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A VANISHING VISCOSITY FLUIDIZED BED MODEL

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Abstract. The evolution of one-dimensional fluidized beds may be modeled in form of a system of partial differential equations of the compressible Navier-Stokes type where the viscosity depends on the density, which may vanish, the source term is nonlinear, and the constitutive law for the pressure blows up for finite values of the density. A finite-differences scheme is used to solve an approximated problem in Lagrangian coordinates, which we show to be equivalent to the corresponding problem in Eulerian coordinates. We then prove compactness and convergence properties of the sequence of solutions of the approximated problems and partially identify the limit as a solution of the original problem.

1. Introduction

Two-phase systems where a dense phase of small particles is fluidized within a gas flow appear in many industrial applications, among which the fluidized bed combustor is probably the most important. In this article we study a mathematical problem which modelizes the dynamical aspects of a fluidized bed, which we assume to follow the general rules of continuum mechanics. For the spatial one dimensional case, the system of equations strongly resembles to the compressible Navier-Stokes equations, although the constitutive laws for the pressure and stress tensor are of different nature.

Let us briefly comment the main steps in the deduction of the model. In the interior of a domain $\Omega \subset \mathbb{R}^3$, a collection of solid particles of constant

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density, ρ_s, (solid-phase) is blown by an inflow of gas of constant density, ρ_g, (gas-phase) from the bottom of the boundary, $\partial\Omega$. As we mentioned above, we assume that the system gas-solid behaves according to the conservation laws of continuum mechanics and, therefore, averaged velocities corresponding to the gas and solid phase, $\mathbf{v_g}$, $\mathbf{v_s}$, are defined. We further assume that the gas density is much smaller than the solid density,

$$\rho_g \ll \rho_s, \tag{1.1}$$

and that the specific volume of the solid phase, which we denote by ρ, is bounded by a constant ρ^*, i.e., $0 \leq \rho \leq \rho^* < 1$. We may describe the behaviour of the two-phase flow by the following mass and momentum conservation laws, see [4]:

$$\operatorname{div}((1-\rho)\mathbf{v_g} + \rho\mathbf{v_s}) = 0 \quad \text{in } \mathrm{Q_T} := \Omega \times (0,\mathrm{T}), \tag{1.2}$$

$$\rho_t + \operatorname{div} \mathbf{m} = 0, \tag{1.3}$$

$$\mathbf{m}_t + \mathbf{div}\,\big(\frac{\mathbf{m}}{\rho} \times \mathbf{m}\big) - \mathbf{div}\,\tau + \nabla p_c = \rho\mathbf{g} - \nabla p_h, \tag{1.4}$$

in $Q^+ := \{(x,t) \in Q_T : \rho(x,t) > 0\}$, and

$$\rho = 0 \quad \text{and} \quad \mathbf{m} = 0 \quad \text{in } Q_T \backslash Q^+. \tag{1.5}$$

Here, $\mathbf{m} := \rho\mathbf{v_s}$ is the momentum per unit of volume and mass of the solid-phase, p_c is the *collisional pressure* of the solid-phase and p_h is the *hydrodynamical pressure*, due to friction between phases. Finally, $\mathbf{g}$ denotes the gravity force and τ is the stress tensor. Notice that assumption (1.1) implies that mass and momentum conservation equations for the gas-phase are reduced to (1.2).

The pressures p_c and p_h and the stress tensor τ are modelized by constitutive laws. For the hydrodynamical pressure, describing friction forces, different particular choices may be found in the literature, deduced from experimental or theoretical arguments, see [4, 1, 6, 19, 20]. We shall assume that its gradient is proportional to the difference of velocities of both phases, which is commonly accepted in the literature, and to some smooth and positive function, $q(\rho)$,

$$-\nabla p_h = q(\rho)(\mathbf{v_g} - \mathbf{v_s}).$$

For the collisional pressure, $p \equiv p_c$, we also find many different modelizations in the literature, see [4, 18, 6, 5, 19]. In view of the lack of precise knowledge, we assume this pressure to take a form such that the following qualitative properties are satisfied: (i) the pressure is a smooth increasing function of

ρ, with $p(0)=0$; and (ii) the pressure contains a limiting term, avoiding the volumetric fraction, ρ, to reach the threshold value ρ^*. The following example is taken in [27]

$$p(\rho)=A^2\rho^\gamma \exp\big(\frac{\varepsilon_0\rho}{\rho^*-\rho}\big), \quad \gamma>1, \quad A,\varepsilon_0>0, \tag{1.6}$$

although we shall consider more general conditions on p, see (2.2).

For the stress tensor, we assume that it has a similar form than for a Newtonian fluid, see [4, 2, 22, 18],

$$\tau(\rho,\mathbf{v_s}) := \lambda \mathrm{div}\, \mathbf{v_s}\mathbf{I} + 2\nu\mathbf{D}(\mathbf{v_s}),$$

where λ,ν are the kinematic viscosity coefficients, which we suppose that may depend on ρ, $\mathbf{I}$ is the identity tensor and $\mathbf{D}(\mathbf{v_s}) := \frac{1}{2}(\nabla\mathbf{v_s}+\nabla{\mathbf{v_s}}^T)$.

The dimensionless rescaled model may then be written in one space dimension as in [18, 27],

$$\rho_t + m_x = 0 \tag{1.7}$$

$$m_t + \big(\frac{m^2}{\rho}\big)_x - \big(\mu(\rho)\big(\frac{m}{\rho}\big)_x\big)_x + p(\rho)_x = f(\rho,m,t), \tag{1.8}$$

in

$$Q^+ := \{(x,t)\in Q_T : \rho(x,t)>0\}\,, \tag{1.9}$$

where $\mu := \lambda+2\nu$, f modelizes gravity and frictional forces, and

$$\rho = m = 0 \quad \text{in } Q_T\backslash Q^+, \tag{1.10}$$

with $Q_T := \Omega\times(0,T)$, with $\Omega := (0,L)$, and $T,L<\infty$.

Turning to the boundary conditions, we observe that in addition to the fixed boundary, in which we impose the natural boundary condition

$$m(0,\cdot)=m(L,\cdot)=0 \quad \text{in } (0,T), \tag{1.11}$$

we must impose a further condition in the *free boundary*, the unknown set in Q_T given by $\Gamma := \overline{Q^+}\cap\overline{Q^0}$, with $Q^0 := \{(x,t)\in Q_T : \rho(x,t)=0\}$. The compatibility conditions among equations (1.7)-(1.8), which hold in Q^+ and equation (1.10), which hold in Q^0, imply that the normal flux of m and the normal tension must vanish in Γ. Since we already assumed $p(0)=0$, this free boundary condition implies that, to have a continuous transition of ρ between Q^+ and Q^0, we must assume

$$\lim_{\rho\to 0}\tau(\rho) = \lim_{\rho\to 0}\mu(\rho)=0. \tag{1.12}$$

Equations (1.7)-(1.8) are completed with initial conditions

$$\rho(\cdot,0)=\rho_0, \quad m(\cdot,0)=m_0 \quad \text{in } \Omega. \tag{1.13}$$

From now on, we shall refer to problem (1.7)-(1.13) as **Problem P**.

