varepsilon, v_\varepsilon$ are independent of φ_ε). Using now $\xi \in L^2(0, T; \mathcal{V})$ as a test function we get

$$\left| \int_0^T \langle u_{\varepsilon t}, \xi \rangle_{\mathcal{V}', \mathcal{V}} \right| \leq \|\nabla\varphi_\varepsilon(u_\varepsilon)\|_{L^2} \|\nabla\xi\|_{L^2} + \|b(u_\varepsilon)\|_{L^\infty} \|\nabla w_\varepsilon\|_{L^2} \|\nabla\xi\|_{L^2}$$
$$+ \|F(u_\varepsilon, v_\varepsilon)\|_{L^2} \|\xi\|_{L^2},$$

from where we deduce

$$(2.42) \qquad \|u_{\varepsilon t}\|_{L^2(0, T; \mathcal{V}')} \leq c,$$

with c independent of ε. A similar estimate holds for v_ε. From the third equation of (1.1) we get

$$(2.43) \qquad \|\Delta w_\varepsilon\|_{L^\infty(Q_T)} \leq \|v_\varepsilon - u_\varepsilon + C\|_{L^\infty(Q_T)} \leq \text{const.}$$

Therefore, using estimates (2.41)-(2.43) and standard compactness results we can extract subsequences (labeled again by ε) such that

$$(2.44)\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(Q_T), \\ \varphi(u_\varepsilon) \to \xi & \text{weakly in } \varphi(u_D) + L^2(0,T;\mathcal{V}), \\ u_{\varepsilon t} \to u_t & \text{weakly in } L^2(0,T;\mathcal{V}'), \\ \varepsilon^{1/2} u_\varepsilon \to 0 & \text{weakly in } L^2(0,T;\mathcal{V}), \\ w_\varepsilon \rightharpoonup w & \text{weakly } * - \text{weakly in } L^\infty(0,T;W^{2,s}(\Omega)), \text{ for all } s < \infty. \end{cases}$$

From the compact imbedding $L^\infty(\Omega) \subset H^{-1}(\Omega)$ and Corollary 4 (p. 85) of [39] we obtain

$$u_\varepsilon \to u \quad \text{in} \quad \mathcal{C}([0,T],\mathcal{V}').$$

Since φ is continuous and non-decreasing we have that $-\Delta\varphi(\cdot)$ is a maximal monotone graph in $L^2(0,T;\mathcal{V}')$ and, therefore, it is strongly-weakly closed in such space (see, e.g. [8]), from where we deduce

$$(2.45) \qquad\qquad\qquad \xi = \varphi(u).$$

Assume, now, $\mathbf{H}_4$. In order to pass to the limit on $b(u_\varepsilon)$ and $F(u_\varepsilon, v_\varepsilon)$ we shall prove that $u_\varepsilon \to u$ strongly in $L^q(Q_T)$ for all $q < \infty$. To do this we use a modification of the arguments given in [17], [31] or [19]. Defining the space

$$\mathcal{H} = \{u \in L^{2/\alpha}(0,T;W^{\alpha,2/\alpha}(\Omega)), u_t \in L^2(0,T;\mathcal{V}')\}.$$

By (2.41) and (2.42) we have that u_ε is uniformly bounded in $\mathcal{H}$. Then, from the compact imbedding $\mathcal{H} \subset L^{2/\alpha}(Q_T)$ we deduce the existence of a subsequence of u_ε such that

$$u_\varepsilon \to u \quad \text{strongly in} \quad L^{2/\alpha}(Q_T) \quad \text{and} \quad \text{a.e. in} \quad Q_T.$$

This fact together with the weak $*$ convergence of u_ε to u in $L^\infty(Q_T)$ implies that $u_\varepsilon \to u$ strongly in $L^q(Q_T)$ for all $q < \infty$. And similarly for v.

Identification of the limit. Let $\zeta \in L^2(0,T;\mathcal{V})$ be a test function. By (2.44) and (2.45) it is clear that

$$\int_0^T \langle u_{\varepsilon t}, \zeta \rangle \to \int_0^T \langle u_t, \zeta \rangle \quad \text{and} \quad \int_0^T \int_\Omega \nabla\varphi(u_\varepsilon) \cdot \nabla\zeta \to \int_0^T \int_\Omega \nabla\varphi(u) \cdot \nabla\zeta.$$

From the pointwise convergence of $u_\varepsilon, v_\varepsilon$ to u,v in Q_T, we get $F(u_\varepsilon, v_\varepsilon) \to F(u,v)$ a.e. in Q_T, and since F is Lipschitz continuous we obtain

$$\|F(u_\varepsilon, v_\varepsilon)\|_{L^2} \le c\left(\|u_\varepsilon\|_{L^2} + \|v_\varepsilon\|_{L^2} + 1\right) \le \text{const.},$$

so there exists $\tilde{F} \in L^2(Q_T)$ such that $F(u_\varepsilon, v_\varepsilon) \to \tilde{F}$ weakly in $L^2(Q_T)$. Lebesgue's theorem implies $\tilde{F} \equiv F(u,v)$, and therefore

$$\int_0^T \int_\Omega F(u_\varepsilon, v_\varepsilon)\zeta \to \int_0^T \int_\Omega F(u,v)\zeta.$$

Similarly we obtain $b(u_\varepsilon) \to b(u)$ and $b(v_\varepsilon) \to b(v)$ strongly in $L^q(Q_T)$ for all $q < \infty$. Due to the compact imbedding $L^2(0,T;W^{2,2}(\Omega)) \subset L^2(0,T;\mathcal{V})$ we also have $w_\varepsilon \to w$ strongly in $L^2(0,T;\mathcal{V})$. We, finally, deduce

$$(2.46) \qquad\qquad \int_{Q_T} b(u_\varepsilon)\nabla w_\varepsilon \cdot \nabla\zeta \to \int_{Q_T} b(u)\nabla w \cdot \nabla\zeta.$$

So the limit, u, has been identified as the first component of a solution of (1.1). The other components are handled in a very similar way and we skip therefore the proof. In the case in which $\mathbf{H}_4$ does not hold, i.e., when both b and F are linear functions, the passing to the limit is easier because we do not need to ensure the pointwise convergence of $u_\varepsilon, v_\varepsilon$ to u, v. In this situation the identification of the limit is just a consequence of the weak convergences in (2.44) and (2.45). Finally, from [3], Theorem 2.2, we obtain the additional regularity

$$\sqrt{\varphi'(u)}\nabla u, \sqrt{\varphi'(v)}\nabla v \in L^2(Q_T).$$

To finish, notice that due to Theorem 2.2 we have that , for all $\varepsilon > 0$, (2.7) holds. We then deduce that this property also holds in the limit $\varepsilon \to 0$. $\square$

The following regularity result will be used in the uniqueness section, see Theorem 3.4.

COROLLARY 2.3. *In the conditions of Theorem 2.1, let $\varphi \in \mathcal{C}^1([0,\infty))$ be strictly increasing with $\varphi'(0) = 0$. Assume*

$$(2.47) \qquad \int_0^s \frac{b'(\sigma)^2}{\varphi'(\sigma)}d\sigma < \infty \quad and \quad \int_0^s \frac{b(\sigma)b'(\sigma)^2}{\varphi'(\sigma)}d\sigma < \infty \quad for\ all\ s \in [0,\infty).$$

Then there exists a solution of problem (1.1) with the regularity given in Theorem 2.1 and

$$b(u), b(v) \in L^2(0, T; \mathcal{V}).$$

Remark. Let $\varphi(s) := s^m$ and $b(s) := s^\gamma$ with $m > 1, \gamma \geq 1$. If $m < 2\gamma$ then (2.47) is satisfied.

