A PARABOLIC CROSS-DIFFUSION SYSTEM FOR GRANULAR MATERIALS*

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Abstract. A cross-diffusion system of parabolic equations for the relative concentration and the dynamic repose angle of a mixture of two different granular materials in a long rotating drum is analyzed. The main feature of the system is the ability to describe the axial segregation of the two granular components. The existence of global-in-time weak solutions is shown for arbitrary large cross-diffusion by using entropy-type inequalities and approximation arguments. The uniqueness of solutions is proved if cross-diffusion is not too large. Furthermore, we derive a sufficient condition on the parameters to have non-segregation. Finally, numerical simulations illustrate the long-time coarsening of the segregation bands in the drum.

Key words. Strongly nonlinear parabolic system, cross-diffusion, segregation, existence of weak solutions, uniqueness of solutions, entropy-type estimates.

AMS subject classifications. 35K55, 76T25.

1. Introduction.

One important feature of granular materials, consisting of different components, is their ability to segregate under external agitation rather than to further mix [21]. Consider a long cylinder rotating about its longitudinal axis, which is partially filled with a mixture of two different kinds of granular particles. The mixture of grains may exhibit both radial and axial size segregation. Roughly speaking, radial segregation occurs during the first few revolutions of the drum and is often followed by slow axial segregation. Axial segregation leads to either a stable array of concentration bands or, after a very long time, to complete segregation [2, 3, 24].

In this paper we are interested in the existence analysis of a specific model for granular materials derived in [3]. Consider a mixture of two kinds of particles with volume concentrations u_1, $u_2 \in [0,1]$, placed in a horizontal long narrow rotating cylinder of length $L > 0$. Let $u = u_1 - u_2 \in [-1,1]$ be the relative concentration of the mixture. Introduce the so-called dynamic angle of repose θ as the arctangent of the average slope of the free surface of the mixture which is assumed to be flat (see Figure 1.1). The variables u and θ are assumed to be constant in each cross section of the drum and depend therefore only on the axial coordinate $z \in \Omega = (0,L)$ and on the time $t > 0$.

Fig. 1.1. *Relative concentration u and dynamical angle of repose θ in the geometry of the cross section of a rotating drum. The gray region indicates the mixture partially filling the drum. The variables u and θ are assumed to be constant in each cross section.*

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In [3] the following cross-diffusion system for the evolution of u and θ has been derived:

$$u_t - (\nu u_z - (1-u^2)\theta_z)_z = 0, \tag{1.1}$$

$$\theta_t - (\gamma u + \theta)_{zz} + \theta = \mu u \quad \text{in } Q_T := \Omega \times (0,T), \tag{1.2}$$

where the subindices denote partial derivatives. The model (1.1)-(1.2) is obtained by averaging the mass conservation laws for the two components of the granular matter over the cross section of the cylinder, under the main assumptions that the mass of grains in each cross-section of the drum remains constant and that the grains separate predominantly near the surface of the drum, whereas in the bulk of the drum particles are equally advected by the bulk flow (see [3] for details of the derivation).

The positive constant ν is related to the Fickian diffusion constants arising in the surface fluxes of the two materials. The constant $\gamma > 0$ is proportional to the difference of the Fickian diffusivities. Finally, μ is related to the difference of the static repose angles of the two kind of particles.

We impose as in [3, 20] periodic boundary conditions and initial conditions for u and θ, as we are not interested in effects due to the boundary conditions:

$$\begin{array}{lll} u(0,\cdot) = u(L,\cdot), & u_z(0,\cdot) = u_z(L,\cdot) & \multirow{2}{*}{\text{in } (0,T),} \\ \theta(0,\cdot) = \theta(L,\cdot), & \theta_z(0,\cdot) = \theta_z(L,\cdot) & \\ u(\cdot,0) = u_0, & \theta(\cdot,0) = \theta_0 & \text{in } \Omega. \end{array} \tag{1.3}$$

In the physical literature, periodic boundary conditions have been employed in numerical simulations of the dynamics of the granular materials in order to eliminate boundary effects [3, 20]. The subsequent analysis also works for no-flux and Dirichlet boundary conditions (with appropriate changes of the obtained estimates).

We remark that the problem is intrinsically one-dimensional in space since the equations are obtained by averaging over the cross section. For a two-dimensional model we refer, for instance, to [10].

The terms $((1-u^2)\theta_z)_z$ and γu_{zz} in (1.1)-(1.2) are called *cross-diffusion* terms [17]. We remark that segregation effects due to cross-diffusion are well known in population dynamics, and related cross-diffusion systems have been studied in mathematical biology (see, e.g., [18, 22]).

Segregation phenomena of granular material in rotating drums have been intensively investigated in the physical literature. For instance, radial segregation has been investigated numerically using particle methods [10] and analytically using leading-order analysis [6] or shock-wave analysis [14]. Axial segregation has been simulated, for instance, in [2, 3, 20] and analyzed in [3, 5, 16]. For more references, particularly for experimental studies, we refer to the monograph [21] and the review paper [19].

Mathematically, the evolution problem (1.1)-(1.2) has a full and non-symmetric diffusion matrix:

$$A := \begin{pmatrix} \nu & -(1-u^2) \\ \gamma & 1 \end{pmatrix}.$$

Problems with full diffusion matrix also arise, for instance, in semiconductor theory [7], population dynamics [18], and in non-equilibrium thermodynamics [9]. As a consequence, no classical maximum principle arguments and no regularity theory as for single equations are generally available for such kind of problems.

Notice that there are values for u and the parameters ν and γ for which the above matrix A is *not* positive definite in the sense that $x^\top Ax < 0$ may hold for some

x. The ellipticity of the system (1.1)-(1.2) is guaranteed if $4\nu > \gamma$ (and $|u| \le 1$). For these values, the existence of global-in-time solutions of (1.1)-(1.3) can be proved using standard techniques. The question arises if it is possible to prove the existence of global weak solutions for *any* values of the parameters $\nu > 0$ and $\gamma > 0$. In this paper we give a positive answer to this question.

The key of the existence analysis is the observation that the system (1.1)-(1.2) possesses a functional whose time derivative is uniformly bounded in time if $|u| < 1$. Indeed, using the functions $\phi(u)$, where

$$\phi(s) := \frac{\gamma}{2} \log \frac{1+s}{1-s} \qquad \text{for } -1 < s < 1,$$

and θ in the weak formulation of (1.1) and (1.2), respectively, and adding the resulting equations leads to the inequality

$$\frac{d}{dt} \int_0^L \left(\Phi(u) + \frac{1}{2}\theta^2 \right) dz + \int_0^L (\gamma\nu u_z^2 + \theta_z^2) dz = \int_0^L (\mu u\theta - \theta^2) dz \le c, \tag{1.4}$$

where $c > 0$ only depends on μ and L. Here the function $\Phi(s) := \frac{\gamma}{2}(1-s)\log(1-s) + \frac{\gamma}{2}(1+s)\log(1+s) \ge 0$ is the primitive of ϕ such that $\Phi(0) = 0$. Observe that this estimate is purely formal since the values $|u| = 1$ are possible.