As we already mentioned, Problem P is very related to the compressible Navier-Stokes equations. However, two important differences have to be noted. First, the a priori boundedness of the volumetric fraction, ρ, which we shall often call *density*, by analogy with the Navier-Stokes equations. Second, the dependence of viscosity on density (and the related property (1.12)) which is usually taken as a positive constant.

The literature on the Navier-Stokes equations is extensive, see for instance [7]-[12], [23]-[25], [13]-[15], [21, 16, 17]. We emphasize that in [11], the authors prove that the one-dimensional compressible Navier-Stokes problem with discontinuous transition to the vacuum is an ill-posed problem. This motivates, to some extent, our assumption (1.12) since although our pressure term is qualitatively different to that of the Navier-Stokes equations when $\rho \to \rho^*$, they behave alike when $\rho \to 0$, i.e., in the transition to the vacuum. The disadvantage of considering a vanishing viscosity term is that it does not allow us to obtain the usual energy estimates for the velocity, representing this one of the main difficulties of proving the existence of solutions of Problem P.

The rest of the article is organized as follows. In Section 2, we present our precise assumptions and state the main results. These are related to an approximated problem, which we call Problem P_ε, for which we find uniform estimates which imply the compactness of the sequence of approximated solutions. In Section 3, we prove the existence of solutions and obtain uniform estimates of a related problem in Lagrangian coordinates. In Section 4, we prove the equivalence between the problems in Lagrangian and Eulerian coordinates. Finally, in Section 5, we prove our main results.

2. Assumptions and main results

We consider the following assumptions on the data:

(H1) The initial data of the density is such that $\rho_0^{\alpha-1/2} \in H^1(\Omega)$, $\alpha > 1/2$, with $0 \leq \rho_0 < 1$ in Ω and $\|\rho_0\|_{L^1} = 1$.

(H2) The viscosity is assumed to take the form

$$\mu(\rho) = \nu\rho^\alpha, \quad \text{with } \nu > 0. \tag{2.1}$$

(H3) The initial data of the momentum is of the form $m_0 := \rho_0 u_0$, with $u_0 \in L^2(\Omega)$.

(H4) The pressure term is assumed to be such that $p \in C^1([0,1);\mathbb{R}_+)$ is increasing, with $p(0)=0$, and satisfying

$$\int_0^{\rho'} \frac{p(s)}{s^2} ds < \infty \quad \text{and} \quad \int_{\rho'}^1 \Big(\int_{\rho'}^z p(s) ds \Big)^{1/2} dz = \infty, \tag{2.2}$$

for an arbitrary $\rho' \in (0,1)$. The source term, $f \in C([0,1) \times \mathbb{R} \times \mathbb{R}_+)$ is given by

$$f(\rho, m, t) = q(\rho)(M(t) - m), \tag{2.3}$$

where $q \in C([0,1])$ is increasing, with $q(0)=0$ and $q(1) < \infty$, and $M \in C([0,T];\mathbb{R}_+)$ represents the flow of gas entering into the domain through the point $x=0$.

Remark 1. (1) A regularization and passing to the limit technique allows us to prove Theorems 1 and 2 below under weaker assumptions on function f. Indeed, we only need that f satisfies the Carathéodory conditions.

(2) In the rescaling of the previous section, we normalized the specific volume to be $\rho < 1$ instead of $\rho < \rho^*$. This is the reason why we write in (H1) $\rho_0 < 1$ and in (H4) the upper limit of integration of the second integral is 1.

(3) It is not difficult to see that solutions of (1.2)-(1.5) with $\mathbf{m} = \mathbf{0}$ are unidimensional. These static stationary solutions are called *stationary cloud solutions*, and we show in [27] that for a domain large enough, a free boundary arises. Moreover, when laws of the type (1.6) are assumed for the pressure, we show that the regularity of ρ in this boundary is determined by the exponent γ.

This exponent is also important in the study of the linear stability of trivial constant solutions, where the condition $\gamma \geq 3$ is a necessary condition for unstable bubbling regimes to appear (when gas flow speed is greater than a critical rate). We have that stationary cloud solutions satisfy hypothesis (H1) if $\gamma < 2\alpha$. Therefore, if we want to cover the case of a pressure given by (1.6), we must allow α to be larger than $3/2$. Although the lack of precise physical knowledge about the viscosity coefficient led us to take a simple constitutive law satisfying (1.12), the arguments above motivated us to choose (2.1) for a general α, instead of the more obvious linear law.

In order to approximate the solutions of Problem P, we introduce an auxiliary problem consisting of equations (1.7)-(1.8) with function μ replaced by a certain perturbation μ_ε, completed with the following auxiliary conditions

$$u(0,\cdot) = u(L,\cdot) = 0 \quad \text{in } (0,T), \tag{2.4}$$

$$\rho(\cdot,0)=\rho_{0\varepsilon},\quad u(\cdot,0)=u_0\quad\text{in }\Omega. \tag{2.5}$$

We take the perturbed data satisfying the following assumptions:

$(\mathrm{H1})_\varepsilon$ $\rho_{0\varepsilon}^{\alpha-1/2}\in H^1(\Omega)$, $\|\rho_{0\varepsilon}\|_{L^1}=1$, and

$$\rho_{0\varepsilon}^{\alpha-1/2}\to\rho_0^{\alpha-1/2}\quad\text{in }H^1(\Omega)\quad\text{as }\varepsilon\to0. \tag{2.6}$$

In addition, we assume that there exist constants C and ρ^+, independent of ε, such that

$$0<C\varepsilon^{\frac{2}{3}}\le\rho_{0\varepsilon}\le\rho^+<1. \tag{2.7}$$

$(\mathrm{H2})_\varepsilon$ The perturbed viscosity is of the form

$$\mu_\varepsilon(\rho):=\mu(\rho)+\varepsilon=\nu\rho^\alpha+\varepsilon. \tag{2.8}$$

Before stating our results, we give the notion of weak solution of equations (1.7)-(1.10) with boundary and initial data given by (2.4)-(2.5), with viscosity given by (2.8) and with the data satisfying hypothesis $(\mathrm{H1})_\varepsilon$, $(\mathrm{H2})_\varepsilon$, (H3) and (H4). We shall refer to this problem as to **Problem $\mathbf{P}_\varepsilon$**. Since the density shall be a positive function, we formulate Problem P_ε in terms of ρ and $u:=m/\rho$.