Proof. We consider the sequence of solutions $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of problems $(1.1)_\varepsilon$ constructed as in proof of Theorem 2.1. Since $u_\varepsilon \geq c\varepsilon$ we have that $\psi(u_\varepsilon) - \psi(u_{D\varepsilon})$ is an admissible test function for any $\varepsilon > 0$. For simplicity, we suppose $u_D = 0$ in Σ_D and therefore $u_{D\varepsilon} = \varepsilon e^{-\lambda_1 t}$. We have for the diffusion term

$$\int_{Q_T} \nabla\varphi(u_\varepsilon) \cdot \nabla(\psi(u_\varepsilon) - \psi(u_{D\varepsilon})) = \int_{Q_T} \frac{\varphi'(u_\varepsilon)\psi'(u_\varepsilon)}{b'(u_\varepsilon)^2} |\nabla b(u_\varepsilon)|^2 = \int_{Q_T} |\nabla b(u_\varepsilon)|^2,$$

with $\psi(s) = \int_0^s b'(\sigma)^2/\varphi'(\sigma)d\sigma$. For the drift term we get

$$\int_{Q_T} b(u_\varepsilon)\nabla w_\varepsilon \cdot \nabla(\psi(u_\varepsilon) - \psi(u_{D\varepsilon})) = \int_{Q_T} \nabla w_\varepsilon \cdot \nabla B(u_\varepsilon) = \int_{Q_T} (v_\varepsilon - u_\varepsilon + C)B(u_\varepsilon)$$
$$+ \int_{\Sigma_D} B(u_{D\varepsilon})\nabla w_\varepsilon \cdot \nu,$$

with $B(s) := \int_0^s b(\sigma)\psi'(\sigma)d\sigma$. For the time derivative term we obtain

$$\int_{Q_T} u_{\varepsilon t}(\psi(u_\varepsilon) - \psi(u_{D\varepsilon})) = \varepsilon\lambda_1 \int_{Q_T} u_\varepsilon \psi'(u_{D\varepsilon}) + \int_\Omega (G(u_\varepsilon(T)) - G(u_{\varepsilon 0}))$$
$$- \int_\Omega (u_\varepsilon(T)\psi(u_{D\varepsilon}(T)) - u_{\varepsilon 0}\psi(u_{D\varepsilon}(0))),$$

with $G(s) := \int_0^s \psi(\sigma) d\sigma$. We have then

$$\int_{Q_T} |\nabla b(u_\varepsilon)|^2 \leq \int_\Omega G(u_{\varepsilon 0}) + u_\varepsilon(T) \psi(u_{D\varepsilon}(T)) + \int_{Q_T} (v_\varepsilon - u_\varepsilon + C) B(u_\varepsilon)$$

$$+ \int_{\Sigma_D} B(u_{D\varepsilon}) \nabla w_\varepsilon \cdot \nu + \int_{Q_T} F(u_\varepsilon, v_\varepsilon)(\psi(u_\varepsilon) - \psi(u_{D\varepsilon})).$$

Using the $L^\infty(Q_T)$ uniform bounds of $u_\varepsilon, v_\varepsilon$ and property (2.47) we deduce

$$\int_{Q_T} |\nabla b(u_\varepsilon)|^2 \leq c \quad \text{for all } \varepsilon > 0.$$

We find then by (2.44) that $b(u_\varepsilon) \to b(u)$ weakly in $L^2(0, T; \mathcal{V})$, with u the first component of a solution of (1.1). We may follow the same argument to deduce the property for v. $\square$

Remark. The technique we have used is also applicable when $F(u, v)$ is a maximal monotone graph, see [15] for a likely system but without transport terms. We also point out that functions $\varphi(u)$ and $\varphi(v)$ as well as $b(u)$ and $b(v)$ may be different as long as they fulfill the assumptions given on the data.

3. Uniqueness of solutions. We present in this section three theorems on the uniqueness question for problem (1.1) which share a *duality technique* in their proofs, i.e., the searching of suitable test functions which allow to deduce the uniqueness property.

The first result is obtained by using a technique introduced by Antontsev, Díaz and Domansky [2] for a system of two-phase filtration in porous medium. Here we assume $(b'(s))^2 \leq c\varphi'(s)$, which holds in the case when diffusion and transport are both linear or in the case in which they are degenerate in a suitable way. It is worth noting that this type of condition also arises as sufficient condition to ensure the existence of strong solutions of (1.2), see [6].

The second result uses a technique introduced by Rulla [37] to study the Stefan problem with prescribed convection. In this case we only assume φ non-decreasing, but an entropy type condition for the electric field on the Dirichlet boundary must be introduced: $\nabla w \cdot \nu = 0$ on Σ_D. Conditions of this type are already classical in the literature of hyperbolic equations, see [28], and they arise as natural conditions which allow to select a unique solution (the so-called entropy solution) when uniqueness fails for weak solutions.

Our last theorem applies to the case in which problem (1.1) has strong solutions in the following sense: $b(u), b(v) \in L^1(0, T; \mathcal{W})$, with

$$(3.1) \qquad \mathcal{W} := \left\{ h \in W^{1,p}(\Omega) : h = 0 \text{ on } \Gamma_D \right\},$$

and with $p > N$ if $N \geq 2$ and $p = 1$ if $N = 1$. To obtain this result we used a method due to Kalashnikov [26] which consists of making a comparaison between an arbitrary weak solution of (1.1) and the weak solution constructed as the limit of a sequence of solutions of regularized problems, see proof of Theorem 2.1. Our result is strongly based on the technique introduced by Díaz and Kersner [13] to study a one dimensional scalar equation.

In the sequel we shall assume that the component w of solutions is non-trivial in the sense that $\|\nabla w\|_{L^2(Q_T)} \neq 0$. On the contrary, the system reduces to the equation

$u_t - \Delta\varphi(u) = F(u, u-C)$, in fact simpler than (1.2) whichis , as we already mentioned, well understood.

THEOREM 3.1. *Assume* $\mathbf{H_1}$-$\mathbf{H_3}$ *and suppose that there exists a constant* $M > 0$ *such that*

$$(3.2) \qquad\qquad (b'(s))^2 \le M\varphi'(s) \quad \text{for all } s > 0$$

and

$$(3.3) \qquad\qquad \left(\frac{\partial}{\partial s_i} F(s_1, s_2)\right)^2 \le M\varphi'(s_i), \quad \text{for all } s_i > 0, \quad i = 1, 2.$$

Then problem (1.1) *has a unique weak solution.*

Proof. Suppose that (u_1, v_1, w_1) and (u_2, v_2, w_2) are two weak solutions of (1.1) and define $(u, v, w) := (u_1 - u_2, v_1 - v_2, w_1 - w_2)$, $F_i := F(u_i, v_i)$, $i = 1, 2$ and $\hat{F} := F_1 - F_2$. Then (u, v, w) satisfies

$$(3.4) \quad \begin{cases} u_t - \Delta\left(\varphi(u_1) - \varphi(u_2)\right) + div\left(b(u_1)\nabla w + (b(u_1) - b(u_2))\nabla w_2\right) = \hat{F}, \\ v_t - \Delta\left(\varphi(v_1) - \varphi(v_2)\right) - div\left(b(v_1)\nabla w + (b(v_1) - b(v_2))\nabla w_2\right) = \hat{F}, \\ -\Delta w + u - v = 0, \end{cases}$$

in Q_T, with auxiliary conditions

$$(3.5) \quad \begin{cases} \nabla\varphi(u_i) \cdot \nu = 0, & \nabla\varphi(v_i) \cdot \nu = 0, & \nabla w \cdot \nu = 0 & \text{on } \Sigma_N, \\ \varphi(u_i) = \varphi(u_D), & \varphi(v_i) = \varphi(v_D), & w = 0 & \text{on } \Sigma_D, \\ u(\cdot, 0) = 0, & v(\cdot, 0) = 0 & & \text{in } \Omega, \end{cases}$$