The estimate (1.4) has an important consequence. With the change of unknowns $u = g(v)$, where g is the inverse of ϕ, i.e. $g : \mathbb{R} \to (-1, 1)$ is given by

$$g(s) := \frac{e^{2s/\gamma} - 1}{e^{2s/\gamma} + 1}, \tag{1.5}$$

the system (1.1)-(1.2) becomes, for $|u| < 1$,

$$g(v)_t - (\nu g'(v)v_z - (1 - g(v)^2)\theta_z)_z = 0, \tag{1.6}$$

$$\theta_t - (\gamma g'(v)v_z + \theta_z)_z + \theta = \mu g(v). \tag{1.7}$$

Since $\gamma g' = 1 - g^2$, the diffusion matrix of the transformed problem

$$B := \begin{pmatrix} \nu g'(v) & -(1 - g(v)^2) \\ \gamma g'(v) & 1 \end{pmatrix} \tag{1.8}$$

satisfies for *any* values of $\nu > 0$ and $\gamma > 0$:

$$(x, y)B(x, y)^\top = \nu g'(v)|x|^2 + |y|^2 \ge 0 \quad \forall x, y \in \mathbb{R}.$$

The fact that the above transformation of variables leads to a system of elliptic equations for all values of the parameters can be related to some analytical work on more general equations. Indeed, this fact is in some sense related to the equivalence between the existence of an entropy and the symmetrizability of hyperbolic conservation laws or parabolic systems [8, 15]. Using the definition of the (generalized) 'entropy'

$$\eta(s) := g(s)s - \chi(s) + \chi(0) \tag{1.9}$$

from [4] (first used in [1]), where $\chi' = g$, gives $\eta(v) = \Phi(g(v)) = \Phi(u)$, with Φ as above. In this sense, the functional $\Phi(u(t)) + \theta(t)^2/2$ can be interpreted as an 'entropy' for the system (1.1)-(1.2) as long as $|u| < 1$. However, notice that the matrix B is *not* symmetric but satisfies the inequality $x^\top Bx > 0$ for all $x \neq 0$, which is sufficient for our existence analysis. The question if this observation leads to an existence theory for elliptic systems with general full diffusion matrices is under investigation [12].

In order to make the above 'entropy' estimate rigorous, we have to overcome the difficulties near the points where $|u| = 1$. For the transformed problem (1.6)-(1.7) this difficulty translates into the fact that the matrix B does not satisfies the *uniform* positive definiteness condition. Therefore, we have to approximate (1.6)-(1.7) appropriately, see section 2.

Our main existence result is as follows:

THEOREM 1.1. *Let γ, $\nu > 0$, $\mu \geq 0$ and u_0, $\theta_0 \in L^2(\Omega)$ with $-1 \leq u_0 \leq 1$ in Ω. For any $T > 0$, there exists a weak solution (u, θ) of (1.1)-(1.2) such that*

$$\begin{aligned} u, \theta \in H^1(0,T;(H^1_{\text{per}}(\Omega))') \cap L^2(0,T;H^1_{\text{per}}(\Omega)), \\ -1 \leq u \leq 1 \quad \text{in } Q_T = \Omega \times (0,T). \end{aligned} \tag{1.10}$$

As explained above, the main difficulties of the proof of this theorem are that the system (1.1)-(1.2) is generally not positive definite and no maximum principle to show $|u| \leq 1$ is available. Nevertheless, we are able to prove the existence of solutions for *any* values of ν and γ and thus for *arbitrary* large cross-diffusion.

The proof consists of three steps. First, instead of using the transformation g, we make a change of unknowns which takes into account the singular points $|u| = 1$ (section 2.1). Then the parabolic problem is discretized in time by a recursive sequence of elliptic equations which can be solved each by Schauder's fixed point theorem (section 2.2). Finally, a priori bounds independent of the time discretization parameter are obtained from an inequality similar to (1.4), and standard compactness results lead to the existence of a solution of the original problem (1.1)-(1.2) (section 2.3). The bound on u can be proved by using Stampacchia's truncation method in the approximate problem.

We notice that for $\gamma = 0$, the diffusion matrix for (1.1)-(1.2) becomes tridiagonal and thus, the problem can be solved by methods, for instance, employed in chemotaxis problems [11].

Besides of the existence analysis we show two additional results. We prove the uniqueness of solutions in a slightly smaller class of functions if the cross-diffusion is not too large (section 3).

THEOREM 1.2. *Let $\gamma < 4\nu$. Then, under the assumptions of Theorem 1.1 there exists at most one solution (u, θ) of (1.1)-(1.2) in the class of functions satisfying (1.10) and $\theta \in L^\infty(0,T;H^1_{\text{per}}(\Omega))$.*

Furthermore, we derive a sufficient condition on the parameters in order to get non-segregation, i.e. convergence of the transient solutions to the constant steady-state given by

$$\bar{u} = \frac{1}{L}\int_0^L u_0(z)dz, \quad \bar{\theta} = \frac{1}{L}\int_0^L \theta_0(z)dz.$$

The rate of convergence turns out to be exponential (Section 4).

THEOREM 1.3. *Let the assumptions of Theorem 1.1 hold and assume that $|u_0| \leq c < 1$ in Ω for some $c < 1$, $\mu\bar{u} = \bar{\theta}$ and*

$$\frac{\nu\gamma}{\mu^2} > \frac{L^4}{8(L^2+1)}. \tag{1.11}$$

Then there exist constants $c_0 > 0$, depending on u_0, θ_0, and constants $\delta_1, \delta_2 > 0$, depending on the parameters, such that for all $t > 0$,

$$\|u(t) - \bar{u}\|_{L^2(\Omega)} \le c_0 e^{-\delta_1 t}, \quad \|\theta(t) - \bar{\theta}\|_{L^2(\Omega)} \le c_0 e^{-\delta_2 t}.$$

The constants c_0 and δ_1, δ_2 are defined in (4.1) and (4.4), respectively. The proof of the above result is based on careful estimates using the 'entropy' (1.9). Aranson et al. [3] have motivated from linear stability theory that the condition $\mu > \nu$ is necessary to have size segregation. The assumption (1.11) shows that the condition $\mu > \nu$ needs *not* to be sufficient. In fact, there are parameter values for which *both* $\mu > \nu$ *and* (1.11) hold, i.e., the granular materials are not segregating (see section 5).

Clearly, the dynamics of granular segregation pattern is of much larger interest for the applications than non-segregation conditions. Therefore, our result has to be understood as a first step in the understanding of the segregation dynamics.

Finally, we present in Section 5 some numerical examples illustrating the segregation or non-segregation behavior.

2. Proof of Theorem 1.1.

2.1. Ideas of the proof. In this section we present and explain the approximations needed in the proof of Theorem 1.1. As already mentioned in the introduction, the function g provides an 'entropy' estimate only if $|u| < 1$. Since $u = \pm 1$ is possible, we use another change of unknowns which includes the points $u = \pm 1$. Let the assumptions of Theorem 1.1 hold and let $\alpha > 1$. Define the transformation $u = g_\alpha(v)$ with $g_\alpha : [-s_\alpha, s_\alpha] \to [-1, 1]$, given by

$$g_\alpha(s) := \alpha \frac{e^{2\alpha s/\gamma} - 1}{e^{2\alpha s/\gamma} + 1} \quad \text{and} \quad s_\alpha := \frac{\gamma}{2\alpha} \log \frac{\alpha + 1}{\alpha - 1}. \tag{2.1}$$

Observe that for $\alpha \to 1$, g_α equals g on $\mathbb{R}$, see (1.5). As the range of g_α is $[-1, 1]$, the critical points $u = \pm 1$ are included in that transformation. In the following we fix some $\alpha > 1$ and write again g for g_α.