Definition 1. A pair (ρ,u) is called a *weak solution* of Problem P_ε if $\rho:\overline{Q_T}\to\mathbb{R}_+$, and $u:\overline{Q_T}\to\mathbb{R}$ are such that

$$\rho\in H^1(0,T;L^2(\Omega))\cap C(\overline{Q_T}),\ \ \rho^{\alpha-1/2}\in L^\infty(0,T;H^1(\Omega)),\ \ p(\rho)\in L^\infty(Q_T),$$
$$u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)),\ \ u_t\in L^2(0,T;H^{-1}(\Omega)),$$
$$0<\rho_-\le\rho\le\rho_+<1\quad\text{a.e. in }Q_T, \tag{2.9}$$

for some constants ρ_- and ρ_+ (which may depend on ε). The equations and auxiliary data are satisfied in the sense

$$\rho_t+(\rho u)_x=0\quad\text{a.e. in }Q_T, \tag{2.10}$$

$$\int_{Q_T}\big(\rho u(\varphi_t+u\varphi_x)-\mu_\varepsilon(\rho)u_x\varphi_x+p(\rho)\varphi_x+f(\rho,m,t)\varphi\big)=0, \tag{2.11}$$

for all $\varphi\in C([0,T];C_0^1(\overline{\Omega}))\cap H^1(0,T;L^2(\Omega))$, and

$$\lim_{t\to0}||\rho(\cdot,t)-\rho_{0\varepsilon}||_{C^0(\Omega)}=\lim_{t\to0}||u(\cdot,t)-u_0||_{H^{-1}(\Omega)}=0. \tag{2.12}$$

We are concerned with the compactness properties of the sequence of solutions $(\rho_\varepsilon,u_\varepsilon)$ of Problem P_ε. We expect the limit of this sequence to be a solution of Problem P corresponding to the unperturbed data (ρ_0,m_0) and μ, in which the density ρ may vanish in the interior of the domain.

Although we give here statements for general $\alpha > 1/2$, we shall particularize the proofs for $\alpha = 1$. This is done for the aim of clarity, as the proof for the general case does not use new arguments and only introduces a rather more obscure notation. We give some details on the different aspects of both proofs in Remark 4.

In the following statements, which are the main results of this article, we assume the existence of solutions of Problem P_ε, which we prove as an auxiliary result in Section 4.

Theorem 1. *Assume* $(H1)_\varepsilon, (H2)_\varepsilon$, $(H3)$ *and* $(H4)$ *and let* $(\rho_\varepsilon, u_\varepsilon)$ *be a weak solution of Problem* P_ε. *Then we have estimates of the norms*

$$||(\rho_{\varepsilon x})^{\alpha-1/2}||_{L^\infty(L^2)}, \quad ||(\rho_\varepsilon^\alpha)_t||_{L^2(L^1)}, \quad ||\rho_\varepsilon^{1/2}u_\varepsilon||_{L^\infty(L^2)}, \quad ||(\rho_\varepsilon^\alpha+\varepsilon)^{1/2}u_{\varepsilon x}||_{L^2}, \tag{2.13}$$

which are independent of ε. *In addition, the constant* ρ_+ *of* (2.9) *may also be fixed independently of* ε.

Here, we introduced the notation $\|\cdot\|_{L^p(L^q)}$ for $\|\cdot\|_{L^p(0,T;L^q(\Omega))}$ and $\|\cdot\|_{L^p}$ for $\|\cdot\|_{L^p(Q_T)}$. The above estimates allow us to pass to the limit in the weak formulation of Problem P_ε. We obtain the following result of convergence to a solution of Problem P.

Theorem 2. *Assume* $(H1)$-$(H4)$. *Consider the sequence* $(\rho_\varepsilon, u_\varepsilon)$ *of solutions of Problems* P_ε *and set* $m_\varepsilon := \rho_\varepsilon u_\varepsilon$, *for any* $\varepsilon > 0$. *Then there exist functions* $\rho \in C(\overline{Q_T})$, m, ζ, $\Gamma \in L^2(Q_T)$ *and* $\chi^2 \in \Re(Q_T)$ *(the set of Radon measures) such that, up to a subsequence,*

$$\rho_\varepsilon \to \rho \text{ uniformly in } Q_T, \quad \text{with } \rho < \rho_+ < 1 \quad \text{in } \overline{Q_T}, \tag{2.14}$$

$$m_\varepsilon \rightharpoonup m, \qquad \rho_\varepsilon^\alpha\Big(\frac{m_\varepsilon}{\rho_\varepsilon}\Big)_x \rightharpoonup \zeta, \qquad \frac{m_\varepsilon}{\sqrt{\rho_\varepsilon}} \rightharpoonup \Gamma \qquad \text{in } L^2(Q_T), \tag{2.15}$$

and

$$\frac{m_\varepsilon^2}{\rho_\varepsilon} \rightharpoonup \chi^2 \quad \text{in } \Re(Q_T). \tag{2.16}$$

Consider the sets Q^+ *defined in* (1.9) *and* $Q^* := \{(x,t) \in Q_T : \rho(x,t) = 0\}$. *Then we have*

$$m = 0 \quad \text{and} \quad \zeta = 0 \quad \text{in } Q^*, \tag{2.17}$$

$$\zeta = \rho^\alpha\Big(\frac{m}{\rho}\Big)_x \quad \text{a.e. in } Q^+, \tag{2.18}$$

$$\chi^2 = \frac{m^2}{\rho} \quad \text{a.e. in } Q^+ \quad \text{and} \quad (\chi^2)_x = 0 \quad \text{in Interior}\,(\mathrm{Q}^*). \tag{2.19}$$

In addition, if $\alpha = 1$, then

$$m_\varepsilon \to m \quad \text{in } L^2(Q_T). \tag{2.20}$$

Remark 2. (1) Note that in the passing to the limit we obtain compactness properties for ρ_ε and m_ε, but not for u_ε. This is interesting from the point of view of the physical model since the relevant conserved physical quantities of the problem are precisely the density and the momentum. Velocity only appears as an auxiliary quantity.

(2) We are able to identify the limit of the diffusion term in the whole domain because we identified this limit almost everywhere, and the limit is an element of $L^2(Q_T)$. However, in the case of the convective term, we are only able to identify the limit almost everywhere because we do not obtain L^p regularity, for $p > 1$, from the sequence of solutions of Problem P_ε, since we only have an estimate of the sequence in $L^1(Q_T)$.