$i = 1, 2$. Taking smooth test functions ψ, ξ, η with homogeneous mixed boundary conditions for problem (3.4)-(3.5), integrating by parts and adding the resulting integral identities we obtain

$$\int_\Omega \psi(T)u(T) + \xi(T)v(T) = \int_{Q_T} u\left(\psi_t + A_u\Delta\psi + \mathbf{B}_u \cdot \nabla\psi + \eta + F_u(\psi + \xi)\right)$$

$$+ \int_{Q_T} v\left(\xi_t + A_v\Delta\xi - \mathbf{B}_v\nabla\xi - \eta + F_v(\psi + \xi)\right)$$

$$(3.6) \qquad\qquad - \int_{Q_T} w\, div\left(b(u_1)\nabla\psi - b(v_1)\nabla\xi + \nabla\eta\right),$$

with $A_z := \int_0^1 \varphi'(sz_1 + (1-s)z_2)ds$, $\mathbf{B}_z := \nabla w_2 \int_0^1 b'(sz_1 + (1-s)z_2)ds$ for $z = u, v$ and $F_u := \int_0^1 \frac{\partial F}{\partial u}(su_1 + (1-s)u_2, v_1)ds$ with a similar definition for F_v. Notice that since $b \in \mathcal{C}^1([0, \infty))$, $u_i, v_i \in L^\infty(Q_T)$, F is Lipschitz continuous and $\nabla w_2 \in L^\infty(Q_T)$ we have that $\mathbf{B}_z, F_z$ are bounded in $L^\infty(Q_T)$, for $z = u, v$. We define the differential operators

$$\begin{aligned} \mathcal{L}_1(\psi, \xi, \eta) &:= \psi_t + A_u^\varepsilon\Delta\psi + \mathbf{B}_u \cdot \nabla\psi + \eta + F_u(\psi + \xi), \\ \mathcal{L}_2(\psi, \xi, \eta) &:= \xi_t + A_v^\varepsilon\Delta\xi - \mathbf{B}_v \cdot \nabla\xi - \eta + F_v(\psi + \xi), \\ \mathcal{L}_3(\psi, \xi, \eta) &:= div\left(b(u_1)\nabla\psi - b(v_1)\nabla\xi + \nabla\eta\right), \end{aligned}$$

with $A_u^\varepsilon := A_u + \varepsilon$ and $\varepsilon > 0$, (and a similar definition for A_v^ε) and set the following

problem to choose the test functions:

$$(3.7) \quad \begin{cases} \mathcal{L}_1(\psi, \xi, \eta) = u & \text{in } Q_T, \\ \mathcal{L}_2(\psi, \xi, \eta) = v & \text{in } Q_T, \\ \mathcal{L}_3(\psi, \xi, \eta) = 0 & \text{in } Q_T, \\ \nabla\psi \cdot \nu = \nabla\xi \cdot \nu = \nabla\eta \cdot \nu = 0 & \text{on } \Sigma_N, \\ \psi = \xi = \eta = 0 & \text{on } \Sigma_D, \\ \psi(T) = \xi(T) = 0 & \text{in } \Omega. \end{cases}$$

LEMMA 3.2. *Problem* (3.7) *has a unique solution with the regularity of test functions of* (1.1), *see* (2.4) *and* (2.5). *Moreover,*

$$\psi, \xi, \eta \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

and there exists a positive constant $C(T)$ *independent of* ε *such that*

$$(3.8) \quad \varepsilon \int_{Q_T} \left(|\Delta\psi|^2 + |\Delta\xi|^2 \right) \le C(T).$$

Continuation of proof of Theorem 3.1. Introducing in (3.6) the test functions provided by Lemma 3.2 we get

$$\int_{Q_T} \left(u^2 + v^2 \right) = \varepsilon \int_{Q_T} (u\Delta\psi + v\Delta\xi).$$

Young's inequality and (3.8) leads to

$$\int_{Q_T} \left(u^2 + v^2 \right) \le \sqrt{\varepsilon} \left(\int_{Q_T} \left(u^2 + v^2 \right) + C(T) \right).$$

Hence, taking the limit $\varepsilon \to 0$, we conclude that $u \equiv v \equiv 0$ a.e. in Q_T, which also implies $w \equiv 0$ a.e. in Q_T. □

Proof of Lemma 3.2.

Step **1**. *A prori estimates.* Multiplying the third equation of (3.7) by η and using the regularity $u_i, v_i \in L^\infty(Q_T)$ and the continuity of b we get

$$(3.9) \quad \int_\Omega |\nabla\eta|^2 \le c_0(T) \int_\Omega \left(|\nabla\psi|^2 + |\nabla\xi|^2 \right)$$

with $c_0(T) \ge 0$. Thanks to (3.2) we can estimate

$$(3.10) \quad \int_\Omega (\mathbf{B}_u \cdot \nabla\psi) \Delta\psi \le \delta \int_\Omega A_u^\varepsilon |\Delta\psi|^2 + \frac{M}{\delta} \int_\Omega |\nabla\psi|^2,$$

for $\delta > 0$. A similar estimate holds, due to (3.3), for $\int_\Omega F_u (\psi + \xi) \Delta\psi$. Multiplying the first equation of (3.7) by $\Delta\psi$ and using (3.10), the analogous expression for the F term we obtain, for a suitable δ

$$(3.11) \quad -\frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla\psi|^2 + \int_\Omega A_u^\varepsilon |\Delta\psi|^2 \le c \left(\int_\Omega \left(|\nabla\psi|^2 + |\nabla\xi|^2 \right) \right.$$

$$(3.12) \quad \left. + \int_\Omega |\nabla\eta|^2 + \int_{Q_T} u^2 \right).$$

From the second equation of (3.7) we obtain a similar inequality for ξ which, being added to (3.11) and taking into account that $A_u^\varepsilon, A_v^\varepsilon > \varepsilon$ and (3.9) allows us to deduce

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\nabla\psi|^2 + |\nabla\xi|^2\right) + \frac{\varepsilon}{2}\int_\Omega \left(|\Delta\psi|^2 + |\Delta\xi|^2\right) \leq c\left(\int_\Omega \left(|\nabla\psi|^2 + |\nabla\xi|^2\right)\right.$$

$$\text{(3.13)} \hspace{6cm} \left. + \int_\Omega \left(u^2 + v^2\right)\right),$$

with $c(T)$ independent of ε. On one hand, we deduce from Gronwall's Lemma that

$$\text{(3.14)} \hspace{3cm} \int_\Omega \left(|\nabla\psi(t)|^2 + |\nabla\xi(t)|^2\right) \leq c_1(T)e^{Tc(T)},$$

with $c_1(T)$ independent of ε. Note that both c and c_1 are increasing and uniformly continuous functions of T. They just depend on norms of the data. On the other hand, integrating (3.13) in $(0,T)$ and using (3.14) we obtain

$$\text{(3.15)} \hspace{3cm} \frac{\varepsilon}{2}\int_{Q_T} \left(|\Delta\psi|^2 + |\Delta\xi|^2\right) \leq c_2(T)e^{Tc(T)},$$

with $c_2(T)$ independent of ε. So we deduced (3.8). Finally, from the third equation of (3.7) we have that

$$\Delta\eta = \nabla b(v_1)\cdot\nabla\xi + b(v_1)\Delta\xi - \nabla b(u_1)\cdot\nabla\psi - b(u_1)\Delta\psi,$$

and from (3.2) and the regularity $\sqrt{\varphi'(v_1)}\nabla v_1 \in L^2(Q_T)$, see Theorem 2.1, we obtain

$$\text{(3.16)} \hspace{2cm} \int_\Omega |\nabla b(v_1)|^2 = \int_\Omega b'(v_1)^2 |\nabla v_1|^2 \leq c\int_\Omega \varphi'(v_1) |\nabla v_1|^2 \leq \text{const.}$$

Hence, using Hölder and Young's inequalities and estimates (3.9) and (3.16) we obtain the $L^2(0,T;H^2(\Omega))$ regularity of η.