With this change of unknown we obtain the system (1.6)-(1.7), with periodic boundary conditions for v and θ and initial conditions

$$v(\cdot, 0) = v_0 := g^{-1}(u_0), \quad \theta(\cdot, 0) = \theta_0 \qquad \text{in } \Omega. \tag{2.2}$$

The new diffusion matrix B is given by (1.8). It holds for any $(x, y) \in \mathbb{R}^2$

$$\begin{aligned}(x, y)B(x, y)^\top &= \nu g'(v)x^2 + y^2 + \left(\gamma g'(v) - (1 - g(v)^2)\right) xy \\ &= \nu g'(v)x^2 + y^2 + (\alpha^2 - 1)xy.\end{aligned}$$

Clearly, if $\alpha = 1$ the matrix satisfies $x^\top Bx > 0$ for all $x \neq 0$, and it seems reasonable that this will be also the case for $\alpha > 1$ sufficiently close to one. In fact, let (v, θ) be a weak solution to (1.1)-(1.2) and use v and θ as test functions in the weak formulation of (1.6)-(1.7), respectively, to obtain the identity

$$\begin{aligned}&\int_\Omega \left(G(v(t)) + \frac{1}{2}\theta(t)^2 \right) dz + \int_0^t \int_\Omega \left(\nu g'(v)^2 v_z^2 + \theta_z^2 + \theta^2 \right) dzdt \\ &\qquad = \int_\Omega \left(G(v_0) + \frac{1}{2}\theta_0^2 \right) dz - (\alpha^2 - 1) \int_0^t \int_\Omega v_z \theta_z dzdt + \int_0^t \int_\Omega \mu g(v)\theta dzdt,\end{aligned}$$

where G is defined by $G'(s) = sg'(s)$ and $G(0) = 0$, i.e.

$$G(s) = \frac{2\alpha s}{\gamma} \frac{e^{2\alpha s/\gamma}}{e^{2\alpha s/\gamma}+1} + \log \frac{2}{e^{2\alpha s/\gamma}+1}. \tag{2.3}$$

Since $|g|$ is bounded by one and $g' \geq (\alpha^2-1)/\gamma$ in $[-s_\alpha, s_\alpha]$, see Lemma 2.2, we can estimate

$$\begin{aligned} &\int_\Omega \left(G(v(t)) + \frac{1}{2}\theta(t)^2 \right) dz + \int_0^t \int_\Omega \left(\frac{\nu}{\gamma}(\alpha^2-1)v_z^2 + \theta_z^2 \right) dzdt \\ &\leq \int_\Omega \left(G(v_0) + \frac{1}{2}\theta_0^2 \right) dz - (\alpha^2-1) \int_0^t \int_\Omega v_z \theta_z dzdt + \int_0^t \int_\Omega (\mu|\theta| - \theta^2) dzdt, \end{aligned} \tag{2.4}$$

as long as $-s_\alpha \leq v \leq s_\alpha$ in Q_t. Choosing $\alpha > 1$ small enough and applying Young's inequality, it is possible to control the second integral on the right-hand side by the integrals on the left-hand side. This gives the estimates $v_z \in L^2(0,T;L^2(\Omega))$ and $\theta \in L^2(0,T;H^1_{\rm per}(\Omega))$. The inequality (2.4) is made rigorous in Lemma 2.6 for a time-discretized version of (1.6)-(1.7).

Still there remain two difficulties: the elliptic operator corresponding to (1.6)-(1.7) is not uniformly elliptic (since g' is only positive, but not uniformly positive in $\mathbb{R}$), and we have to deal with time derivatives in $g(v)$ (instead of having time *and* space derivatives in v). The first difficulty can be overcome by adding a small number $\varepsilon > 0$ to the diffusion term containing $\nu g'(v)$ and to pass to the limit $\varepsilon \to 0$ after solving the approximate problem. To overcome the second difficulty we approximate the system by a semi-discrete problem in time (backward Euler method). This method is also interesting from a numerical point of view, see, e.g., [13].

2.2. A semi-discrete problem. The main objective of this section is to prove that for given $\tau > 0$ and $(\tilde{w}, \tilde{\theta}) \in (H^1_{\rm per}(\Omega))^2$, there exists a solution $(w,\xi) \in (H^1_{\rm per}(\Omega))^2$, satisfying $-s_\alpha \leq w \leq s_\alpha$ in Ω, of the problem

$$\frac{1}{\tau}(g(w) - g(\tilde{w})) - \left(\nu g'(w)w_z - (1-g(w)^2)\xi_z\right)_z = 0, \tag{2.5}$$

$$\frac{1}{\tau}(\xi - \tilde{\theta}) - (\gamma g'(w)w_z + \xi_z)_z + \xi = \mu g(w) \quad \text{in } \Omega. \tag{2.6}$$

This system is a time-discretized version of (1.6)-(1.7). The function $g(s)$ is defined as in (2.1) but we allow for arguments $s \in \mathbb{R}$. We shall use the following notion of weak solution.

Definition 2.1. *The pair (w,ξ) is called a* weak solution *of (2.5)-(2.6) if $(w,\xi) \in (H^1_{\rm per}(\Omega))^2$, $-s_\alpha \leq w \leq s_\alpha$ in Ω, the initial conditions in (1.3) are satisfied in the sense of $(H^1_{\rm per}(\Omega))'$, and for every $(\varphi,\psi) \in (H^1_{\rm per}(\Omega))^2$ we have*

$$\frac{1}{\tau}\int_\Omega (g(w) - g(\tilde{w}))\varphi dz + \int_\Omega \left(\nu g'(w)w_z - (1-g(w)^2)\xi_z\right)\varphi_z dz = 0, \tag{2.7}$$

$$\frac{1}{\tau}\int_\Omega (\xi - \tilde{\theta})\psi dz + \int_\Omega (\gamma g'(w)w_z + \xi_z)\psi_z dz + \int_\Omega \xi\psi dz = \mu \int_\Omega g(w)\psi dz. \tag{2.8}$$

As explained in Section 2.1, we approximate the system (2.5)-(2.6) by a system where an additional ellipticity constant $\varepsilon > 0$ is introduced: Find $(w,\xi) \in (H^1_{\rm per}(\Omega))^2$

such that in Ω

$$\frac{1}{\tau}(g(w)-g(\tilde{w}))-\big((\nu g'(w)+\varepsilon)w_z-(1-g(w)^2)_+\xi_z\big)_z+\varepsilon w=0, \tag{2.9}$$

$$\frac{1}{\tau}(\xi-\tilde{\theta})-(\gamma g'(w)w_z+\xi_z)_z+\xi=\mu g(w), \tag{2.10}$$

where $s_+=\max\{0,s\}$.

The following properties of the function g can be easily shown.

LEMMA 2.2. *The function $g:\mathbb{R}\to(-\alpha,\alpha)$ defined by (2.1) satisfies $g\in C^\infty(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$ and*

$$0<g'\le \alpha^2/\gamma \quad \text{in } \mathbb{R}, \qquad g'\ge(\alpha^2-1)/\gamma \quad \text{in } [-s_\alpha,s_\alpha]. \tag{2.11}$$

Fix $\alpha>1$ such that $2(\alpha^2-1)\le\nu/2\gamma$ and define $h_1,h_2:\mathbb{R}\to\mathbb{R}$ by

$$h_1:=\nu g'-\delta|\gamma g'-(1-g^2)_+|, \quad h_2:=1-\frac{1}{\delta}|\gamma g'-(1-g^2)_+|,$$

with $2(\alpha^2-1)\le\delta\le\nu/2\gamma$. Then

$$h_1>0, \quad h_2\ge 1/2 \quad \text{in } \mathbb{R}, \quad \text{and} \quad h_1\ge\frac{\nu}{2\gamma}(\alpha^2-1) \quad \text{in } [-s_\alpha,s_\alpha]. \tag{2.12}$$

We prove the existence of a solution of (2.9)-(2.10) using Schauder's fixed point theorem. In order to define the fixed-point operator, we consider first the following linearized problem: Let $(\hat{w},\hat{\xi})\in(L^2(\Omega))^2$ be given and find $(w,\xi)\in(H^1_{\rm per}(\Omega))^2$ such that

$$-\big((\nu g'(\hat{w})+\varepsilon)w_z-(1-g(\hat{w})^2)_+\xi_z\big)_z+\varepsilon w=\frac{1}{\tau}(g(\tilde{w})-g(\hat{w})), \tag{2.13}$$

$$-(\gamma g'(\hat{w})w_z+\xi_z)_z+\xi=\mu g(\hat{w})+\frac{1}{\tau}(\tilde{\theta}-\hat{\xi}) \tag{2.14}$$

in Ω. The definition of a weak solution of problem (2.13)-(2.14) is similar to Definition 2.1.

