Applications of a Local Energy Method to Systems of PDE's Involving Free Boundaries

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Abstract. We present a method of analysis for free boundary problems which is based on local energy estimates. This method allows us to deal with a great variety of problems formulated in a very general form, where *generality* stands for:

- No global information, like boundary conditions or boundedness of the domain, is needed.
- No monotonicity assumption on the nonlinearities is required, as the comparison principle is not invoked.
- Coefficients may depend on space and time variables and only a weak regularity is required.
- No restriction on the space dimension is assumed.

In this article we first show how the energy method applies to a simple example, proving the well known property of finite speed of propagation for the porous medium equation. We then give an outline of how the method works in more complicated situations: the occurrence of dead cores for the mangroves' problem and the finite speed of propagation along the characteristics for an evolution convection-diffusion equation.

1. Introduction

In this article we describe by means of three examples the so called *energy method for free boundary problems*. The method was introduced by Antontsev in [1], and was later extended and rendered to a mathematical rigorous form by Antontsev, Díaz and Shmarev [2] and Díaz and Verón [6]. Other contributions have been given by Bernis to PDE's of arbitrary order, see [3]. See also some applications of the method to systems of equations in [9, 5, 7, 8]. The idea of the method is to find a differential inequality which is satisfied by the natural energies associated to the problem and to deduce from its resolution some qualitative properties of the solutions of the original problem. Among these properties we find the finite speed of propagation and the formation of dead cores. Both are related to a degeneration introduced in the problem when the solution attains certain value, usually normalized to be the zero value. *Finite speed of propagation* means that solutions

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corresponding to compact supported initial datum remain with compact support at least for some time, meanwhile *dead core* means that even if the initial datum is strictly positive, a region where the zero level is attained may appear in finite time. Both phenomena are purely nonlinear.

2. The Porous Medium Equation and Finite Speed of Propagation

We consider the Cauchy problem for the porous medium equation,

$$v_t - (v^m)_{xx} = 0, \quad v(\cdot,0) = v_0\,, \tag{1}$$

with $m > 1$ and $v_0 \geq 0$ of compact support in $\mathbb{R}$. It is well known that the support of $v(\cdot,t)$ remains bounded for all $t > 0$. This is a peculiar behaviour of solutions of some type of parabolic problems which degenerate in some subset of the domain, and which is called the *finite speed of propagation* property. The aim of this section is to show how the energy method works for this simple example. Introducing the change of unknown $u := v^m$ and defining $p := 1/m$, (1) transforms into

$$(u^p)_t - u_{xx} = 0, \quad u(\cdot,0) = u_0 := v_0^m\,. \tag{2}$$

Let us denote by B_r the ball $\{x \in \Omega \colon |x - x_0| < r\}$, for any $r > 0$. Let $x_0 \in \mathbb{R}$ and $\rho_0 > 0$ be such that $u_0(x_0) = 0$ in B_{ρ_0}. Multiplying (2) by u and integrating in $B_\rho \times (0,t)$, with $\rho < \rho_0$ and $t > 0$, we obtain

$$\tfrac{p}{p+1}\int_{B_\rho} u(t)^{p+1} + \int_0^t \int_{B_\rho} |u_x|^2 = \int_0^t \int_{\partial B_\rho} u u_x n\,, \tag{3}$$

with n the outer normal to B_ρ. We define the *energy functions*

$$b(\rho,t) := \sup_{0\leq\tau\leq t} \int_{B_\rho} u(\tau)^{p+1} \quad \text{and} \quad E(\rho,t) := \int_0^t \int_{B_\rho} |u_x|^2\,,$$

which are clearly non-negative and non-decreasing with respect to ρ and t. Applying Hölder's inequality in (3), taking into account the increasing character of the different terms with respect to t and using that $E_\rho(\rho,t) = \int_0^t \int_{\partial B_\rho} |u_x|^2$, with E_ρ the partial derivative of E with respect to ρ, we obtain

$$\tfrac{p}{p+1} b(\rho,t) + E(\rho,t) \leq \left(\int_0^t \int_{\partial B_\rho} u^2 \right)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}} =: I(E_\rho)^{\frac{1}{2}}\,. \tag{4}$$

To estimate I, which is a key step of the method, we apply a simple version of an interpolation-trace inequality introduced in a general setting in [6].