3. LAGRANGIAN COORDINATES. PROBLEM PL_ε

Introducing the Lagrangian mass coordinates

$$X(x,t) = \int_0^x \rho(z,t)dz, \quad \hat{t}(x,t) = t \quad \text{for } (x,t) \in Q_T,$$

and making the identification $\hat{t} \equiv t$, we may rewrite formally equations (1.7)-(1.8) (with μ replaced by μ_ε) for the new unknowns $W(X,t) = 1/\rho(x,t)$ and $U(X,t) = u(x,t)$ as

$$W_t - U_X = 0, \tag{3.1}$$

$$U_t - (\beta(W)U_X - \overline{p}(W))_X = F(W,U,t), \tag{3.2}$$

in $D_T := I \times (0,T)$, with $I = (0,1)$ (due to the normalization of the initial mass, see hypothesis $(\mathrm{H1})_\varepsilon$). Here we introduced the functions $\beta(s) := \mu_\varepsilon(1/s)/s$, $\overline{p}(s) := p(1/s)$ and $F(s,\sigma,t) := sf(1/s,\sigma/s,t)$. The boundary and initial data are

$$U(0,\cdot) = U(1,\cdot) = 0 \quad \text{a.e. in } (0,T), \tag{3.3}$$

$$W(\cdot,0) = W_{0\varepsilon}, \quad U(\cdot,0) = U^0 \quad \text{a.e. in } I. \tag{3.4}$$

Assumptions on $(\mathrm{H1})_\varepsilon$ and (H3) on the initial data are replaced by

$(\mathrm{HL1})_\varepsilon$ $W_{0\varepsilon} \in H^1(I)$, $U^0 \in L^2(I)$ and there exist constants C and ρ^+, independent of ε, such that

$$0 < \frac{1}{\rho^+} \le W_{0\varepsilon} \le C\varepsilon^{-2/3}. \tag{3.5}$$

The definition of weak solution for this problem is similar than that of Problem P_ε. We write it for the case $\alpha = 1$.

Definition 2. A pair (W, U) is called a *weak solution* of Problem PL_ε if $W : \overline{D_T} \to \mathbb{R}_+$ and $U : \overline{D_T} \to \mathbb{R}$ are such that

$$W \in L^\infty(0,T;H^1(I)) \cap H^1(0,T;L^2(I)) \cap C(\overline{D_T}), \quad \overline{p}(W) \in L^\infty(D_T), \tag{3.6}$$

$$U \in L^\infty(0,T;L^2(I)) \cap L^2(0,T;H_0^1(I)) \cap H^1(0,T;H^{-1}(I)), \tag{3.7}$$

$$0 < 1/\rho_+ \leq W \leq 1/\rho_- < 1 \quad \text{a.e. in } D_T, \tag{3.8}$$

for some positive constants ρ_- and ρ_+. Equations (3.1)-(3.2) are satisfied in the following sense

$$W_t = U_X \quad \text{a.e. in } D_T, \tag{3.9}$$

$$\int_{D_T} \big(U\Phi_t - \beta(W)U_X\Phi_X + \overline{p}(W)\Phi_X + F(W,U,t)\Phi\big) = 0, \tag{3.10}$$

for all $\Phi \in C([0,T];C_0^1(\overline{I})) \cap H^1(0,T;L^2(I))$. The solutions verify the initial conditions in the sense:

$$\lim_{t\to 0} ||W(\cdot,t) - W_{0\varepsilon}||_{C^0(I)} = \lim_{t\to 0} ||U(\cdot,t) - U^0||_{H^{-1}(I)} = 0. \tag{3.11}$$

Theorem 3. *Assume* $(HL1)_\varepsilon, (H2)_\varepsilon$, $(H3)$ *and* $(H4)$. *Then there exists a weak solution of Problem* PL_ε. *In addition,* ρ_+ *may be chosen independent of* ε.

To obtain the existence of solutions of Problem PL_ε, we first consider a spatial discretization of equations (3.1)-(3.2), leading to a system of ordinary differential equations in time. The local solution of the system is a straightforward consequence of the Theorem of Piccard–Lipschitz. To prove the existence of a global solution, we obtain uniform estimates of suitable norms of the solution. In particular, we deduce the discrete energy inequalities corresponding to the natural energy inequalities of the continuum problem. We finally extend the solution of the system of ODE's to space by piecewise linear approximations, showing that the estimates are independent of the discretization parameter. We then pass to the limit and identify it as a solution of Problem PL_ε.

We introduce a finite difference scheme to discretize Problem PL_ε, see [7]. For any given integer N, let $h = 1/N$. We define $X_k = kh$, for $k \in \{0, \ldots, N\}$, and $X_{k+1/2} = (X_k + X_{k+1})/2$, for $k \in \{0, \ldots, N-1\}$. We denote by $U_k(t)$ and $W_{k+1/2}(t)$ the approximations of $U(X_k,t)$ and $W(X_{k+1/2},t)$, respectively. Define the spatial difference operator δ by

$$\delta g_k = \frac{g_{k+1/2}-g_{k-1/2}}{h}, \qquad \delta g_{k+1/2} = \frac{g_{k+1}-g_k}{h}.$$

We set the following system of ordinary differential equations:

$$\dot{W}_{k+1/2} = \delta U_{k+1/2}, \qquad k\in\{0,\dots,N-1\}, \tag{3.12}$$

$$\dot{U}_k = \delta(\beta\delta U - \overline{p})_k + f_k, \quad k\in\{1,\dots,N-1\}, \tag{3.13}$$

in $(0,T)$, where we use the notation $\beta_{k+1/2}\equiv\beta(W_{k+1/2})$, $\overline{p}_{k+1/2}\equiv\overline{p}(W_{k+1/2})$ and

$$f_k(t)\equiv F((W_{k+1/2}+W_{k-1/2})/2, U_k, t).$$

We complete equations (3.12)-(3.13) fixing the end-points values as

$$U_0(t) = U_N(t) = 0, \tag{3.14}$$

and the initial values as

$$W_{k+1/2}(0) = W_{0\varepsilon}(X_{k+1/2}) \quad \text{and} \quad U_k(0) = \int_{X_{k-1}}^{X_k} U^0. \tag{3.15}$$