LEMMA 2.3. *Let $(\tilde{w},\tilde{\theta})\in(H^1_{\rm per}(\Omega))^2$ and $(\hat{w},\hat{\xi})\in(L^2(\Omega))^2$ be given. Then there exists a unique weak solution of problem (2.13)-(2.14).*

Proof. We define the bilinear form $a:(H^1_{\rm per}(\Omega))^2\times(H^1_{\rm per}(\Omega))^2\to\mathbb{R}$,

$$a((w,\xi),(\varphi,\psi)):=\int_\Omega\big[\big((\nu g'(\hat{w})+\varepsilon)w_z-(1-g(\hat{w})^2)_+\xi_z\big)\varphi_z+\varepsilon w\varphi\big]\,dz + \int_\Omega\big((\gamma g'(\hat{w})w_z+\xi_z)\psi_z+\xi\psi\big)dz,$$

and the linear functional $f:(L^2(\Omega))^2\to\mathbb{R}$,

$$f(\varphi,\psi):=\frac{1}{\tau}\int_\Omega\big((g(\tilde{w})-g(\hat{w}))\varphi+(\tilde{\theta}-\hat{\xi})\psi\big)+\mu\int_\Omega g(\hat{w})\psi.$$

In order to apply the Lemma of Lax-Milgram, we have to check that a is continuous and coercive in $(H^1_{\rm per}(\Omega))^2\times(H^1_{\rm per}(\Omega))^2$ and that f is continuous in $(L^2(\Omega))^2$. The continuity of a and f follows easily from the pointwise bounds of g and g' and the regularity of $\tilde{w}$, $\tilde{\theta}$, $\hat{w}$, and $\hat{\xi}$. For the coercivity of a we estimate

$$\begin{aligned} a((w,\xi),(w,\xi)) &= \int_\Omega \big((\nu g'(\hat{w})+\varepsilon)|w_z|^2 + |\xi_z|^2 + \varepsilon|w|^2 + |\xi|^2\big)dz \\ &\quad + \int_\Omega \big((\gamma g'(\hat{w}) - (1-g(\hat{w})^2)_+)w_z\xi_z\big)dz \\ &\geq \int_\Omega \big((\varepsilon + h_1(\hat{w}))|w_z|^2 + h_2(\hat{w})|\xi_z|^2 + \varepsilon|w|^2 + |\xi|^2\big)dz, \end{aligned}$$

using Young's inequality, where the functions h_1 and h_2 are defined in Lemma 2.2. The bounds (2.12) then imply that

$$a((w,\xi),(w,\xi)) \geq \min\{\varepsilon, 1/2\}\left(\|w\|^2_{H^1_{\rm per}(\Omega)} + \|\xi\|^2_{H^1_{\rm per}(\Omega)}\right),$$

and the coercivity of a is proved. □

LEMMA 2.4. *Let $(\tilde{w},\tilde{\theta}) \in (H^1_{\rm per}(\Omega))^2$. Then there exists a unique weak solution of problem (2.9)-(2.10).*

Proof. We use the Schauder fixed point theorem. For this define the map $S : (L^2(\Omega))^2 \to (L^2(\Omega))^2$ by $S(\hat{w},\hat{\xi}) = (w,\xi)$, where (w,ξ) is the weak solution of (2.13)-(2.14). We have to check that S is continuous and compact and that the set

$$\Lambda := \left\{u \in (L^2(\Omega))^2 : u = \lambda S(u)\right\},$$

for $\lambda \in [0,1]$, is bounded. The continuity of S follows by standard arguments. The compactness of S is just a consequence of the compactness of the embedding $H^1_{\rm per}(\Omega) \subset L^2(\Omega)$.

It remains to show that Λ is bounded. If $\lambda = 0$ then $\Lambda = \{(0,0)\}$ is trivially bounded. For $\lambda \in (0,1]$, the equation $S(\hat{w},\hat{\xi}) = \frac{1}{\lambda}(\hat{w},\hat{\xi})$ is equivalent to

$$\int_\Omega \Big(\big((\nu g'(\hat{w})+\varepsilon)\hat{w}_z - (1-g(\hat{w})^2)_+\hat{\xi}_z\big)_z\varphi_z + \varepsilon\hat{w}\varphi\Big)\,dz = \frac{\lambda}{\tau}\int_\Omega (g(\tilde{w}) - g(\hat{w}))\varphi dz,$$

$$\int_\Omega \Big((\gamma g'(\hat{w})\hat{w}_z + \hat{\xi}_z)\psi_z + \hat{\xi}\psi\Big)\,dz = \lambda\int_\Omega \left(\mu g(\hat{w}) + \frac{1}{\tau}(\tilde{\theta} - \hat{\xi})\right)\psi dz$$

Using $(\varphi,\psi) = (\hat{w},\hat{\xi})$ as a test function, adding the resulting integral identities and applying Young's inequality as in (2.12), we obtain

$$\begin{aligned} \int_\Omega \big((\varepsilon + h_1(\hat{w}))|\hat{w}_z|^2 + h_2(\hat{w})|\hat{\xi}_z|^2 + \varepsilon|\hat{w}|^2 + |\hat{\xi}|^2)\big)dz &= \frac{\lambda}{\tau}\int_\Omega (g(\tilde{w}) - g(\hat{w}))\hat{w}dz \\ &\quad + \lambda\int_\Omega \big(\mu g(\hat{w}) + \frac{1}{\tau}(\tilde{\theta} - \hat{\xi})\big)\hat{\xi}dz. \end{aligned}$$

Using again Young's inequality on the right-hand side of this equation and employing the estimate (2.12), we deduce

$$\begin{aligned} \int_\Omega \big(\varepsilon(|\hat{w}_z|^2 + |\hat{w}|^2) + |\hat{\xi}_z|^2 + |\hat{\xi}|^2\big)dz &\leq \frac{\lambda^2}{\tau^2\varepsilon}\int_\Omega (g(\tilde{w}) - g(\hat{w}))^2dz + \frac{2\lambda^2}{\tau^2}\int_\Omega \tilde{\theta}^2dz \\ &\quad + 2(\lambda\mu)^2\int_\Omega |g(\hat{w})|^2dz, \end{aligned}$$

and since $g \in L^\infty(\mathbb{R})$, the assertion follows. □

In the following we derive uniform bounds for the solution of (2.9)-(2.10) which allow to pass to the limit $\varepsilon \to 0$. This proves the existence of a solution of (2.5)-(2.6). We need the following auxiliary result whose proof is standard.

LEMMA 2.5. *Let $\varphi \in C(\mathbb{R})$ be non-decreasing with $\varphi(0) = 0$ and define $\Phi \in C^1(\mathbb{R})$ by $\Phi(s) := \int_0^s g'(\sigma)\varphi(\sigma)d\sigma$. Then it holds for all $s, t \in \mathbb{R}$*

$$\Phi(s) - \Phi(t) \le (g(s) - g(t))\varphi(s). \tag{2.15}$$

LEMMA 2.6. *Let $(\tilde{w}, \tilde{\xi}) \in (H^1_{\rm per}(\Omega))^2$ be such that $-s_\alpha \le \tilde{w} \le s_\alpha$ in Ω and let $(w_\varepsilon, \xi_\varepsilon) \in (H^1_{\rm per}(\Omega))^2$ be a solution of (2.9)-(2.10). Then the following estimates hold:*

$$-s_\alpha \le w_\varepsilon \le s_\alpha \quad \text{in } \Omega, \tag{2.16}$$

$$\begin{aligned}&\int_\Omega \Big(G(w_\varepsilon) + \frac{1}{2}\xi_\varepsilon^2\Big)dz + C\tau \int_\Omega (|w_{\varepsilon z}|^2 + |\xi_{\varepsilon z}|^2 + |\xi_\varepsilon|^2)dz \\ &\quad \le \int_\Omega \Big(G(\tilde{w}) + \frac{1}{2}\tilde{\xi}^2\Big)dz + C'\tau,\end{aligned} \tag{2.17}$$

for some positive constants C, C' independent of ε and τ, and for G defined in (2.3).