Lemma 2.1. *Let $\varphi \in H^1(x_0 - \rho, x_0 + \rho)$, for $x_0 \in \mathbb{R}$ and $\rho > 0$. Then*

$$|\varphi(x_0 - \rho)| + |\varphi(x_0 + \rho)| \leq L_0 \left(\|\varphi_x\|_2 + \rho^{-\delta} \|\varphi\|_{p+1} \right)^\gamma \|\varphi\|_r^{1-\gamma}\,, \tag{5}$$

with $L_0 > 0$, $p > 0$, $\gamma := \frac{2}{2+r}$, $\delta := \frac{p+3}{2(p+1)}$ and $r \in [1,2]$.

Here we used the notation $\|\cdot\|_s := \|\cdot\|_{L^s(x_0-\rho,x_0+\rho)}$. The proof of this lemma is worked out by using some basic inequalities for L^p and Sobolev spaces. Using this lemma together with Young and Hölder inequalities we obtain

$$I \leq ct^{\frac{1-\gamma}{2}}(E+b)^{\beta}\,, \tag{6}$$

with $\beta = \gamma/2+(1-\gamma)/(p+1)$, and c a positive constant. It is important to observe here that $\beta > 1/2$ if and only if $p < 1$, since the argument we use now requires this condition. This is the moment in which the degenerate parabolicity of the porous medium equation comes in play. Applying Young's inequality to (6), we obtain

$$I(E_\rho)^{\frac{1}{2}} \leq \varepsilon(E+b) + c_\varepsilon t^{\frac{1-\gamma}{\kappa}}(E_\rho)^{\frac{1}{\kappa}}\,, \tag{7}$$

for some $\varepsilon > 0$ and $\kappa := 2(1-\beta)$. Combining (4) and (7) and using $b \geq 0$ we get

$$E \leq ct^{\frac{1-\gamma}{\kappa}}(E_\rho)^{\frac{1}{\kappa}}\,,$$

for $0 < \rho < \rho_0$ and $0 < t < T$. Since $\beta > 1/2$, we have $\kappa < 1$ and a direct integration of this differential inequality in (ρ, ρ_0) gives

$$E^{1-\kappa}(\rho,t) \leq E^{1-\kappa}(\rho_0,t) - c(\rho_0-\rho)t^{1-\gamma}\,.$$

Therefore we deduce that choosing $\rho \leq c\rho_0 - E^{1-\kappa}(\rho_0,t)t^{\gamma-1} =: r(t)$, which is possible till certain instant $t^* > 0$ then $E(\rho,t) = 0$ for all $t \in (0,t^*)$ and ρ such that $\rho \leq r(t)$. And the result follows, i.e. $u(\cdot,t) = 0$ in $B_{r(t)}$ for all $t \in (0,t^*)$, or in other words, the support of $u(\cdot,t)$ remains bounded in $\mathbb{R}$ at least while $t < t^*$.

3. The Mangroves' Problem and Formation of Dead Cores

Mangroves grow on saturated soils or muds which are subject to regular inundation by tidal water with salt concentration c_w close to that of sea water. The mangrove roots take up fresh water from the saline soil and leave behind most of the salt, resulting in a net flow of water downward from the soil surface, which carries salt with it. In the absence of lateral flow, the steady state salinity profile in the root zone must be such that the salinity around the roots is higher than c_w, and that the concentration gradient is large enough so that the advective downward flow of salt is balanced by the diffusive flow of salt back up to the surface. In [7] we studied the following dimensionless model for the mangroves' problem

$$\begin{cases} v_t + (vq)_x - v_{xx} + f(x,v) = 0 \\ q_x + f(x,v) = 0 \end{cases} \quad \text{in } Q_T := (0,d)\times(0,T)\,, \tag{8}$$

with

$$v(0,\cdot) = v_D, \quad v_x(d,\cdot) = q(d,\cdot) = 0 \quad \text{in } (0,T), \quad \text{and } v(\cdot,0) = v_0 \quad \text{in } (0,d)\,. \tag{9}$$

Here $v := 1-u$ and $0 \leq u \leq 1$ denotes the water salt concentration, q the water discharge and $f(x,v)$ a function modeling the fresh water uptake by the roots of mangroves, typically given by $f(x,v) := k(x)v^p$, with $k \geq 0$ the root distribution (a bounded function) and p a positive constant. The variable $x \in \Omega$ denotes depth. The equations of (8) state the conservation of mass of salt and water.