The existence of local solution of problem (3.12)-(3.15) is obtained by the Theorem of Piccard–Lipschitz. To prove the global existence we need to show that the Lipschitz constant corresponding to system (3.12)-(3.13) is finite for any $T<\infty$. To this end it is sufficient to prove

$$\begin{gathered} 1 < W_- \le W_{k+1/2} \le W_+ < \infty, \\ -\infty < U_- \le U_k \le U_+ < \infty, \end{gathered} \tag{3.16}$$

in $(0,T)$, for any $T>0$ and for certain constants W_-, W_+, U_- and U_+. These estimates are obtained via energy estimates involving the following energy functionals:

$$\begin{aligned} E^p(\sigma) &:= \int_\sigma^\infty \overline{p}(s)ds = \int_0^{1/\sigma}\frac{p(s)}{s^2}ds, \\ E^\beta(\sigma) &:= \int_{W'}^\sigma \beta(s)ds = \Big[\varepsilon\log s - \frac{1}{s}\Big]_{W'}^\sigma, \end{aligned} \tag{3.17}$$

for an arbitrary constant $W'>1$. Observe that since p and β are positive, we have that E^p is decreasing and E^β is increasing. In addition, hypothesis (H4) implies that, for any $z'>1$,

$$\lim_{z\to 1}\int_z^{z'} E^p(\sigma)d\sigma = \infty. \tag{3.18}$$

We also have, for any $\varepsilon > 0$,

$$\lim_{\sigma\to\infty} E^\beta(\sigma) = \infty. \tag{3.19}$$

Notice that (3.19) implies that if $\varepsilon > 0$ and $E^\beta(\sigma) < \infty$, then $\sigma \leq C < \infty$. However, for $\varepsilon = 0$ we have $\lim_{\sigma\to\infty} E^\beta(\sigma) = 1/W' < \infty$, and therefore the constant C above depends on ε.

In the following, we shall use the notation $E^p_{k+1/2}(t)$ for $E^p(W_{k+1/2}(t))$ and similarly for E^β.

Lemma 1. *There exists a constant C independent of h and ε such that*

$$\sum_{k=1}^{N-1}\Big(U_k^2(T)h+\int_0^T (U_k^2+\beta_{k+1/2}(\delta U_{k+1/2})^2)h\Big)+\sum_{k=0}^{N-1} E^p_{k+1/2}(T)h \leq C(1+T), \tag{3.20}$$

and

$$\sum_{k=1}^{N-1}(\delta E_k^\beta)^2(T)h \leq C(1+T). \tag{3.21}$$

Proof. Multiplying equation (3.12) by $U_k h$ and summing in $k = 1, 2, \ldots, N$, using (3.14) and the summation by parts formulae, integrating in $(0, T)$ and using

$$\sum_{k=1}^{N-1}\delta\overline{p}_k U_k = -\sum_{k=0}^{N-1}\overline{p}_{k+1/2}\delta U_{k+1/2} = -\sum_{k=0}^{N-1}\overline{p}_{k+1/2}\dot{W}_{k+1/2} = \frac{d}{dt}\sum_{k=0}^{N-1}E^p_{k+1/2}h,$$

we deduce

$$\mathcal{E}_1 \leq \Big(T + \sum_{k=1}^{N-1}U_k^2(0)h + \sum_{k=0}^{N-1}E^p_{k+1/2}(0)h\Big), \tag{3.22}$$

where $\mathcal{E}_1$ is the left hand side of (3.20). We shall see at the end of this proof that the constant at the right of (3.22) is independent of h and ε.

To prove the second energy inequality (3.21), we observe that combining equations (3.12) and (3.13) we obtain

$$\delta\dot{E}_k^\beta = \delta(\beta\delta U)_k = \dot{U}_k + \delta\overline{p}_k - f_k. \tag{3.23}$$

Multiplying by $\delta E_k^\beta h$, summing in $k = \{1, \ldots, N-1\}$ and integrating in $(0, T)$, we get

$$\frac{1}{2}\sum_{k=1}^{N-1}\int_0^T \frac{d}{dt}(\delta E_k^\beta)^2 h = \sum_{k=1}^{N-1}\int_0^T(\dot{U}_k + \delta\overline{p}_k - f_k)\delta E_k^\beta h. \tag{3.24}$$

We have

$$\sum_{k=1}^{N-1}\int_0^T \dot{U}_k\delta E_k^\beta h=\sum_{k=1}^{N-1}\Big[U_k\delta E_k^\beta\Big]_0^T h-\sum_{k=1}^{N-1}\int_0^T U_k\delta \dot{E_k^\beta} h, \tag{3.25}$$

but, by (3.23)

$$\sum_{k=1}^{N-1}\int_0^T U_k\delta \dot{E}_k^\beta h=\sum_{k=1}^{N-1}\int_0^T U_k\delta(\beta\delta U)_k h. \tag{3.26}$$

Summation by parts in the right of (3.26) gives

$$\sum_{k=1}^{N-1}\int_0^T U_k\delta(\beta\delta U)_k h=\sum_{k=0}^{N-1}\int_0^T \beta_{k+1/2}(\delta U_{k+1/2})^2 h. \tag{3.27}$$

Therefore, from (3.25)-(3.27) we get

$$\sum_{k=1}^{N-1}\int_0^T \dot{U}_k\delta E_k^\beta h\leq\sum_{k=1}^{N-1}\Big(\frac{1}{4}(\delta E_k^\beta)^2(T)+2U_k^2(T)-U_k(0)\delta E_k^\beta(0)h\Big)h \\ +\sum_{k=0}^{N-1}\int_0^T \beta_{k+1/2}(\delta U_{k+1/2})^2 h. \tag{3.28}$$

On the other hand, since $\overline{p}$ is decreasing and E^β is increasing we have

$$\int_0^T\sum_{k=1}^{N-1}\delta\overline{p}_k\delta E_k^\beta h\leq 0. \tag{3.29}$$

We have, then from (3.24), (3.28) and (3.29)

$$\frac{1}{4}\sum_{k=1}^{N-1}(\delta E_k^\beta)^2(T)h\leq\sum_{k=1}^{N-1}\Big((\delta E_k^\beta)^2(0)+2(U_k^2(T)+U_k^2(0))\Big)h \\ +\frac{1}{2}\sum_{k=1}^{N-1}\int_0^T\Big(2\beta_{k+1/2}(\delta U_{k+1/2})^2+(\delta E_k^\beta)^2\Big)h.$$

Using (3.20) and Gronwall's inequality we deduce

$$\sum_{k=1}^{N-1}(\delta E_k^\beta)^2(T)h\leq C\Big(T+\sum_{k=1}^{N-1}\Big(U_k^2(0)+(\delta E_k^\beta(0))^2\Big)h+\sum_{k=0}^{N-1}E_{k+1/2}^p(0)h\Big). \tag{3.30}$$

Let us finally show that the constants in (3.22) and (3.30) do not depend on h neither on ε. On one hand we have, by definition of E^p, see (3.17)

$$E^p_{k+1/2}(0) = \int_{W_{0\varepsilon}(X_{k+1/2})}^{\infty} \bar{p}(s)ds = \int_0^{1/W_{0\varepsilon}(X_{k+1/2})} \frac{p(s)}{s^2}ds \le \int_0^{\rho^+} \frac{p(s)}{s^2}ds \le C,$$

with C independent of h and ε, and where we used assumptions (H4) and $(\mathrm{HL1})_\varepsilon$. On the other hand, we may write

$$\delta E^\beta_k(0) = \frac{1}{h}\int_{W_{0\varepsilon}(X_{k-1/2})}^{W_{0\varepsilon}(X_{k+1/2})} \frac{d}{ds}E^\beta(s)ds = \frac{1}{h}\int_{W_{0\varepsilon}(X_{k-1/2})}^{W_{0\varepsilon}(X_{k+1/2})} \beta(s)ds.$$