In addition, there exists a subsequence of $(w_\varepsilon, \xi_\varepsilon)$ (not relabeled) such that $(w_\varepsilon, \xi_\varepsilon) \rightharpoonup (w, \xi)$ weakly in $(H^1_{\rm per}(\Omega))^2$ and strongly in $(L^2(\Omega))^2$ as $\varepsilon \to 0$, and (w, ξ) is a weak solution of problem (2.5)-(2.6).

Proof. We use $\varphi(w_\varepsilon) := \max(w_\varepsilon - s_\alpha, 0)$ as a test function in the weak formulation of (2.9). Since φ is increasing and $\varphi(0) = 0$ we can employ Lemma 2.5. Let Φ be defined as in Lemma 2.5. Then, together with the identities $(1 - g(s)^2)_+\varphi'(s) = 0$ for all $s \in \mathbb{R}$ and $\Phi(\tilde{w}) = 0$, we obtain

$$0 \ge \frac{1}{\tau}\int_\Omega (g(w_\varepsilon) - g(\tilde{w}))\varphi(w_\varepsilon)dx \ge \int_\Omega (\Phi(w_\varepsilon) - \Phi(\tilde{w}))dx = \int_\Omega \Phi(w_\varepsilon)dx.$$

This implies $\Phi(w_\varepsilon) = 0$ and therefore $w_\varepsilon \le s_\alpha$ in Ω. In a similar way we deduce $w_\varepsilon \ge -s_\alpha$ in Ω. Observe that these bounds imply that $(1 - g(w_\varepsilon)^2)_+ = 1 - g(w_\varepsilon)^2$ in Ω.

Now we use $(w_\varepsilon, \xi_\varepsilon)$ as a test function in the weak formulation of problem (2.9)-(2.10). Adding the corresponding integral identities and using property (2.15) we get, after multiplication by τ,

$$\begin{aligned}&\int_\Omega \Big(G(w_\varepsilon) + \frac{1}{2}\xi_\varepsilon^2\Big)dz + \tau\int_\Omega \big(h_1(w_\varepsilon)|w_{\varepsilon z}|^2 + h_2(w_\varepsilon)|\xi_{\varepsilon z}|^2 + |\xi_\varepsilon|^2\big)dz \\ &\qquad\qquad \le \mu\tau\int_\Omega g(w_\varepsilon)\xi_\varepsilon dz + \int_\Omega \Big(G(\tilde{w}) + \frac{1}{2}\tilde{\xi}^2\Big)dz.\end{aligned}$$

Applying Young's inequality and the bounds (2.11) and (2.12) for g', h_1 and h_2, we deduce (2.17).

Finally, the uniform estimates (2.16) and (2.17) imply the existence of a subsequence (not relabeled) of $(w_\varepsilon, \xi_\varepsilon)$ and of a pair $(w, \xi) \in (H^1_{\rm per}(\Omega))^2$ such that, as $\varepsilon \to 0$,

$$w_\varepsilon \overset{*}{\rightharpoonup} w \quad \text{weakly* in } L^\infty(\Omega), \tag{2.18}$$

$$w_{\varepsilon z} \rightharpoonup w_z \quad \text{weakly in } L^2(\Omega), \tag{2.19}$$

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{weakly in } H^1_{\rm per}(\Omega).$$

In fact, the convergences (2.18) and (2.19) imply $w_\varepsilon \rightharpoonup w$ weakly in $H^1_{\rm per}(\Omega)$ and thus, by the compactness of the embedding $H^1_{\rm per}(\Omega) \subset L^2(\Omega)$, we deduce for a subsequence, as $\varepsilon \to 0$, $w_\varepsilon \to w$ and $\xi_\varepsilon \to \xi$ strongly in $L^2(\Omega)$ and a.e. in Ω. These convergence results and the continuity of g and g' allow us to pass to the limit $\varepsilon \to 0$ in the weak formulation of problem (2.9)-(2.10) and to identify (w, ξ) as a weak solution of (2.5)-(2.6). □

2.3. End of the proof of Theorem 1.1. Let $T > 0$ and $N \in \mathbb{N}$ be given and let $\tau = T/N$ be the time step. We define recursively pairs $(v^k, \theta^k) \in (H^1_{\rm per}(\Omega))^2$, $k = 1, \ldots, N$, as the weak solution of the problem (2.5)-(2.6) corresponding to the data $(\tilde w, \tilde\theta) = (v^{k-1}, \theta^{k-1})$, and with $(v^0, \theta^0) = (v_0, \theta_0)$. Then we define the piecewise constant functions

$$v^\tau(x,t) := v^k(x) \quad \text{and} \quad \theta^\tau(x,t) := \theta^k(x) \quad \text{if } (x,t) \in \Omega \times ((k-1)\tau, k\tau],$$

for $k = 1, \ldots, N$, and introduce the discrete entropies

$$\eta^k := \int_\Omega \Big(G(v^k) + \frac{1}{2}|\theta^k|^2 \Big) dz, \quad \eta^\tau(t) := \int_\Omega \Big(G(v^\tau(\cdot,t)) + \frac{1}{2}|\theta^\tau(\cdot,t)|^2 \Big) dz. \tag{2.20}$$

We have the following consequence of Lemma 2.6.

Corollary 2.7. *There exist uniform bounds with respect to τ for the norms*

$$\|\eta^\tau\|_{L^\infty(0,T)}, \ \|v^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}, \ \|g(v^\tau)\|_{L^2(0,T;H^1_{\rm per}(\Omega))} \ \text{and} \ \|\theta^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}.$$

In addition,

$$-s_\alpha \le v^\tau \le s_\alpha \quad \text{in } Q_T = \Omega \times (0,T). \tag{2.21}$$

Proof. From the 'entropy' inequality (2.17) we obtain

$$\eta^m - \eta^0 = \sum_{k=1}^m (\eta^k - \eta^{k-1}) \le C' m\tau - C\tau \sum_{k=1}^m \int_\Omega (|v^k_z|^2 + |\theta^k_z|^2 + |\theta^k|^2) dz,$$

for $m = 1, \ldots, N$. Taking the maximum over m yields

$$\|\eta^\tau\|_{L^\infty(0,T)} + C \int_{Q_T} \big(|v^\tau_z|^2 + |\theta^\tau_z|^2 + |\theta^\tau|^2 \big) dz dt \le \eta^0 + C'T.$$

Since both g and g' are smooth and bounded we also deduce the estimate for the norm $\|g(v^\tau)\|_{L^2(0,T;H^1_{\rm per}(\Omega))}$. Finally, (2.21) follows directly from (2.16). □

We need uniform estimates of the time derivatives. For this, we introduce the shift operator and linear interpolations in time. For $t \in ((k-1)\tau, k\tau]$, $k = 1, \ldots, N$, we define $\sigma_\tau v^\tau(\cdot,t) := v^{k-1}$ and $\sigma_\tau \theta^\tau(\cdot,t) := \theta^{k-1}$ in Ω. Setting $\delta t := (t/\tau - (k-1)) \in [0,1]$, we introduce

$$\tilde g^\tau := g(\sigma_\tau v^\tau) + \delta t \big(g(v^\tau) - g(\sigma_\tau v^\tau) \big), \quad \tilde\theta^\tau := \sigma_\tau \theta^\tau + \delta t \big(\theta^\tau - \sigma_\tau \theta^\tau \big) \tag{2.22}$$

in Q_T.