The question is whether or not the water surrounding the roots of mangroves may reach a threshold level of salinization, $v = 0$, in finite time. We know that if $p = 1$, i.e. f is a linear function on the second variable, then $v = 0$ is not possible in finite time. This is the same situation than that of the heat equation with a linear source term. However, if $p < 1$, then f is non-Lipschitz continuous and a different behaviour of v may appear. In fact, we prove, see theorem 3.1, that even when v_0 is positive, a *dead core* (region with $v = 0$) may appear in finite time. To apply the energy method we consider the set

$$\mathcal{P}(t) := \{(x,\tau) \colon\ |x - x_0| < R(\tau;t), \quad \tau \in (t,T)\}, \quad \text{for } t \in (0,T),$$

with $R(\tau;t) := (\tau - t)^\nu$, $0 < \nu < 1$ to be fixed, $\alpha > 0$ and $x_0 \in (0,d)$ such that $R(T;0) < x_0 < d - R(T;0)$, implying $\mathcal{P}(t) \subset Q_T$ for all $t \in (0,T)$. For brevity, we shall write $\mathcal{P}$ instead of $\mathcal{P}(t)$. We decompose the boundary of $\mathcal{P}$ as

$$\partial\mathcal{P}(t) := \partial_f\mathcal{P}(t) \cup \partial_l\mathcal{P}(t),$$

with $\partial_f\mathcal{P}(t) := \{(x,T) \in \partial\mathcal{P}\}$ and $\partial_l\mathcal{P}(t) := \{(x,\tau) \in \partial\mathcal{P} \colon t < \tau < T\}$. Finally, we define the *local energy functions*

$$E(t) := \int_{\mathcal{P}(t)} |v_x|^2 \, dx \, d\tau \quad \text{and} \quad C(t) := \int_{\mathcal{P}(t)} v^{p+1} \, dx \, d\tau. \tag{10}$$

Observe that E and C are non-increasing functions with respect to t. We make the following assumption on f: there exist positive constants k_0, k_1 such that

$$0 < k_0 s^{p+1} \le s f(\cdot, s) \le k_1 s^{p+1} \quad \text{for } s \in [0,1] \tag{11}$$

in $\mathcal{P}(t)$ for a.e. $t \in (0,T)$, with $p \in (0,1)$ and $k_0 > k_1/2$.

Theorem 3.1. *Let (v,q) be a solution of (8)–(9) and assume (11). Then there exists a constant $M > 0$ such that if $E(0) + C(0) \le M$ then $v \equiv 0$ in $\mathcal{P}(t^*)$, for some $t^* \in (0,T)$.*

Let us observe that testing the first equation of (8) with v and using the second equation of (8) leads to the following estimate

$$E(0) + C(0) \le \int_\Omega v_0^2(x) dx + \int_0^T v_D(t) v_x(0,t) dt.$$

In some situations, for instance when $v_D \ge v_0$ the maximum principle implies $v_x(0,\cdot) \le 0$, allowing us to obtain an estimate of $E(0) + C(0)$ only in terms of v_0.

Proof of Theorem 3.1. The proof consists of three steps.
Step 1. Multiplying the first equation of (8) by v and integrating in $\mathcal{P}$ gives

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{2}(v^2)_t + \tfrac{1}{2}((v^2 q)_x + v^2 q_x) + (|v_x|^2 - (v v_x)_x) + v f(x, 1-v) \right\} dx d\tau = 0.$$

Using the divergence theorem, the second equation of (8) and (11) we find

$$\int_{\mathcal{P}} |v_x|^2\, dxd\tau + k_0 \int_{\mathcal{P}} v^{p+1}\, dxd\tau \le \int_{\partial_l \mathcal{P}} v v_x n_x\, dxd\tau - \\ - \tfrac{1}{2} \int_{\partial_l \mathcal{P}} v^2 (n_\tau + q n_x)\, dxd\tau + \tfrac{k_1}{2} \int_{\mathcal{P}} v^{p+2}\,,$$

with (n_x, n_τ) the unitary outward normal vector to $\mathcal{P}$. Using $v \le 1$, $q \le dk_1$,