Observe that convergence (2.6) and definition (3.4) implies

$$\sum_{k=1}^{N-1} \Big|\frac{R_{0\varepsilon}(X_{k+1/2}) - R_{0\varepsilon}(X_{k-1/2})}{h}\Big|^2 h \le C,$$

with C independent of ε and h. Therefore, using the primitive function of β, see (3.17), and the bound (3.5), we obtain

$$\begin{aligned}\sum_{k=1}^{N-1}(\delta E^\beta_k(0))^2 h &\le \sum_{k=1}^{N-1}\frac{1}{h}\big(\varepsilon|\log R_{0\varepsilon}(X_{k+1/2}) - \log R_{0\varepsilon}(X_{k-1/2})| \\ &\quad + |R_{0\varepsilon}(X_{k+1/2}) - R_{0\varepsilon}(X_{k-1/2})|)\big)^2 \\ &\le \sum_{k=1}^{N-1} 2(\frac{\varepsilon}{\rho^-}+1)^2\Big|\frac{R_{0\varepsilon}(X_{k+1/2}) - R_{0\varepsilon}(X_{k-1/2})}{h}\Big|^2 h \le C,\end{aligned}$$

with C independent of h and ε, and where we used $(\mathrm{HL1})_\varepsilon$ and that ε is a small number (say $\varepsilon < 1$). $\square$

Lemma 2. *There exist constants W_-, independent of h and ε, and W_+, independent of h, such that*

$$1 < W_- \le W_{k+1/2} \le W_+ < \infty \quad \text{in } (0,T), \quad \text{for any } k \in \{0,\dots,N-1\}.$$

Proof. Inequality (3.20) of Lemma 1 implies that for any $t \in (0,T]$ we can find at least one $k \in \{0,\dots,N-1\}$ such that

$$E^p_{k+1/2}(t) \le C(1+T). \tag{3.31}$$

In a similar way, equation (3.12) implies

$$\sum_{k=0}^{N-1} W_{k+1/2}(t)h = \sum_{k=0}^{N-1} W_{k+1/2}(0)h,$$

so we also can find k such that $E^{\beta}_{k+1/2}(t) \leq C(1+T)$, where we denote by C any constant independent of h and ε. Using a discrete version of the embedding $H^1 \subset L^\infty$ and inequality (3.21) we obtain

$$\sup_{k\in\{0,\ldots,N-1\}} E^{\beta}_{k+1/2}(t) \leq C(1+T).$$

By (3.19), we deduce the existence of a constant, say W_+, which depends on ε, such that $W_{k+1/2} \leq W_+$ for any $k \in \{0,\ldots,N-1\}$.

To obtain the lower bound we introduce the function $G : (1,\infty) \to \mathbb{R}_+$ given by

$$G(\sigma) := \begin{cases} \displaystyle\int_{W'}^{\sigma} \beta(s)\sqrt{E^p(s)}ds & \text{if } \sigma < W', \\ 0 & \text{if } \sigma > W'. \end{cases}$$

It is clear that G is increasing and differentiable in $(1,\infty)\backslash\{W'\}$. Moreover, the uniform bound of β in $(1,W')$ implies

$$\lim_{\sigma\to 1^+} G(\sigma) = -\infty, \tag{3.32}$$

uniformly in ε, see (3.18). The Theorem of the Mean Value ensures that for any $\sigma_1,\sigma_2 > 1$ there exists $\bar\sigma \in (\sigma_1,\sigma_2)$ such that $|G(\sigma_2)-G(\sigma_1)| \leq \sqrt{E^p(\bar\sigma)}|E^\beta(\sigma_2)-E^\beta(\sigma_1)|$. On one hand, (3.31) implies that $G(W_{k+1/2})(t) \equiv G_{k+1/2} \leq C(1+T)$. On the other, for any $l > k$ there exists

$$\overline{W}_j \in \big(\min\{W_{j-1/2},W_{j+1/2}\},\max\{W_{j-1/2},W_{j+1/2}\}\big)$$

such that

$$\begin{aligned} |G_{l+1/2}-G_{k+1/2}| &\leq \sum_{j=k+1}^{l} |G_{j+1/2}-G_{j-1/2}| \leq \sum_{j=k+1}^{l} \sqrt{E^p(\overline{W}_j)}|\delta E^{\beta}_j|h \\ &\leq \sum_{j=k+1}^{l} \left(E^p_{j-1/2}+E^p_{j+1/2}\right)h + \sum_{j=k+1}^{l} (\delta E^\beta_j)^2 h \leq C(1+T), \end{aligned}$$

with C independent of h and ε, by Lemma 1. Here we used that E^p is increasing. In a similar way we prove this bound for $l > k$, and the assertion follows from (3.32). □

Proof of Theorem 3. To obtain the existence of solutions of Problem PL_ε we consider the following piecewise linear approximations of W and U:

$$W^h(X,\cdot) = \begin{cases} W_{1/2} & X \in [0, X_{1/2}], \\ \frac{X - X_{k-1/2}}{h} W_{k+1/2} + \frac{X_{k+1/2} - X}{h} W_{k-1/2}, & X \in [X_{k-1/2}, X_{k+1/2}], \\ W_{N-1/2} & X \in [X_{N-1/2}, X_N], \end{cases}$$

and, for any $X \in [X_k, X_{k+1}]$, with $k \in \{0, \ldots, N-1\}$,

$$U^h(X,t) = \frac{X - X_k}{h} U_{k+1}(t) + \frac{X_{k+1} - X}{h} U_k(t).$$

Lemma 2 allows us to obtain the following estimate, uniform in h,

$$1 < W_- \le W^h \le W_+ < \infty \quad \text{in } Q_T. \tag{3.33}$$

Using (3.33) and Lemma 1, we also obtain estimates independent of h of the norms

$$||U^h||_{L^\infty(L^2)}, \quad ||U^h_X||_{L^2}, \quad \text{and} \quad ||W^h_X||_{L^\infty(L^2)}. \tag{3.34}$$