Step **2**. *Existence of solutions of (3.7).* We use fixed point argument. Consider the set

$$K := \left\{h \in L^2(0,T^*;\mathcal{V}) : \|h\|_{L^2(0,T^*;\mathcal{V})} \leq R\right\},$$

where T^* and R will be suitably chosen. K is convex and weakly compact in $L^2(0,T^*;\mathcal{V})$. We define $Q : K \subset L^2(0,T^*;\mathcal{V}) \to L^2(0,T^*;\mathcal{V})$ by $Q(\hat\eta) := \eta$, where η is the unique solution of $\mathcal{L}_3(\hat\psi,\hat\xi,\eta) = 0$, being $(\hat\psi,\hat\xi)$ the unique solution of

$$\text{(3.17)} \hspace{3cm} \begin{cases} \mathcal{L}_1(\hat\psi,\hat\xi,\hat\eta) = u, \\ \mathcal{L}_2(\hat\psi,\hat\xi,\hat\eta) = v, \end{cases}$$

with the same auxiliary conditions as in (3.7). Since $u, v, \nabla\hat\eta \in L^2(Q_{T^*})$ we can justify the a priori estimates in Step **1** and deduce that any solution of (3.17) satisfies

$$\text{(3.18)} \hspace{2cm} \hat\psi, \hat\xi \in H^1(0,T^*;L^2(\Omega)) \cap L^\infty(0,T^*;H^1(\Omega)) \cap L^2(0,T^*;H^2(\Omega)).$$

Uniqueness of solutions of (3.17) follows from (3.18) and linearity of the differential operators. Existence of solutions of (3.17) is proven by uncoupling the problem and applying again a fixed point technique. Assume for the moment that such a solution

exists and, therefore, it is unique and satisfies (3.18). We easily deduce that the solution of $\mathcal{L}_3(\hat{\psi}, \hat{\xi}, \eta) = 0$ satisfies $\eta \in L^2(0, T^*; H^2(\Omega))$. Notice that if $\hat{\eta}$ is a fixed point of Q then $(\hat{\psi}, \hat{\xi}, \hat{\eta})$ is a local solution of (3.7). To prove the existence of such a fixed point we shall show

(i) $Q(K) \subset K$, for suitable $R, T^* > 0$,

(ii) Q is weakly-weakly sequentially continuous in $L^2(0, T^*; \mathcal{V})$,

and apply the fixed point theorem [4]. (i) follows from the previous a priori estimates. From (3.9) we find

$$\|\eta\|_{L^2(0,T^*;\mathcal{V})} \le c_0(T^*) \left(\left\|\hat{\psi}\right\|_{L^2(0,T^*;\mathcal{V})} + \left\|\hat{\xi}\right\|_{L^2(0,T^*;\mathcal{V})} \right),$$

and from (3.11) and the corresponding estimate for ξ we obtain

$$\left\|\hat{\psi}\right\|_{L^2(0,T^*;\mathcal{V})} + \left\|\hat{\xi}\right\|_{L^2(0,T^*;\mathcal{V})} \le c_1(T^*) \|\hat{\eta}\|_{L^2(0,T^*;\mathcal{V})} e^{cT^*}.$$

It follows that

$$\|Q(\hat{\eta})\|_{L^2(0,T^*;\mathcal{V})} \le c_3(T^*) e^{cT^*} R.$$

Notice that, as we already mentioned, functions $c_i(T^*)$ are uniformly continuous and non.decreasing with $c_i(0) = 0$ (they depend on the norms of the data in Q_T) and therefore we can take T^* small enough to obtain $c_3(T^*) e^{cT^*} \le 1$, deducing $Q(K) \subset K$. In fact, these properties of c_i imply the global existence (once the local existence is proven). (ii) is a direct consequence of the linearity and regularity and we omit the proof. This finishes the proof of the existence of a fixed point and, therefore, of a local solution of (3.7). We already mentioned why, in fact, the solution is global. Finally, the uniqueness of solutions is again a consequence of the linearity of the problem and the regularity of the solution. To finish, notice that the proof of existence of solutions of (3.17) may be performed in a similar way. □

Following, we state the second result on uniqueness of solutions of (1.1). The main feature of this theorem is that it allows to consider a nonlinear diffusion, φ, not necessarily strictly increasing. However, we need to assume that an entropy type condition on the electric field holds on the Dirichlet boundary.

THEOREM 3.3. *Assume* $\mathbf{H_1}$-$\mathbf{H_3}$ *and let* $b(s) = s$. *If*

(3.19) $$\nabla w \cdot \nu = 0 \quad on \quad \Gamma_D \times (0, T),$$

and

(3.20) $$|F(s_1, \sigma_1) - F(s_2, \sigma_2)| \le c\left[(\varphi(s_1) - \varphi(s_2)) + (\varphi(\sigma_1) - \varphi(\sigma_2))\right],$$

for all $s_i, \sigma_i \ge 0$, $i = 1, 2$, *then problem* (1.1) *has a unique solution in the class of weak solutions such that*

$$w \in L^\infty(0, T; W^{2,\infty}(\Omega)).$$

Remark. The equality in (3.19) is a consequence of the different sign that transport terms have in u and v-equations. Indeed, suppose that there exist two solutions

$(u_1, \tilde{v}, w_1)$ and $(u_2, \tilde{v}, w_2)$. Then, under the conditions of Theorem 3.3, with the equality sign in (3.19) replaced by $\geq$, uniqueness follows.

Proof of Theorem 3.3. As in proof of Theorem 3.1 we obtain the following identity

$$
\int_{Q_T} u_t \psi + v_t \xi = \int_{Q_T} (\varphi(u_1) - \varphi(u_2)) \Delta\psi + u\nabla w_2 \cdot \nabla\psi + u\eta
$$

$$
+ \int_{Q_T} (\varphi(v_1) - \varphi(v_2)) \Delta\xi - v\nabla w_2 \cdot \nabla\xi - v\eta
$$

$$
- \int_{Q_T} w\,div\,(u_1\nabla\psi - v_1\nabla\xi + \nabla\eta) - \int_{Q_T} (F_1 - F_2)(\psi + \xi).
$$

We choose the test functions as solutions of the problem

$$
(3.21) \quad
\begin{cases}
-\Delta\psi(t) = u(t) & \text{in } \Omega, \\
-\Delta\xi(t) = v(t) & \text{in } \Omega, \\
-\Delta\eta(t) = div\,(v_1(t)\nabla\xi(t) - u_1(t)\nabla\psi(t)) & \text{in } \Omega, \\
\nabla\psi \cdot \nu = \nabla\xi \cdot \nu = \nabla\eta \cdot \nu = 0 & \text{on } \Sigma_N, \\
\psi = \xi = \eta = 0 & \text{on } \Sigma_D,
\end{cases}
$$

for a.e. $t \in (0, T)$. Existence, uniqueness and regularity of solutions is a consequence of the theory of linear elliptic equations. In particular, since $u, v \in \mathcal{C}([0, T]; \mathcal{V}')$ we deduce $\psi, \xi \in \mathcal{C}([0, T]; \mathcal{V})$. Then, $u(\cdot, 0) = v(\cdot, 0) = 0$ in Ω and the homogeneous boundary conditions for ψ and ξ imply

$$
(3.22) \qquad\qquad \nabla\psi(\cdot, 0) = \nabla\xi(\cdot, 0) = 0 \quad \text{in } L^2(\Omega).
$$

Using these test functions we get

$$
(3.23) \quad
\begin{aligned}
\frac{1}{2}\int_\Omega \left(|\nabla\psi(T)|^2 + |\nabla\xi(T)|^2\right) &+ \int_{Q_T} [u(\varphi(u_1) - \varphi(u_2)) + v(\varphi(v_1) - \varphi(v_2))] \\
&= \int_{Q_T} \nabla w_2 \cdot (u\nabla\psi - v\nabla\xi) \\
&+ \int_{Q_T} [\nabla(\psi - \xi) \cdot \nabla\eta - (F_1 - F_2)(\psi + \xi)].
\end{aligned}
$$