$$E(t) + (k_0 - \tfrac{k_1}{2}) C(t) \le \tfrac{1+dk_1}{2} \int_t^T [v^2] d\tau + \int_{\partial_l \mathcal{P}} |v||v_x|\, dxd\tau\,, \tag{12}$$

where we introduced the notation $[v] := |v(x_0 + R(\tau;t),\tau)| + |v(x_0 - R(\tau;t),\tau)|$.
Step 2. Our aim is to estimate the right hand side of (12) by means of E, C and their derivatives. First notice that $E_t(t) = \int_t^T [|v_x|^2] R_t(\tau;t) d\tau$, with the subindex t denoting derivative with respect to t. We use Hölder's inequality to get

$$\int_{\partial \mathcal{P}} |v||v_x|\, dxd\tau \le \left(\int_t^T -R_t [|v_x|^2]\, d\tau \right)^{1/2} \left(\int_t^T (-R_t)^{-1} [v^2]\, d\tau \right)^{1/2} = \\ =: (-E_t)^{1/2} I_1 \le (-(E+C)_t)^{1/2} I_1\,. \tag{13}$$

To handle $I_1(t)$ and $I_2(t) := \int_t^T [v^2]$ of (12) we apply the interpolation-trace inequality stated in lemma 2.1 to the function $\varphi := v(\cdot,t)$. We take $r < 2$ and find, by applying Hölder's inequality with exponent $\theta := \frac{1-p}{2-r}$

$$\|v(\cdot,t)\|_r \le \|v(\cdot,t)\|_2^{\frac{2}{r\theta'}} \, \|v(\cdot,t)\|_{p+1}^{\frac{p+1}{r\theta}}\,. \tag{14}$$

Combining (5) and (14) and using $v \le 1$ we get

$$[v^2] \le [v]^2 \le L_0^2 m(R) (\|v_x\|_2^2 + \|v\|_{p+1}^{p+1})^\gamma |Q_T|^{\frac{2(1-\gamma)}{r\theta'}} \, \|v\|_{p+1}^{\frac{2(1-\gamma)(p+1)}{r\theta}} \tag{15}$$

with $m(R) := \max\left\{1, R^{-2\delta\gamma}\right\}$. We then deduce from (15)

$$I_1 \le L_0 |Q_T|^{\frac{1-\gamma}{r\theta'}} \left(\int_t^T m(R)(-R_t)^{-1} (\|v_x\|_2^2 + \|v\|_{p+1}^{p+1})^{\gamma + \frac{2(1-\gamma)}{r\theta}} d\tau \right)^{1/2}\,. \tag{16}$$

Due to the crucial assumption $p < 1$, it is compatible to choose $r < 2$ and $r \ge 4/(3-p)$. Then we obtain that μ given by $\mu^{-1} := \gamma + (2(1-\gamma)/r\theta)$ satisfies $\mu \ge 1$. Using Hölder's inequality with exponent μ and substituting R we obtain from (16)

$$I_1 \le \Lambda (E+C)^{\frac{\gamma}{2} + \frac{1-\gamma}{r\theta}}\,, \tag{17}$$

with $\Lambda(t) := L_0 |Q_T|^{\frac{1-\gamma}{r\theta'}} \nu^{-1/2} \left(\int_t^T (\tau - t)^{\mu'(1-\nu-2\delta\nu\gamma)} d\tau \right)^{1/2\mu'}$. Function Λ is finite whenever we choose $\nu < (\mu+1)/(\mu(1+2\delta))$ which is always possible since the only restriction assumed on ν is $0 < \nu < 1$. Gathering (13) and (17) we get

$$\int_{\partial \mathcal{P}} |v||v_x|\, dxd\tau \le \Lambda(t) (-(E+C)_t)^{1/2} (E+C)^{\frac{\gamma}{2} + \frac{1-\gamma}{r}}\,. \tag{18}$$