LEMMA 2.8. *There exist uniform bounds with respect to τ for the norms*

$$\|\tilde{g}_t^\tau\|_{L^2(0,T;(H^1_{\rm per}(\Omega))')}, \quad \|\tilde{g}^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))\cap L^\infty(Q_T)},$$
$$\|\tilde{\theta}_t^\tau\|_{L^2(0,T;(H^1_{\rm per}(\Omega))')} \quad \text{and} \quad \|\tilde{\theta}^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}.$$

Proof. From the definition (2.22) of $\tilde{g}^\tau$ and equation (2.5) we compute

$$\tilde{g}_t^\tau = \frac{1}{\tau}\big(g(v^\tau) - g(\sigma_\tau v^\tau)\big) = \big(\nu g'(v^\tau)v_z^\tau - (1-g(v^\tau)^2)\theta_z^\tau\big)_z.$$

Using the boundedness of g' in $\mathbb{R}$ and Corollary 2.7 we obtain a uniform bound for $\|\tilde{g}_t^\tau\|_{L^2((0,T;H^1_{\rm per})')}$. Moreover, since g is bounded, it is clear that $\tilde{g}^\tau \in L^\infty(Q_T)$ for any $\tau \geq 0$. We also have

$$\tilde{g}_z^\tau = \delta t g'(v^\tau)v_z^\tau + (1-\delta t)g'(\sigma_\tau v^\tau)(\sigma_\tau v^\tau)_z. \tag{2.23}$$

Since $(\sigma_\tau v^\tau)_z = \sigma_\tau v_z^\tau$, the $L^\infty(Q_T)$ bound for $\tilde{g}^\tau$ together with (2.23) and Corollary 2.7 implies a uniform bound for $\|\tilde{g}^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}$. In a similar way we obtain uniform estimates for $\tilde{\theta}^\tau$. □

Proof of Theorem 1.1. The functions v^τ, θ^τ, $\tilde{g}^\tau$, $\tilde{\theta}^\tau$ satisfy the weak formulation

$$\int_0^T \langle \tilde{g}_t^\tau, \varphi\rangle dt + \int_{Q_T} \big(\nu g'(v^\tau)v_z^\tau - (1-g(v^\tau)^2)\theta_z^\tau\big)\varphi_z dzdt = 0, \tag{2.24}$$

$$\int_0^T \langle \tilde{\theta}_t, \psi\rangle dt + \int_{Q_T} \big(\gamma g'(v^\tau)v_z^\tau + \theta_z^\tau\big)\psi_z dzdt + \int_{Q_T} \theta^\tau \psi dydt = \mu \int_{Q_T} g(v^\tau)\psi dzdt, \tag{2.25}$$

for any $\varphi, \psi \in L^2(0,T;H^1_{\rm per}(\Omega))$. The estimates of Lemma 2.8 allow us to extract a subsequence (not relabeled) such that, as $\tau \to 0$,

$$\tilde{g}_t^\tau \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;(H^1_{\rm per}(\Omega))'), \tag{2.26}$$
$$\tilde{g}^\tau \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)), \tag{2.27}$$
$$\tilde{g}^\tau \overset{*}{\rightharpoonup} u \quad \text{weakly* in } L^\infty(Q_T),$$
$$\tilde{\theta}_t^\tau \rightharpoonup \theta_t \quad \text{weakly in } L^2(0,T;(H^1_{\rm per}(\Omega))'), \tag{2.28}$$
$$\tilde{\theta}^\tau \rightharpoonup \theta \quad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)). \tag{2.29}$$

The compact embedding $H^1_{\rm per}(\Omega) \subset L^\infty(\Omega)$, the convergence results (2.26)-(2.29) and Aubin's lemma [23] imply, up to a subsequence,

$$\tilde{g}^\tau \to u \quad \text{strongly in } L^2(0,T;L^\infty(\Omega)), \tag{2.30}$$
$$\tilde{\theta}^\tau \to \theta \quad \text{strongly in } L^2(0,T;L^\infty(\Omega)).$$

Moreover, Corollary 2.7 yields the existence of a subsequence such that

$$v^\tau \rightharpoonup v \quad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)),$$
$$v^\tau \overset{*}{\rightharpoonup} v \quad \text{weakly* in } L^\infty(Q_T),$$
$$g(v^\tau) \rightharpoonup \hat{u} \quad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)), \tag{2.31}$$
$$\theta^\tau \rightharpoonup \hat{\theta} \quad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)).$$

It holds $\tilde{g}^\tau - g(v^\tau) = \tau(\delta t - 1)\tilde{g}^\tau_t$, and therefore, by Lemma 2.8,

$$\|\tilde{g}^\tau - g(v^\tau)\|_{L^2(0,T;(H^1_{\rm per})')} \to 0 \quad \text{as } \tau \to 0. \tag{2.32}$$

Hence, $u = \hat{u}$. In a similar way we obtain $\theta = \hat{\theta}$. Finally,

$$\begin{aligned}
&\|g(v^\tau) - u\|_{L^1(0,T;L^2(\Omega))} \\
&\le \|g(v^\tau) - \tilde{g}^\tau\|_{L^1(0,T;L^2(\Omega))} + \|\tilde{g}^\tau - u\|_{L^1(0,T;L^2(\Omega))} \\
&\le \|g(v^\tau) - \tilde{g}^\tau\|^{1/2}_{L^1(0,T;(H^1_{\rm per}(\Omega))')}\|g(v^\tau) - \tilde{g}^\tau\|^{1/2}_{L^1(0,T;H^1_{\rm per}(\Omega))} \\
&\quad + \|\tilde{g}^\tau - u\|_{L^1(0,T;L^2(\Omega))} \\
&\le C\|g(v^\tau) - \tilde{g}^\tau\|^{1/2}_{L^2(0,T;(H^1_{\rm per}(\Omega))')} + \|\tilde{g}^\tau - u\|_{L^1(0,T;L^2(\Omega))} \\
&\to 0,
\end{aligned} \tag{2.33}$$

as $\tau \to 0$. Therefore, $g(v^\tau) \to u$ strongly in $L^1(0,T;L^2(\Omega))$ and a.e. in Q_T. Now, letting $\tau \to 0$ in (2.24)-(2.25), we obtain, for $\varphi, \psi \in L^2(0,T;H^1_{\rm per}(\Omega))$,

$$\int_0^T \langle u_t, \varphi\rangle dt + \int_{Q_T} \big((\nu u_z - (1-u^2)\theta_z\big)\varphi_z dzdt = 0, \tag{2.34}$$

$$\int_0^T \langle \theta_t, \psi\rangle dt + \int_{Q_T} \big(\gamma u_z + \theta_z\big)\psi_z dzdt + \int_{Q_T} \theta\psi dz = \mu \int_{Q_T} u\psi dzdt. \tag{2.35}$$

This proves Theorem 1.1. □

3. Proof of Theorem 1.2. Let (u_1, θ_1) and (u_2, θ_2) be two weak solutions of (1.1)-(1.3) with the same initial data, satisfying (1.10) and $\theta_1 \in L^\infty(0,T;H^1_{\rm per}(\Omega))$. Set $Q_t = \Omega \times (0,t)$. The equations satisfied by $u = u_1 - u_2$ and $\theta = \theta_1 - \theta_2$ read

$$u_t - \nu u_{zz} + \theta_{zz} = \big((u_1+u_2)u\theta_{1z} + u_2^2\theta_z\big)_z, \tag{3.1}$$

$$\theta_t - \theta_{zz} + \theta = \gamma u_{zz} + \mu u. \tag{3.2}$$

Take u and θ as test functions in the weak formulations of (3.1) and (3.2), respectively, and add (3.2), multiplied by some number $a > 0$, and (3.1) to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u(t)^2 + a\theta(t)^2)dz + \int_{Q_t} (\nu u_z^2 + a\theta_z^2 + a\theta^2)dzdt \\
&= \int_{Q_t} (1 - a\gamma - u_2^2)u_z\theta_z dzdt + a\mu\int_{Q_t} u\theta dzdt - \int_{Q_t} (u_1+u_2)u\theta_{1z}u_z dzdt.
\end{aligned} \tag{3.3}$$