Now we perform the arguments to handle the terms involving u. The terms involving v are similarly treated after a change of sign. Due to the choice of the test functions

$$
\int_{Q_T} u\nabla w_2 \cdot \nabla\psi = -\int_{Q_T} \Delta\psi\nabla w_2 \cdot \nabla\psi.
$$

As in [37], let us show that (3.19) implies

$$
(3.24) \qquad -\int_{Q_T} \Delta\psi\nabla w_2 \cdot \nabla\psi \leq \frac{1}{2}\|w_2\|_{L^\infty(W^{2,\infty})}\int_{Q_T} |\nabla\psi|^2.
$$

Integrating formally by parts the left hand side of (3.24) we get

$$
(3.25)\int_\Omega -\Delta\psi\,(\nabla w_2 \cdot \nabla\psi) = \int_\Omega \nabla\psi \cdot \nabla(\nabla w_2 \cdot \nabla\psi) - \int_{\partial\Omega} (\nabla w_2 \cdot \nabla\psi)(\nabla\psi \cdot \nu).
$$

See [37] for a rigorous derivation of this identity. Using that $\psi = 0$ on Γ_D implies that $\nabla\psi$ has the same direction as ν on Γ_D we obtain

$$(3.26) \qquad \int_{\partial\Omega} (\nabla w_2 \cdot \nabla\psi)(\nabla\psi \cdot \nu) = \int_{\Gamma_D} |\nabla\psi|^2 \, \nabla w_2 \cdot \nu.$$

Denoting by $H(\cdot)$ the Hessian matrix we get after integrating by parts

$$\int_\Omega \nabla\psi \cdot \nabla(\nabla w_2 \cdot \nabla\psi) = \int_\Omega \nabla\psi : H(w_2) : \nabla\psi - \frac{1}{2}\int_\Omega \Delta w_2 \, |\nabla\psi|^2$$

$$(3.27) \qquad\qquad\qquad + \frac{1}{2}\int_{\Gamma_D} |\nabla\psi|^2 \, \nabla w_2 \cdot \nu.$$

Substituting (3.26) and (3.27) in (3.25) leads to

$$-\int_\Omega \Delta\psi \, (\nabla w_2 \cdot \nabla\psi) = \int_\Omega \nabla\psi : H(w_2) : \nabla\psi - \frac{1}{2}\int_\Omega \Delta w_2 \, |\nabla\psi|^2$$

$$-\frac{1}{2}\int_{\Gamma_D} |\nabla\psi|^2 \, \nabla w_2 \cdot \nu,$$

and using $\nabla w_2 \cdot \nu \geq 0$ on Γ_D (as a particular case of (3.19)) and the regularity assumed on w_2 we deduce (3.24). For problem (3.21) estimate (3.9) holds and then we have

$$(3.28) \qquad \int_{Q_T} (\nabla\psi \cdot \nabla\eta - \nabla\xi \cdot \nabla\eta) \leq c \int_{Q_T} \left(|\nabla\psi|^2 + |\nabla\xi|^2 \right).$$

Finally, Hölder's, Young's and Poincaré's inequalities together with (3.20) gives

$$-\int_{Q_T} (F_1 - F_2)(\psi + \xi) \leq \varepsilon c \int_{Q_T} \left[(\varphi(u_1) - \varphi(u_2))^2 + (\varphi(v_1) - \varphi(v_2))^2 \right]$$

$$(3.29) \qquad\qquad\qquad + \frac{c}{\varepsilon}\int_{Q_T} \left(|\nabla\psi|^2 + |\nabla\xi|^2 \right),$$

for all $\varepsilon > 0$. Then, using φ Lipschitz continuous and non-decreasing, substituting estimates (3.24) (and the corresponding for v), (3.28) and (3.29) in (3.23) and choosing ε, δ small enough we obtain

$$(3.30) \qquad \int_\Omega \left(|\nabla\psi(T)|^2 + |\nabla\xi(T)|^2 \right) \leq c \int_{Q_T} \left(|\nabla\psi|^2 + |\nabla\xi|^2 \right).$$

Gronwall's inequality and (3.22) imply $\nabla\psi \equiv \nabla\xi \equiv 0$ a.e. in Q_T, from where the assertion follows. $\square$

We finally present our third result. The main assumption is on the regularity of the solution constructed in Theorem 2.1: we suppose $\nabla b(u), \nabla b(v) \in L^1(0, T; L^p(\Omega))$, with p given in (3.1). As a consequence of Corollary 2.3 this regularity property is satisified in the following example:

$$\varphi(s) := s^m, \quad b(s) := s^\gamma \quad \text{with } 1 < m < 2\gamma \text{ and } \gamma \geq 1,$$

in space dimension one.

THEOREM 3.4. *Assume* $\mathbf{H}_1$–$\mathbf{H}_3$ *and suppose that there exists an open set* $\tilde{B} \subset \Gamma_D$ *such that the* $(N-1)-$*dimensional Haussdorf measure of* $\tilde{B}$ *and* Γ_D *coincides. Suppose*

$$\varphi \in C^2((0, \infty)), \quad \text{with} \quad \varphi'(0) = 0$$

*and assume the existence of a positive constant c and a convex function $\mu \in C^0([0, \infty))$
$\cap\, C^2((0, \infty))$ such that $\mu(0) = 0$,*

(3.31) $$0 < \mu'(r) \le \varphi'(r) \quad and \quad \varphi(r) \le c\mu(r) \quad for \quad r > 0.$$

Then problem (1.1) *has a unique solution in the class of weak solutions satisfying*

$$b(u), b(v) \in L^1(0, T; \mathcal{W}),$$

with $\mathcal{W}$ given by (3.1).

Proof. Consider, as in proof of Theorem 2.1, the sequence of regularized problems
$(1.1)_\varepsilon$ in which we approximate solutions of the degenerate problem (1.1) by taking
the perturbed auxiliary conditions given by (2.39), remaining the other conditions
unchanged. We know from Theorem 2.2 that for each $\varepsilon > 0$ problem $(1.1)_\varepsilon$ has, at
least, a weak solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ with the additional regularity stated in the men-
tioned theorem and converging to a weak solution (u, v, w) of (1.1) (Theorem 2.1).
Moreover, there exist positive constants λ and c, independent of φ and ε, such that

(3.32) $$c \ge u_\varepsilon, v_\varepsilon \ge \varepsilon e^{-\lambda t} \quad \text{a.e. in } Q_T$$

and

(3.33) $$\|\nabla w_\varepsilon\|_{L^\infty(Q_T)} \le c.$$

Suppose that (u_2, v_2, w_2) is another weak solution of (1.1) and define $(U_\varepsilon, V_\varepsilon, W_\varepsilon) :=
(u_\varepsilon - u_2, v_\varepsilon - v_2, w_\varepsilon - w_2)$ and $F_\varepsilon := F(u_\varepsilon, v_\varepsilon) - F(u_2, v_2)$. Then $(U_\varepsilon, V_\varepsilon, W_\varepsilon)$ satisfy