Assumption $(\mathrm{HL1})_\varepsilon$ allows us to get uniform in h estimates for $||U^{0h}||_{L^2}$ and $||W^h_{0\varepsilon X}||_{L^2}$. Therefore, there exist functions W and U such that, up to a subsequence (not relabeled),

$$W^h \overset{*}{\rightharpoonup} W \quad \text{in } L^\infty(D_T), \tag{3.35}$$

$$W^h_X \overset{*}{\rightharpoonup} W_X \quad \text{in } L^\infty(0,T;L^2(I)), \tag{3.36}$$

$$U^h \overset{*}{\rightharpoonup} U \quad \text{in } L^\infty(0,T;L^2(I)), \tag{3.37}$$

$$U^h_X \rightharpoonup U_X \quad \text{in } L^2(D_T). \tag{3.38}$$

Equation (3.12) implies that $W^h_t(X_{k+1/2},\cdot) = U^h_X(X_{k+1/2},\cdot)$, and therefore we obtain $\|W^h_t\|_{L^2} \le 2\|U^h_X\|_{L^2}$. Hence, up to a subsequence

$$W^h_t \rightharpoonup W_t \quad \text{in} \quad L^2(D_T). \tag{3.39}$$

Convergences (3.35), (3.36), (3.39) and Corollary 4, page 85 of [26] imply

$$W^h \to W \text{ in } C(\overline{D_T}). \tag{3.40}$$

Property (3.33), convergences (3.37), (3.40) and the continuity of β, $\overline{p}$ and F imply

$$\beta^h := \beta(W^h) \to \beta(W) \quad \text{and} \quad \overline{p}^h := \overline{p}(W^h) \to \overline{p}(W) \quad \text{in } C(\overline{D_T}), \tag{3.41}$$

$$F^h := F(W^h, U^h, \cdot) \rightharpoonup F(W,U,\cdot) \quad \text{in } L^2(D_T). \tag{3.42}$$

Convergences (3.38) and (3.41) give

$$\beta^h U_X^h \rightharpoonup \beta(W)U_X \quad \text{in } L^2(D_T). \tag{3.43}$$

We now identify (W,U) as a weak solution of the problem PL_ε. Let us consider a test functions $\phi \in C^\infty(D_T)$ with compact support in $I \times [0,T)$. We use the notation

$$\begin{aligned} \phi_k(t) &:= \phi(X_k,t) \quad \text{for } k \in \{1,\dots,N-1\},\\ \phi_{k+1/2}(t) &:= \phi(X_{k+1/2},t) \quad \text{for } k \in \{0,\dots,N-1\},\\ U_k^h(t) := U^h(X_k,t), &\quad W_{k+1/2}^h(t) = W(X_{k+1/2},t), \quad \text{etc.} \end{aligned}$$

Multiplying equations (3.12) and (3.13) by $\phi_{k+1/2}(t)h$ and $\phi_k(t)h$, respectively, summing in k and integrating, we obtain that (W^h,U^h) satisfy

$$\sum_{k=0}^{N-1}\Big(W_{k+1/2}^h(0)\phi_{k+1/2}(0)h + \int_0^T W_{k+1/2}^h\dot{\phi}h\Big) - \sum_{k=1}^{N-1}\int_0^T U_k^h\delta\phi_k h = 0, \tag{3.44}$$

$$\begin{aligned} \sum_{k=1}^{N-1}\Big(U_k^h(0)\phi_k(0)h + \int_0^T U_k^h\dot{\phi}_k\Big) + \sum_{k=0}^{N-1}\int_0^T \left(\overline{p}^h - \beta^h\delta U^h\right)_{k+1/2}\delta\phi_{k+1/2}\\ + \sum_{k=1}^{N-1}\int_0^T f_k^h\phi_k = 0, \end{aligned} \tag{3.45}$$

where $f_k^h = F((W_{k+1/2}^h + W_{k-1/2}^h)/2, U_k^h, \cdot)$. For the continuous unknowns, the convergence results (3.35)-(3.43) imply that

$$\int_0^1 W^h(\cdot,0)\phi(\cdot,0) + \int_{D_T}(W^h\phi_t - U^h\phi_X), \tag{3.46}$$

$$\int_0^1 U^h(\cdot,0)\phi(\cdot,0) + \int_{D_T}(U^h\phi_t + (\overline{p}^h - \beta^h U_X^h)\phi_X + f^h\phi), \tag{3.47}$$

converge to the corresponding expressions with W^h and U^h replaced by W and U. So, for example, we have, as $h \to 0$,

$$\int_{D_T}\beta^h U_X^h\phi_X \to \int_{D_T}\beta(W)U_X\phi_X.$$

Therefore, if we show

$$\sum_{k=0}^{N-1}\int_0^T (\beta^h\delta U^h)_{k+1/2}\delta\phi_{k+1/2} \to \int_{D_T}\beta^h U_X^h\phi_X \quad \text{as } h \to 0, \tag{3.48}$$

and the same convergences for the corresponding terms of (3.46)-(3.47) then the identification of the limit is achieved. We show here how to obtain (3.48). The other terms of equations (3.44)-(3.45) are treated in a similar way. First observe that (3.34), the Lipschitz continuity of β in the interval (W_-, W_+), see (3.33), and

$$\left|W^h(X,t) - W^h_{k+1/2}(t)\right| = \left|\int_{X_{k+1/2}}^{X} W^h_X(z,t)dz\right| \le h^{1/2}\|W^h_X(\cdot,t)\|_{L^2},$$

imply

$$\left|\beta^h_{k+1/2} - \beta(W^h(X,\cdot))\right| \le Ch^{1/2} \quad \text{in } (0,T),$$

for any $X \in [X_{k-1/2}, X_{k+1/2}]$. Therefore,

$$\begin{aligned}
\int_{D_T} \beta^h U^h_X \phi_X &= \int_0^T \sum_{k=0}^{N-1} \int_{X_k}^{X_{k+1}} \beta^h U^h_X \phi_X = \int_0^T \sum_{k=0}^{N-1} \int_{X_k}^{X_{k+1}} \beta^h U^h_X \phi_X \\
&= \int_0^T \sum_{k=0}^{N-1} \int_{X_k}^{X_{k+1}} \big(\beta^h_{k+1/2} + O(h^{1/2})\big)\delta U^h_{k+1/2}\big(\delta\phi_{k+1/2} + O(h)\big) \\
&= \int_0^T \sum_{k=0}^{N-1} h\big(\beta^h_{k+1/2} + O(h^{1/2})\big)\delta U^h_{k+1/2}\big(\delta\phi_{k+1/2} + O(h)\big) \\
&= \int_0^T \sum_{k=0}^{N-1} \big(h\beta^h_{k+1/2}\delta U^h_{k+1/2}\delta\phi_{k+1/2} + O(h^{1/2})\big),
\end{aligned} \tag{3.49}$$

where, as usual, we introduced the notation $O(h)$ for a function of h, O, such that $|O(h)| \le Ch$ as $h \to 0$. From (3.49) we obtain

$$\int_0^T \sum_{k=0}^{N-1} h\beta^h_{k+1/2}\delta U^h_{k+1/2}\delta\phi_{k+1/2} \to \int_{D_T} \beta U_X \phi_X \text{ as } h \to 0,$$

which is (3.48).