We apply Young's inequality to the second integral on the right-hand side:

$$a\mu\int_{Q_t} u\theta dzdt \le \frac{a\mu^2}{2}\int_{Q_t} u^2 dzdt + \frac{a}{2}\int_{Q_t} \theta^2 dzdt.$$

For the third integral on the right-hand side of (3.3) we use the Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty(\Omega)} \le C_0\|u\|^{1/2}_{H^1(\Omega)}\|u\|^{1/2}_{L^2(\Omega)} \quad \forall u \in H^1(0,L)$$

and the Young inequality

$$x^{1/2}y^{3/2} \le \frac{\varepsilon}{2}x^2 + C(\varepsilon)y^2 \quad \forall x, y \ge 0,\ \varepsilon > 0.$$

Then, with the abbreviation $C_1 = 2C_0\|\theta_{1z}\|_{L^\infty(0,T;L^2(\Omega))} < \infty$ and $|u_1|$, $|u_2| \le 1$,

$$\begin{aligned}
&\int_{Q_t} (u_1 + u_2)u\theta_{1z}u_z dzdt \\
&\le 2\|u\|_{L^2(0,t;L^\infty(\Omega))}\|\theta_{1z}\|_{L^\infty(0,t;L^2(\Omega))}\|u_z\|_{L^2(0,t;L^2(\Omega))} \\
&\le C_1\|u\|^{1/2}_{L^2(0,t;L^2(\Omega))}\left(\|u\|^2_{L^2(Q_t)} + \|u_z\|^2_{L^2(Q_t)}\right)^{1/4}\|u_z\|_{L^2(0,t;L^2(\Omega))} \\
&\le C_1\left(\|u\|_{L^2(Q_t)}\|u_z\|_{L^2(Q_t)} + \|u\|^{1/2}_{L^2(Q_t)}\|u_z\|^{3/2}_{L^2(Q_t)}\right) \\
&\le \frac{\varepsilon}{2}\|u_z\|^2_{L^2(Q_t)} + \frac{C_1^2}{2\varepsilon}\|u\|^2_{L^2(Q_t)} + \frac{\varepsilon}{2}\|u_z\|^2_{L^2(Q_t)} + C(\varepsilon)C_1^4\|u\|^2_{L^2(Q_t)}.
\end{aligned}$$

With these inequalities we can estimate (3.3) as

$$\begin{aligned}
&\frac{1}{2}\left(\|u(t)\|^2_{L^2(\Omega)} + a\|\theta(t)\|^2_{L^2(\Omega)}\right) + \frac{a}{2}\|\theta\|^2_{L^2(Q_t)} \\
&\le -\int_{Q_t}\left(-(|1 - a\gamma| + 1)|u_z||\theta_z| + (\nu - \varepsilon)u_z^2 + a\theta_z^2\right) \\
&\quad + \left(\frac{a\mu^2}{2} + \frac{C_1^2}{2\varepsilon} + C(\varepsilon)C_1^4\right)\|u\|^2_{L^2(Q_t)}. \qquad (3.4)
\end{aligned}$$

It can be easily seen that the quadratic form

$$A(x, y) = -(|1 - a\gamma| + 1)xy + (\nu - \varepsilon)x^2 + ay^2, \quad x, y \ge 0,$$

is positive definite if we choose $a = 1/\gamma$ and $\varepsilon = \nu - \gamma/4 > 0$ (since $\gamma < 4\nu$ by assumption). Then Gronwall's lemma applied to (3.4) implies that $u(t) = \theta(t) = 0$ in Ω for any $t > 0$. This proves Theorem 1.2. ☐

4. Proof of Theorem 1.3. Let (u, θ) be a weak solution of (1.1)-(1.3) given by Theorem 1.1. Let $\alpha > 1$ and set

$$c_0 = \frac{1}{2}\int_0^L \left(\gamma(u_0 + 1)\ln\frac{1 + u_0}{1 + \bar{u}} + \gamma(1 - u_0)\ln\frac{1 - u_0}{1 - \bar{u}} + (\theta_0 - \bar{\theta})\right)dz. \qquad (4.1)$$

Notice that c_0 is well defined even if $u_0(z) = \pm 1$. For the proof of Theorem 1.3 we need the following lemma:

LEMMA 4.1. *Define the function* $\psi : [-1, 1] \to \mathbb{R}$ *by*

$$\psi(u) = \frac{\gamma}{2\alpha}\ln\left(\frac{\alpha + u}{\alpha + \bar{u}}\,\frac{\alpha - \bar{u}}{\alpha - u}\right).$$

Then the function $\Psi : [-1, 1] \to \mathbb{R}$*, defined by*

$$\Psi(u) = \frac{\gamma}{2\alpha}(\alpha + u)\ln\frac{\alpha + u}{\alpha + \bar{u}} + \frac{\gamma}{2\alpha}(\alpha - u)\ln\frac{\alpha - u}{\alpha - \bar{u}},$$

satisfies for all $u \in [-1,1]$,

$$\Psi'(u) = \psi(u), \quad \Psi''(u) = \frac{\gamma}{\alpha^2 - u^2}, \quad \Psi(u) \geq \frac{\gamma}{2\alpha^2}(u-\bar{u})^2.$$

The lemma follows from Taylor expansion around $\bar{u}$:

$$\Psi(u) = \Psi(\bar{u}) + \Psi'(\bar{u})(u-\bar{u}) + \frac{1}{2}\Psi''(\xi)(u-\bar{u})^2 \geq \frac{\gamma}{2\alpha^2}(u-\bar{u})^2.$$

Proof of Theorem 1.3. We use $\psi(u) \in L^\infty(Q_T) \cap L^2(0,T;H^1_{\rm per}(\Omega))$ and $\theta - \bar{\theta} \in L^2(0,T;H^1_{\rm per}(\Omega))$ as test functions in the weak formulation of (1.1)-(1.2), respectively, and add the resulting equations:

$$\begin{aligned}
&\int_\Omega \left(\Psi(u(t)) + \frac{1}{2}(\theta(t)-\bar{\theta})^2\right) dz + \int_{Q_t} (\nu\psi'(u)u_z^2 + \theta_z^2) dzdt \qquad (4.2)\\
&= \int_\Omega \left(\Psi(u_0) + \frac{1}{2}(\theta_0-\bar{\theta})^2\right) dz + \int_{Q_t} ((1-u^2)\psi'(u) - \gamma)u_z\theta_z dzdt \\
&\quad + \int_{Q_t} (\mu u - \theta)(\theta - \bar{\theta}) dzdt.
\end{aligned}$$

For the second integral on the right-hand side we use Young's inequality:

$$\begin{aligned}
&\int_{Q_t} ((1-u^2)\psi'(u) - \gamma)u_z\theta_z dzdt = \gamma \int_{Q_t} \frac{1-\alpha^2}{\alpha^2-u^2} u_z\theta_z dzdt \\
&\leq \frac{\nu\gamma}{2}(\alpha^2-1)^{1/2} \int_{Q_t} \frac{u_z^2}{\alpha^2-u^2} dzdt + \frac{\gamma}{2\nu}(\alpha^2-1)^{3/2} \int_{Q_t} \frac{\theta_z^2}{\alpha^2-u^2} dzdt \\
&\leq \frac{\nu\gamma}{2}(\alpha^2-1)^{1/2} \int_{Q_t} \frac{u_z^2}{\alpha^2-u^2} dzdt + \frac{\gamma}{2\nu}(\alpha^2-1)^{1/2} \int_{Q_t} \theta_z^2 dzdt.
\end{aligned}$$