In a similar way, but choosing $r=2$ in (5), we get the following estimate

$$I_2 \leq L_0 \Gamma(E+C)\,, \tag{19}$$

with $\Gamma^2(t) := \int_t^T (\tau - t)^{-\delta\nu} d\tau < \infty$ if $\nu < 1/\delta$.
Step 3. From (12), (18) and (19) we deduce

$$c_0(E+C) \leq \Lambda(-(E+C)_t)^{1/2}(E+C)^{\frac{\gamma}{2}+\frac{1-\gamma}{r}}\,,$$

with $c_0 \leq k_0 - \frac{k_1}{2} - \frac{1+dk_1}{2} L_0 \Gamma(t)$. Notice that making $T-t$ small enough, say $T-t \leq \varepsilon$, we can ensure $c_0 > 0$. Making the assumption, to force a contradiction, that $E(t)+C(t) > 0$ for all $t \in [0,T]$, we arrive at the inequality

$$c_0^2\,(E(t)+C(t))^{2(1-\frac{\gamma}{2}-\frac{1-\gamma}{r})} \leq -\Lambda(t)^2 (E+C)_t(t)\,. \tag{20}$$

Due again to $p<1$ we find $\sigma := 2\left(1-\frac{\gamma}{2}-\frac{1-\gamma}{r}\right) < 1$. We assume $T > \varepsilon$ and restrict t to take values on $(T-\varepsilon, T)$ (so $T-t \leq \varepsilon$ is fulfilled). Integrating (20) in $t \in (T-\varepsilon, t^*)$ with $t^* \in (T-\varepsilon, T)$ we obtain

$$(E+C)^{1-\sigma}(t^*) \leq (E+C)^{1-\sigma}(T-\varepsilon) - (1-\sigma)c_0^2 \int_{T-\varepsilon}^{t^*} \Lambda(t)^{-2} dt\,.$$

Therefore, since $E+C$ is non-increasing we have that if the initial energy satisfies

$$(E+C)^{1-\sigma}(0) \leq (1-\sigma)c_0^2 \int_{T-\varepsilon}^{t^*} \Lambda(t)^{-2} dt =: M^{1-\sigma}$$

then $E(t^*)+C(t^*)=0$ and therefore $v=0$ in $\mathcal{P}(t^*)$.

4. The Boussinesq System and Finite Speed of Propagation Along the Characteristics

The Boussinesq system of hydrodynamics equations arises from a zero order approximation to the coupling between the Navier-Stokes equations and the thermodynamic equation. The presence of density gradients in a fluid means that gravitational potential energy can be converted into motion through the action of bouyant forces. Density differences are induced, for instance, by gradients of temperature arising by heating non uniformly the fluid. In the Boussinesq approximation of a large class of flows problems, thermodynamical coefficients, such as viscosity, specific heat and thermal conductivity, can be assumed as constants leading to a coupled system with linear second order operators in the Navier-Stokes equations and in the heat conduction equation. However, there are some fluids, such as lubricants or some plasma flows, for which this is no longer an accurate assumption. In this case the following problem must be considered, see [4]

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}(\mu(\theta)D(\mathbf{u})) = \mathbf{F}(\theta) \\ \theta_t + \mathbf{u}\cdot\nabla\theta - \Delta\varphi(\theta) = 0 \end{cases} \quad \text{in } Q_T\,, \tag{21}$$

with $\operatorname{div}\mathbf{u} = 0$ in Q_T, and

$$\begin{aligned} \mathbf{u} = \mathbf{0}, \qquad \varphi(\theta) = 0 \qquad \text{on } \Sigma_T, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, \quad \theta(\cdot,0) = \theta_0 \quad \text{on } \Omega\,. \end{aligned} \tag{22}$$

We shall study the finite speed of propagation of the temperature variable, θ, along the characteristics defined by $\mathbf{u}$. We suppose that φ is given by $\varphi(s) := s^m$ with $m > 1$, although more general functions may be considered, see [8]. Assume that the velocity component of a solution of (21)–(22) is such that $\mathbf{u} \in C([0,T]; C^1_\sigma(\Omega))$, where σ denotes free divergence. Since $\mathbf{u} = 0$ on Σ_T, we may consider the prolongation of $\mathbf{u}$ by zero to the whole $\mathbb{R}^N$, and formulate the problem

$$\begin{cases} \dfrac{\partial \mathbf{X}}{\partial t}(\mathbf{x},t) = \mathbf{u}(\mathbf{X}(\mathbf{x},t),t) & \text{in } \mathbb{R}^N \times (0,T), \\ \mathbf{X}(\mathbf{x},0) = \mathbf{x} & \text{in } \mathbb{R}^N\,, \end{cases}$$

which admits a unique solution $\mathbf{X} \in C^1(\mathbb{R}^N \times [0,T])$. In the following we shall suppose that θ_0 vanishes in some ball B_{ρ_0} centered in $\mathbf{x}_0$ and compactly imbedded in Ω. The following property is a consequence of the continuity of $\mathbf{X}$:

$$\begin{cases} \text{there exist } \hat{t} > 0 \text{ and } \rho_1 > \rho_0 \text{ such that if } t < \hat{t} \text{ and } \rho < \rho_1 \\ \text{then } \mathbf{X}(B_\rho, t) \in \Omega\,. \end{cases} \tag{23}$$