(3.34) $$\begin{cases} U_{\varepsilon t} - \Delta\left(\varphi(u_\varepsilon) - \varphi(u_2)\right) + div\left(b(u_\varepsilon)\nabla W_\varepsilon + (b(u_\varepsilon) - b(u_2))\nabla w_2\right) = F_\varepsilon, \\ V_{\varepsilon t} - \Delta\left(\varphi(v_\varepsilon) - \varphi(v_2)\right) - div\left(b(v_\varepsilon)\nabla W_\varepsilon + (b(v_\varepsilon) - b(v_2))\nabla w_2\right) = F_\varepsilon, \\ -\Delta W_\varepsilon + U_\varepsilon - V_\varepsilon = 0, \end{cases}$$

in Q_T, and the auxiliary conditions

(3.35) $$\begin{cases} \varphi(u_{D\varepsilon}) = \varphi(u_D) + \varphi(\varepsilon e^{-\lambda_1 t}), \quad \varphi(v_{D\varepsilon}) = \varphi(v_D) + \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_D, \\ \varphi(u_{D2}) = \varphi(u_D), \quad \varphi(v_{D2}) = \varphi(v_D), \quad W_{D\varepsilon} = 0 & \text{on } \Sigma_D, \\ \nabla\varphi(u_{N\varepsilon}) \cdot \nu = \nabla\varphi(v_{N\varepsilon}) \cdot \nu = \nabla W_\varepsilon \cdot \nu = 0 & \text{on } \Sigma_N, \\ \nabla\varphi(u_2) \cdot \nu = \nabla\varphi(v_2) \cdot \nu = 0 & \text{on } \Sigma_N, \\ U_\varepsilon(\cdot, 0) = V_\varepsilon(\cdot, 0) = \varepsilon & \text{in } \Omega. \end{cases}$$

Taking smooth test functions ψ, ξ, η with homogeneous mixed boundary conditions
for problem (3.34)-(3.35) we get

$$\int_\Omega \psi(T)U_\varepsilon(T) + \xi(T)V_\varepsilon(T) = \varepsilon \int_\Omega \psi(0) + \xi(0) - \int_{\Sigma_D} \varphi(\varepsilon e^{-\lambda_1 t})\left(\nabla\psi + \nabla\xi\right) \cdot \nu$$

$$+ \int_{Q_T} U_\varepsilon\left(\psi_t + A_u^\varepsilon\Delta\psi + \mathbf{B}_u^\varepsilon \cdot \nabla\psi + \eta + F_u^\varepsilon(\psi + \xi)\right)$$

$$+ \int_{Q_T} V_\varepsilon\left(\xi_t + A_v^\varepsilon\Delta\xi - \mathbf{B}_v^\varepsilon\nabla\xi - \eta + F_v^\varepsilon(\psi + \xi)\right)$$

(3.36) $$- \int_{Q_T} W_\varepsilon div\left(b(u_\varepsilon)\nabla\psi - b(v_\varepsilon)\nabla\xi + \nabla\eta\right),$$

with $A_z^\varepsilon := \int_0^1 \varphi'(sz_\varepsilon + (1-s)z_2)ds$, $\mathbf{B}_z^\varepsilon := \nabla w_2 \int_0^1 b'(sz_\varepsilon + (1-s)z_2)ds$ for $z = u, v$
and $F_u^\varepsilon := \int_0^1 \frac{\partial F}{\partial u}(su_\varepsilon + (1-s)u_2, v_\varepsilon)ds$ with a similar definition for F_v^ε, where here

and in the sequel we denote by z any of the functions u or v. Due to $b \in \mathcal{C}^1([0, \infty))$, F Lipschitz continuous, (3.32) and (3.33) we have

$$(3.37) \qquad \|\mathbf{B}_z^\varepsilon\|_{L^\infty(Q_T)}, \|F_z^\varepsilon\|_{L^\infty(Q_T)} \le k_0,$$

with k_0 independent of ε. In addition, using (3.31) we deduce the existence of a constant

$$(3.38) \qquad k(\varepsilon) := \varepsilon^{-1} e^{\lambda T} \mu(\varepsilon e^{-\lambda T})$$

such that

$$(3.39) \qquad 0 < k(\varepsilon) \le A_z^\varepsilon \le k_0 \quad \text{in } Q_T.$$

We consider sequences of $\mathcal{C}^\infty(Q_T)$ functions such that

$$A_z^{\varepsilon,n} \to A_z^\varepsilon, \quad \mathbf{B}_z^{\varepsilon,n} \to \mathbf{B}_z^\varepsilon, \quad F_z^{\varepsilon,n} \to F_z^\varepsilon, \quad \text{y} \quad b_z^n \to b(z_\varepsilon),$$

strongly in $L^2(Q_T)$ when $n \to \infty$. We choose $A_z^{\varepsilon,n}$ monotone decreasing on n and $\mathbf{B}_z^{\varepsilon,n}$, $F_z^{\varepsilon,n}$ and b_z^n monotone increasing on n. Because of (3.39), (3.37) and the $L^\infty(Q_T)$ regularity of solutions of (1.1) we deduce

$$(3.40) \qquad k(\varepsilon) \le A_z^{\varepsilon,n} \le k_0, \quad \text{and} \quad \|\mathbf{B}_z^{\varepsilon,n}\|_{L^\infty(Q_T)}, \|F_z^{\varepsilon,n}\|_{L^\infty(Q_T)}, \|b_z^{\varepsilon,n}\|_{L^\infty(Q_T)} \le k_0$$

in Q_T. We rewrite identity (3.36) as

$$
\begin{aligned}
\int_\Omega \psi(T) U_\varepsilon(T) + \xi(T) V_\varepsilon(T) ={}& \varepsilon \int_\Omega \psi(0) + \xi(0) - \int_{\Sigma_D} \varphi(\varepsilon e^{-\lambda_1 t}) (\nabla\psi + \nabla\xi) \cdot \nu \\
&+ \int_{Q_T} U_\varepsilon \left[(A_u^\varepsilon - A_u^{\varepsilon,n}) \Delta\psi + (\mathbf{B}_u^\varepsilon - \mathbf{B}_u^{\varepsilon,n}) \cdot \nabla\psi \right] \\
&+ \int_{Q_T} V_\varepsilon \left[(A_v^\varepsilon - A_v^{\varepsilon,n}) \Delta\xi - (\mathbf{B}_v^\varepsilon - \mathbf{B}_v^{\varepsilon,n}) \nabla\xi \right] \\
&+ \int_{Q_T} U_\varepsilon (F_u^\varepsilon - F_u^{\varepsilon,n}) (\psi + \xi) + V_\varepsilon (F_v^\varepsilon - F_v^{\varepsilon,n}) (\psi + \xi) \\
&+ \int_{Q_T} \nabla W_\varepsilon \cdot ((b(u_\varepsilon) - b_u^n) \nabla\psi - (b(v_\varepsilon) - b_v^n) \nabla\xi) \\
&- \int_{Q_T} W_\varepsilon \, div \, (b_u^n \nabla\psi - b_v^n \nabla\xi + \nabla\eta) \\
&+ \int_{Q_T} U_\varepsilon (\psi_t + A_u^{\varepsilon,n} \Delta\psi + \mathbf{B}_u^{\varepsilon,n} \cdot \nabla\psi + \eta + F_u^{\varepsilon,n} (\psi + \xi)) \\
&+ \int_{Q_T} V_\varepsilon (\xi_t + A_v^{\varepsilon,n} \Delta\xi - \mathbf{B}_v^{\varepsilon,n} \cdot \nabla\xi - \eta + F_v^{\varepsilon,n} (\psi + \xi)) \\
(3.41) \qquad ={}& : I_1 + \cdots + I_9,
\end{aligned}
$$

and set the following problem to choose the test functions:

$$(3.42) \qquad \begin{cases} \psi_t + A_u^{\varepsilon,n} \Delta\psi + \mathbf{B}_u^{\varepsilon,n} \cdot \nabla\psi + \eta + F_u^{\varepsilon,n} (\psi + \xi) = 0 & \text{in } Q_T, \\ \xi_t + A_v^{\varepsilon,n} \Delta\xi - \mathbf{B}_v^{\varepsilon,n} \cdot \nabla\xi - \eta + F_v^{\varepsilon,n} (\psi + \xi) = 0 & \text{in } Q_T, \\ \Delta\eta + div \, (b_u^n \nabla\psi - b_v^n \nabla\xi) = 0 & \text{in } Q_T, \\ \psi = \xi = \eta = 0 & \text{on } \Sigma_D, \\ \nabla\psi \cdot \nu = \nabla\xi \cdot \nu = \nabla\eta \cdot \nu = 0 & \text{on } \Sigma_N, \\ \psi(\cdot, T) = \chi_\delta^u, \quad \xi(\cdot, T) = \chi_\delta^v & \text{in } \Omega, \end{cases}$$

with $\chi_\delta^z \in \mathcal{C}_0^\infty(\Omega)$ satisfying $dist(\Sigma_D, supp(\chi_\delta^z)) \geq \delta$ and χ_δ^z uniformly bounded in $L^1(\Omega)$ for all $\delta > 0$.