Since $\mu\bar{u} = \bar{\theta}$, the last integral on the right-hand side of (4.2) becomes

$$\begin{aligned}
\int_{Q_t} (\mu u - \theta)(\theta - \bar{\theta}) dzdt &= \mu \int_{Q_t} (u-\bar{u})(\theta-\bar{\theta}) dzdt - \int_{Q_t} (\theta-\bar{\theta})^2 dzdt \\
&\leq \frac{\mu^2\delta}{2} \int_{Q_t} (u-\bar{u})^2 dzdt + \left(\frac{1}{2\delta} - 1\right) \int_{Q_t} (\theta-\bar{\theta})^2 dzdt,
\end{aligned}$$

where we choose

$$\frac{L^2}{2(L^2+2)} < \delta < \frac{4\nu\gamma}{\mu^2 L^2}.$$

This is possible by assumption (1.11). We employ Lemma 4.1 to estimate the first integral on the left-hand side of (4.2):

$$\int_\Omega \left(\Psi(u(t)) + \frac{1}{2}(\theta(t)-\bar{\theta})^2\right) dz \geq \int_\Omega \left(\frac{\gamma}{2\alpha^2}(u(t)-\bar{u})^2 + \frac{1}{2}(\theta(t)-\bar{\theta})^2\right) dz.$$

Finally, the second term on the left-hand side of (4.2) can be estimated by using the Poincaré inequality

$$\|v - \bar{v}\|_{L^2(\Omega)} \leq \frac{L}{\sqrt{2}} \|v_z\|_{L^2(\Omega)} \quad \forall v \in H^1_{\rm per}(\Omega) \text{ with } \bar{v} = \int_0^L v(z)dz.$$

We obtain

$$\int_{Q_t} (\nu\psi'(u)u_z^2 + \theta_z^2)dzdt \geq \int_{Q_t} \left(\frac{2\nu\gamma}{L^2} \frac{(u-\bar{u})^2}{\alpha^2 - u^2} + \frac{2}{L^2}(\theta - \bar{\theta})^2 \right) dzdt.$$

Putting the above estimates together, we infer from (4.2)

$$\begin{aligned} &\int_\Omega \left(\frac{\gamma}{2\alpha^2}(u(t)-\bar{u})^2 + \frac{1}{2}(\theta(t)-\bar{\theta})^2 \right) dz \qquad (4.3) \\ &\leq c_0^2 + \int_{Q_t} \left(\frac{\mu^2\delta}{2} - \frac{2\nu\gamma}{L^2} + \frac{\nu\gamma}{L^2}(\alpha^2-1)^{1/2} \right) \frac{(u-\bar{u})^2}{\alpha^2-u^2} dzdt \\ &\quad + \left(\frac{1}{2\delta} - \frac{L^2+2}{L^2} + \frac{\gamma}{\nu L^2}(\alpha^2-1)^{1/2} \right) \int_{Q_t} (\theta-\bar{\theta})^2 dzdt. \end{aligned}$$

Observing that

$$\frac{(u-\bar{u})^2}{\alpha^2-u^2} \geq \frac{(u-\bar{u})^2}{\alpha^2},$$

we can let $\alpha \to 1$ in (4.3) to obtain

$$\begin{aligned} \frac{1}{2}\int_\Omega \left(\gamma(u(t)-\bar{u})^2 + (\theta(t)-\bar{\theta})^2 \right) dz \leq c_0^2 - \int_{Q_t} \left(\frac{2\nu\gamma}{L^2} - \frac{\mu^2\delta}{2} \right) (u-\bar{u})^2 dzdt \\ - \left(\frac{L^2+2}{L^2} - \frac{1}{2\delta} \right) \int_{Q_t} (\theta-\bar{\theta})^2 dzdt. \end{aligned}$$

Defining

$$\delta_1 = \frac{4\nu}{L^2} - \frac{\mu^2\delta}{\gamma} > 0, \quad \delta_2 = \frac{2(L^2+2)}{L^2} - \frac{1}{\delta} > 0, \qquad (4.4)$$

the theorem follows from Gronwall's lemma. □

5. Numerical examples. In this section we illustrate the long-time coarsening of the segregation bands in the drum by numerical experiments. For the numerical discretization, we use a time-discretized version of (1.6)-(1.7) (backward Euler scheme), as motivated by the existence analysis of Section 2, instead of discretizing directly (1.1)-(1.2). The space discretization is performed by using finite differences. The nonlinear system is solved by a simple fixed-point strategy.

In the following examples, we illustrate the segregation behavior of the component u of the solutions of (1.1)-(1.2). The behavior relies on three important conditions. First, condition (1.11) ensures the convergence of u to a constant steady state. Second, the authors of [3] conjectured that the condition $\mu > \nu$ is a necessary condition to have segregation. This conjecture arises from a linear stability analysis sketched in [3], showing that perturbations of the form $\exp(\lambda t + 2\pi z/\ell)$, where $\lambda \in \mathbb{R}$ and $\ell > 0$ is the wave length of the perturbations, are unstable if $\mu > \nu + 4\pi^2(\nu+\gamma)/\ell^2$. Therefore, this instability is captured only if the length L of the domain satisfies the third condition

$$L > 2\pi\sqrt{(\gamma+\nu)/(\mu-\nu)}. \qquad (5.1)$$

In Figure 5.1 we present the behavior of u in the (z,t)-plane for two different domain lengths. The number of grid points is $N = 50$. The parameters in Figure 5.1

(a) satisfy $\mu > \nu$ and (1.11), but *not* (5.1). We observe convergence of u to a constant steady state. We expect this behavior in view of Theorem 1.3. This example shows that the condition $\mu > \nu$ is *not* sufficient for segregation. The parameters in Figure 5.1 (b) satisfy the segregation condition (5.1) but not (1.11). The granular materials segregate since the length of the cylinder is large enough, as claimed by the linear stability analysis.

(a)	(b)
$L =$	$L =$
1,	30,
$0 \leq$	$0 \leq$
$t \leq$	$t \leq$
0.159.	3049.

FIG. 5.1. $\gamma = 2$, $\mu = 3$, $\nu = 2$, $u_0(z) = 0.8\cos(4\pi z/L)$, $N = 50$.

Figure 5.2 shows that (1.11) is a sufficient but not necessary condition to have non-segregation. Indeed, the parameters are chosen such that (1.11) is not satisfied, but the granular materials do not segregate.

FIG. 5.2. $L = 4$, $\gamma = 2$, $\mu = 2$, $\nu = 3$, $u_0(z) = 0.8\cos(4\pi z/L)$, $N = 50$.

A more detailed view of the same segregation phenomena as above but with a larger number of bands is presented in Figure 5.3. The parameters do not satisfy (1.11) but (5.1) holds. Thus, we expect segregation. The initial short-wave perturbations produce decaying standing waves (Figure 5.3 (a)). The segregated bands emerge, and we observe metastable long-wave bands. Finally, after larger time, the system segregates again (Figure 5.3 (b)). This illustrates the very slow coarsening of the band structure (see [3]).

(a)	(b)
$0 \leq$	$4.8 \leq$
$t \leq$	$t \leq$
1.06.	8.9.

FIG. 5.3. $\gamma = 100$, $\mu = 40$, $\nu = 0.5$, $L = 30$, $u_0(z) = 0.75\cos(80\pi z/L)$, $N = 1000$.

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