Theorem 4.1. *Let $(\mathbf{u},\theta)$ be any solution of (21)–(22) and suppose that $\mathbf{u}$ is locally Lipschitz continuous in Q_T. Then there exists $t^* \in (0,\hat{t})$ and a continuous function $r\colon [0,t^*] \to \mathbb{R}_+$, with $r(0) = \rho_0$ such that*

$$\theta(\mathbf{x},t) \equiv 0 \quad \textit{a.e. in} \quad \{(\mathbf{x},t)\colon \mathbf{x} \in \mathbf{X}(B_{\rho_0},t), \quad t \in (0,t_*)\}\,.$$

Proof. We introduce the change of unknown $v := \theta^m$. Then v satisfies:

$$(v^p)_t + \mathbf{u}\cdot\nabla(v^p) - \Delta v = 0\,,$$

for $p := 1/m$. Multiplying by $v(\cdot,t)$, for $t > 0$ fixed and integrating in $\mathbf{X}(B_\rho,t)$

$$\tfrac{p}{p+1}\int_{\mathbf{X}(B_\rho,t)} \left((v^{p+1})_t + \mathbf{u}\cdot\nabla v^{p+1}\right) + \int_{\mathbf{X}(B_\rho,t)} |\nabla v|^2 = \int_{\partial\mathbf{X}(B_\rho,t)} v\nabla v\cdot\mathbf{n}\,. \tag{24}$$

The Reynolds Transport Lemma asserts that for any regular ψ,

$$\int_{\mathbf{X}(B_\rho,t)} \frac{\partial}{\partial t}\psi(\mathbf{y},t)\,\mathbf{dy} = \frac{d}{dt}\int_{\mathbf{X}(B_\rho,t)} \psi(\mathbf{y},t)\,\mathbf{dy} - \int_{\mathbf{X}(B_\rho,t)} \mathbf{u}(\mathbf{y},t)\cdot\nabla\psi(\mathbf{y},t)\mathbf{dy}\,. \tag{25}$$

Thus, integrating (24) in $(0,t)$ and using (25) and $v(\cdot,0) = 0$ in B_{ρ_0} we obtain

$$\int_{\mathbf{X}(B_\rho,t)} v^{p+1}(t) + \int_0^t \int_{\mathbf{X}(B_\rho,t)} |\nabla v|^2 \le c\int_0^t \int_{\partial\mathbf{X}(B_\rho,t)} v\nabla v\cdot\mathbf{n}\,.$$

Note that this expression is similar to (3) but integrated in a ball transformed along the characteristics defined by $\mathbf{u}$. As usual, we define the energies

$$b(\rho,t) := \sup_{0\le\tau\le t}\int_{\mathbf{X}(B_\rho,t)} v(\tau)^{p+1} \quad \text{and} \quad E(\rho,t) := \int_0^t\int_{\mathbf{X}(B_\rho,t)} |\nabla v|^2\,.$$

Due to the regularity of $\mathbf{X}$, we have $\partial(\mathbf{X}(B_\rho,t)) \equiv \mathbf{X}(\partial B_\rho,t)$ and therefore,

$$E_\rho(\rho,t) = \int_0^t \int_{\partial\mathbf{X}(B_\rho,t)} |\nabla v|^2 \quad \text{for a.e. } \rho > 0\,. \tag{26}$$

To finish the proof we follow the steps given in section 2 using now a version of the interpolation-trace trace inequality for sets transformed along the characteristics, see [8]. We deduce $b + E \leq c(E_\rho)^{1/2}(b+E)^{1/\kappa}$ where $\kappa \in (0,1)$ due to $p < 1$. Integrating this inequality the assertion follows. □

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