LEMMA 3.5. *Problem (3.42) has a unique solution with the regularity of test functions of (1.1), see (2.4) and (2.5). Moreover,*

$$(3.43) \qquad \psi, \xi \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$

$$\eta \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)),$$

and their norms in these spaces are uniformily bounded with respect to n. Finally, there exists a positive constant $C(T)$ independent of ε and δ such that

$$(3.44) \qquad \|\psi\|_{L^\infty(Q_T)}, \|\xi\|_{L^\infty(Q_T)} \leq C(T).$$

Continuation of proof of Theorem 3.4. With the test functions of Lemma 3.5 we have $I_7 = I_8 = I_9 = 0$ in (3.41). Using in the resulting identity the uniform estimates with respect to n provided by Lemma 3.5 we deduce that I_3, I_4, I_5 and I_6 tend to zero when $n \to \infty$. Therefore, identity (3.41) is reduced to

$$(3.45) \int_\Omega \chi_\delta^u U_\varepsilon(T) + \chi_\delta^v V_\varepsilon(T) = \varepsilon \int_\Omega (\psi(0) + \xi(0)) - \int_{\Sigma_D} \varphi(\varepsilon e^{-\lambda_1 t}) (\nabla\psi + \nabla\xi) \cdot \nu.$$

The treatment of the boundary integral deserves the following

LEMMA 3.6. *Let A_ε, $\mathbf{B}_\varepsilon$, $g_\varepsilon \in L^\infty(Q_T)$ with*

$$(3.46) \qquad\qquad k(\varepsilon) < A_\varepsilon,$$

where $k(\varepsilon)$ is given by (3.38). Consider the problem

$$\begin{cases} \psi_t + A_\varepsilon \Delta\psi + \mathbf{B}_\varepsilon \cdot \nabla\psi + g_\varepsilon = 0 & in\ Q_T, \\ \psi = 0 & on\ \Sigma_D, \\ \nabla\psi \cdot \nu = 0 & on\ \Sigma_N, \\ \psi(\cdot, T) = \chi_\delta & in\ \Omega, \end{cases}$$

with $\delta > 0$. Then, there exist a $\delta(\varepsilon) > 0$ and a positive constant c, independent of ε, such that if $\delta < \delta(\varepsilon)$ then

$$(3.47) \qquad -\nabla\psi \cdot \nu \leq c\frac{\|\mathbf{B}_\varepsilon\|_{L^\infty(Q_T)} \|\psi\|_{L^\infty(Q_T)}}{k(\varepsilon)} \quad a.e.\ in \quad \Sigma_D.$$

End of proof of Theorem 3.4. Now we are in a position to pass to the limits $\varepsilon, \delta \to 0$. First note that Lemma 3.5 ensures the existence of uniform in ε, δ estimates of $\|\psi\|_{L^\infty(Q_T)}, \|\xi\|_{L^\infty(Q_T)}$. Therefore

$$(3.48) \qquad \varepsilon \int_\Omega (\psi(0) + \xi(0)) \to 0 \quad as \quad \varepsilon, \delta \to 0.$$

Applying Lemma 3.6 to the two first equations of problem (3.42) and considering the uniform bounds in ε of $\|\mathbf{B}_u^\varepsilon\|_{L^\infty(Q_T)}$, see (3.37), and in ε, δ of $\|\psi\|_{L^\infty(Q_T)}$, see Lemma 3.5, we obtain

$$(3.49) \qquad -\int_{\Sigma_D} \varphi(\varepsilon e^{-\lambda_1 t}) \nabla\psi \cdot \nu \leq c\frac{\varphi(\varepsilon)}{k(\varepsilon)},$$

where we used that φ is non-negative and increasing. Using (3.31) and (3.38) we obtain $\varphi(\varepsilon) \le c\varepsilon k(\varepsilon)$ and from (3.49) we deduce

$$(3.50) \qquad -\int_{\Sigma_D} \varphi(\varepsilon e^{-\lambda_1 t}) \nabla \psi \cdot \nu \le 0 \quad \text{as} \quad \varepsilon, \delta \to 0.$$

A similar argument may be applied to the term involving ξ. To finish, we choose function χ_δ^u as follows: set $\Omega_\delta := \{x \in \Omega : dist\,[\partial\Omega, supp\,(U_\varepsilon(x,T))] > \delta\}$ and define

$$\chi_\delta^u(x,t) := \begin{cases} sign\,\{u(x,t) - u_2(x,T)\} & \text{if } x \in \Omega_\delta, \\ 0 & \text{if } x \in \Omega \backslash \bar{\Omega}_\delta. \end{cases}$$

Then

$$(3.51) \qquad \int_\Omega \chi_\delta^u U_\varepsilon(T) \to \int_\Omega |u(T) - u_2(T)| \quad \text{as} \quad \varepsilon, \delta \to 0,$$

and a similar property holds for the term involving V_ε. Gathering (3.48)-(3.51) we deduce from (3.45)

$$\int_\Omega |u(T) - u_2(T)| + |v(T) - v_2(T)| \le 0,$$

and the assertion follows. $\square$

Proof of Lemma 3.5. The proof follows the same scheme than proof of Lemma 3.2. Therefore, we shall only show how to obtain property (3.44). Set

$$K := \left\{ h \in L^2(0,T^*;\mathcal{V}) \cap L^1(0,T^*;L^\infty(\Omega)) : \|h\|_{L^2(0,T^*;\mathcal{V})} + \|h\|_{L^1(0,T^*;L^\infty(\Omega))} < R \right\},$$

and define $Q : K \to L^2(0,T^*;\mathcal{V}) \cap L^1(0,T^*;L^\infty(\Omega))$ by $Q(\hat{\eta}) = \eta$, being η the solution of

$$(3.52) \qquad \begin{cases} \Delta\eta + div\left(b_u^n \nabla\hat{\psi} - b_v^n \nabla\hat{\xi}\right) = 0 & \text{in } Q_T, \\ \eta = 0 & \text{on } \Sigma_D, \\ \nabla\eta \cdot \nu = 0 & \text{on } \Sigma_N, \end{cases}$$

with $(\hat{\psi}, \hat{\xi})$ solution of

$$(3.53) \qquad \begin{cases} \hat{\psi}_t + A_u^{\varepsilon,n} \Delta\hat{\psi} + \mathbf{B}_u^{\varepsilon,n} \cdot \nabla\hat{\psi} + \hat{\eta} + F_u^{\varepsilon,n}(\hat{\psi} + \hat{\xi}) = 0 & \text{in } Q_T, \\ \hat{\xi}_t + A_v^{\varepsilon,n} \Delta\hat{\xi} - \mathbf{B}_v^{\varepsilon,n} \cdot \nabla\hat{\xi} - \hat{\eta} + F_v^{\varepsilon,n}(\hat{\psi} + \hat{\xi}) = 0 & \text{in } Q_T, \\ \hat{\psi} = \hat{\xi} = 0 & \text{on } \Sigma_D, \\ \nabla\hat{\psi} \cdot \nu = \nabla\hat{\xi} \cdot \nu = 0 & \text{on } \Sigma_N, \\ \hat{\psi}(\cdot,T) = \chi_\delta^u, \quad \hat{\xi}(\cdot,T) = \chi_\delta^v & \text{in } \Omega. \end{cases}$$

By Alexandrov's maximum principle, see [27], the solution of (3.53) satisfy

$$(3.54) \qquad \left\|\hat{\psi}\right\|_{L^\infty(Q_{T^*})} + \left\|\hat{\xi}\right\|_{L^\infty(Q_{T^*})} \le e^{cT^*} \|\hat{\eta}\|_{L^1(0,T^*;L^\infty(\Omega))},$$

with c depending on a bound of $\|F_z^{\varepsilon,n}\|_{L^\infty}$ which we know independent of ε, n, see (3.40). Now define

$$(3.55) \qquad \theta := \eta + b_u^n \hat{\psi} - b_v^n \hat{\xi}.$$

From (3.52) we have that θ satisfies

$$
\begin{cases}
\Delta\theta = div\left(\hat{\psi}\nabla b_u^n - \hat{\xi}\nabla b_v^n\right) & \text{in } Q_T, \\
\theta = 0 & \text{on } \Sigma_D, \\
\nabla\theta \cdot \nu = 0 & \text{on } \Sigma_N.
\end{cases}
$$

By well known results, see [40], we have

$$
\|\theta\|_{L^\infty(\Omega)} \leq c\left(\left\|\hat{\xi}\nabla b_v^n\right\|_{L^p(\Omega)} + \left\|\hat{\psi}\nabla b_u^n\right\|_{L^p(\Omega)}\right),
$$

for p given in (3.1). Then

$$
\|\theta\|_{L^\infty(\Omega)} \leq c\left(\left\|\hat{\xi}\right\|_{L^\infty(\Omega)}\|\nabla b_v^n\|_{L^p(\Omega)} + \left\|\hat{\psi}\right\|_{L^\infty(\Omega)}\|\nabla b_u^n\|_{L^p(\Omega)}\right).
$$

Recalling the definition of θ and using the assumption $b(u), b(v) \in L^1(0,T;\mathcal{W})$ we obtain

$$
(3.56) \qquad \|\eta\|_{L^1(0,T^*;L^\infty(\Omega))} \leq c(T^*)\left(\left\|\hat{\psi}\right\|_{L^\infty(Q_{T^*})} + \left\|\hat{\xi}\right\|_{L^\infty(Q_{T^*})}\right),
$$

with $c(t)$ depending only on $\|\nabla b(z)\|_{L^1(0,t;L^p(\Omega))}$ and $\|b(z)\|_{L^\infty(Q_t)}$, $z = u, v$. Gathering (3.54) and (3.56) we find

$$
\|\eta\|_{L^1(0,T^*;L^\infty(\Omega))} \leq c(T^*)e^{cT^*}\|\hat{\eta}\|_{L^1(0,T^*;L^\infty(\Omega))} < c(T^*)e^{cT^*}R,
$$

and to obtain $Q(K) \subset K$ we only have to choose T^* small enough. The rest of the proof of existence of a fixed point is similar as that in Lemma 3.2. Finally, notice that a continuation argument allows to extend the solution to any $T > 0$. Indeed, estimates (3.54) and (3.56) do not blow up for any finite T^*. $\square$

Proof of Lemma 3.6. Since $\partial\Omega$ is regular, Ω satisfies the exterior sphere condition, i.e., for all $x_0 \in \partial\Omega$ there exists $R_1 > 0$ and $x_1 \in I\!\!R^N\backslash\bar{\Omega}$ such that

$$
B(x_1, R_1) \cap \bar{\Omega} = \{x_0\},
$$

where $B(x_1, R_1) := \left\{x \in I\!\!R^N : |x - x_1| < R_1\right\}$. Let us fix $x_0 \in Interior(\Gamma_D)$. This set is non empty because, by hypothesis, there exists an open set $\tilde{B}$ such that $\tilde{B} \subset \Gamma_D$. Therefore, there exists a small enough $\delta > 0$ such that, by defining $R_2 := \delta + R_1$, we have $B(x_1, R_2) \cap \partial\Omega \subset \Gamma_D$. Moreover, since $dist(\partial\Omega, supp(\chi_\delta)) \geq \delta$, we also have $\chi_\delta \equiv 0$ in $\omega := \Omega \cap B(x_1, R_2)$. We shall use the notation $k_0(\varepsilon) := \|g\|_{L^\infty(Q_T)}$, $k_1(\varepsilon) := \left(\frac{N-1}{R_1} + 1\right)\|\mathbf{B}\|_{L^\infty(Q_T)}$ and $k_2(\varepsilon) := \|\psi\|_{L^\infty(Q_T)}$. We define

$$
\mathcal{L}(\psi) := \psi_t + A_\varepsilon\Delta\psi + \mathbf{B}\cdot\nabla\psi \quad \text{and} \quad w(x,t) := \psi(x,t) + \sigma(r),
$$

with $(x,t) \in \omega \times (0,t)$ and $r := |x - x_0|$. Function $\sigma \in C^2([R_1, R_2])$ will be chosen such that $\sigma''(r) \geq 0$, $\sigma'(r) \leq 0$ and the maximum of w in $\bar{\omega} \times [0,T]$ is attained in $\{x_0\} \times [0,T]$. Assuming these properties we get, due to (3.46)

$$
\mathcal{L}(w) = -g + A_\varepsilon\Delta\sigma + \mathbf{B}\cdot\nabla\sigma \geq k(\varepsilon)\sigma''(r) + k_1(\varepsilon)\sigma'(r) - k_0(\varepsilon).
$$

Choosing $\sigma(r) := \frac{k_0(\varepsilon)}{k_1(\varepsilon)} r + C_2 e^{-\frac{k_1(\varepsilon)}{k(\varepsilon)} r}$, with C_2 an arbitrary constant, we obtain

$$k(\varepsilon)\sigma''(r) + k_1(\varepsilon)\sigma'(r) - k_0(\varepsilon) = 0, \quad \sigma''(r) \geq 0 \quad \text{and}$$

(3.57) $\qquad$ if $\quad C_2 \geq k(\varepsilon)\frac{k_0(\varepsilon)}{k_1^2(\varepsilon)} e^{\frac{k_1(\varepsilon)}{k(\varepsilon)} R_2} \quad$ then $\quad \sigma'(r) \leq 0.$

Taking C_2 with this restriction we have that $\mathcal{L}(w) \geq 0$ in $\bar{\omega} \times [0, T]$ and therefore, by the maximum principle, we deduce that w attains its maximum on the parabolic boundary of $\omega \times [0, T]$. On this boundary the values of w may be estimated as follows:

$$\begin{cases} w(x,t) = \sigma(r) \leq \sigma(R_1) & \text{on } (\Gamma_D \cap \partial\omega) \times [0,T], \\ w(x,t) = \psi(x,t) + \sigma(r) \leq k_2(\varepsilon) + \sigma(R_2) & \text{on } (\partial B(x_1, R_2) \cap \partial\omega) \times [0,T], \\ w(x_0,t) = \sigma(R_1) & \text{on } [0,T], \\ w(x,T) = \sigma(r) + \chi_\delta(x) \leq \sigma(R_1) & \text{in } \omega, \end{cases}$$

where we have used that $\chi_\delta \equiv 0$ in ω. It is a straightforward computation to see that, by making δ small enough, we can choose C_2 such that (3.57) and $\sigma(R_1) = k_2(\varepsilon) + \sigma(R_2)$ hold. As a consequence we obtain that $\nabla w(x_0,t) \cdot \nu \geq 0$ and by the definition of w and taking δ suitably we obtain

$$-\nabla\psi(x_0,t) \cdot \nu \leq c \frac{k_1(\varepsilon)k_2(\varepsilon)}{k(\varepsilon)} \quad \text{in } [0,T].$$

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