Once we have that the limit (W, U) satisfies equations (3.9)-(3.10), the only additional regularity of weak solutions stated in Definition 2 which is not straightforward is that of $U \in H^1(0,T; H^{-1}(I))$. We obtain this regularity using (3.10) for any $\Phi \in L^2(0,T; H^1_0(I))$. We get

$$\int_{D_T} U\Phi_t \le \big(\|\beta(W)\|_{L^\infty}\|U_X\|_{L^2} + \|\overline{p}\|_{L^2}\big)\|\Phi\|_{L^2(H^1_0)} + \|F\|_{L^2}\|\Phi\|_{L^2},$$

and the result follows. Finally, due to the regularity of the solutions and of the initial data, (3.11) holds true. $\square$

Corollary 4. *Let $(W_\varepsilon, U_\varepsilon)$ be the weak solution of Problem PL_ε constructed in Theorem* 3. *Then*

$$\int_0^1 |U_\varepsilon(\cdot,T)|^2 + \int_{D_T} \big(U_\varepsilon^2 + \beta(U_\varepsilon)|U_{\varepsilon X}|^2\big) + \int_0^1 E^p(W_\varepsilon(\cdot,T))$$
$$\leq C\Big(T + \int_0^1 \big(|U^0|^2 + E^p(W_{0\varepsilon})\big)\Big),$$
$$\sup_{[0,T]} \int_0^1 |E^\beta(W_\varepsilon(\cdot,T))_X|^2 \leq C\Big(T + \int_0^1 |E^\beta(W_{0\varepsilon})_X|^2\Big).$$

Proof. Both inequalities are a straightforward consequence of Lemma 1, see (3.22) and (3.30), and the convergence properties shown in the proof of Theorem 3, see (3.35)-(3.43). □

4. Existence of solutions of Problem P_ε

In this section we prove that solutions of Problem PL_ε may be translated, via a change of variables, into solutions of Problem P_ε. In fact, both notions of solution are equivalent but, for brevity, we show the result in just one direction.

The main difficulty is that the natural change of variables, which depends on the solution of PL_ε constructed in the previous section, has a Jacobian matrix which is not continuously differentiable and therefore the usual theorems of change of variables may not be applied. To overcome this difficulty we consider a sequence which approximates the solution of PL_ε and accordingly define a sequence of regular maps approximating the change of variables. We then show that the corresponding sequence in Eulerian coordinates converges to a weak solution of Problem P_ε.

Theorem 5. *Assume that (W,U) is a weak solution of Problem PL_ε. Then there exists a continuous map, $\gamma : \overline{D_T} \to \overline{Q_T}$, which defines a change of variables $\gamma(X,t) := (x(X,t),t) \equiv (x,t)$, such that $0 < \|\det(D\gamma)\|_{L^\infty(D_T)} < \infty$. In addition, the pair $(\rho,u) : \overline{Q_T} \times \overline{Q_T} \to \mathbb{R}_+ \times \mathbb{R}$ defined by*

$$\rho(x,t) := 1/W(\gamma^{-1}(x,t)), \quad u(x,t) := U(\gamma^{-1}(x,t)), \tag{4.1}$$

for $(x,t) \in Q_T$, is a weak solution of Problem P_ε corresponding to the initial data $\big(1/W_{0\varepsilon}(\gamma^{-1}(\cdot,0)), U_{0\varepsilon}(\gamma^{-1}(\cdot,0))\big)$.

Remark 3. Choosing the initial data of Problem PL_ε as $W_{0\varepsilon} = 1/\rho_{0\varepsilon} \circ \gamma(\cdot,0)$ and $U^0 = u_0 \circ \gamma(\cdot,0)$, Theorem 5 provides a solution of Problem P_ε corresponding to the initial data $(\rho_{0\varepsilon}, u_0)$.

Proof. For convenience, we work with functions related to the density in Lagrangian coordinates, $R := 1/W$, instead of W. Let (W, U) be a weak solution of Problem PL_ε. The regularity (3.6)-(3.7) and the bounds (3.8) imply that $R \in L^\infty(0,T;H^1(I)) \cap H^1(0,T;L^2(I)) \cap C(\overline{D_T})$, $p(R) \in L^\infty(D_T)$, and

$$0 < \rho_- \leq R \leq \rho_+ < 1 \quad \text{in} \quad D_T. \tag{4.2}$$

In addition, (R, U) satisfies

$$R_t + R^2 U_X = 0 \quad \text{a.e. in } D_T,$$
$$\int_{D_T} \big(U\Phi_t - (R^2 + \varepsilon R)U_X \Phi_X + p(R)\Phi_X + F(1/R, U, t)\Phi\big) = 0.$$

We split the proof in several steps.

Step 1. Regularization of the solutions of $\mathbf{PL}_\varepsilon$. We consider a regularization $R_n \in C^\infty(D_T)$ of R such that $\|R_n\|_{L^\infty(H^1)} \leq \|R\|_{L^\infty(H^1)}$,

$$R_n \to R \quad \text{in } H^1(0,T;L^2(I)) \cap C(\overline{D_T}), \tag{4.3}$$

satisfies the bounds (4.2) and $\|1/R_n\|_{L^1(I)}(t) = L$ for all $t \in (0,T)$. We define $U_n(X,t) := \int_0^X (1/R_n)_t$. Then using (4.3) we deduce

$$U_n \to U \quad \text{in } L^2(0,T;H_0^1(I)), \tag{4.4}$$

where U is the second component of the weak solution of PL_ε.

Step 2. Regularization of the change of variables. We consider the map $\gamma_n : \overline{D_T} \to \overline{Q_T}$ given by $\gamma_n(X,t) := (y_n(X,t), t)$, with

$$y_n(X,t) = \int_0^X \frac{1}{R_n}(Z,t)dZ.$$

We now check that γ_n defines a smooth change of variables. First, note that

- $\dfrac{\partial y_n}{\partial X} = 1/R_n > 0$ in $\overline{D_T}$, with $y_n(0,\cdot) = 0$ and $y_n(1,\cdot) = L$ in $(0,T)$,
- $\dfrac{\partial y_n}{\partial t} = \displaystyle\int_0^X (1/R_n)_t = U_n$ in D_T, and
- $\dfrac{\partial y_n}{\partial t \partial X} = (1/R_n)_t$ in D_T.

Therefore, $\gamma_n(\overline{D_T}) = \overline{Q_T}$, $D\gamma_n \in C^\infty(D_T)$ and $\det(D\gamma_n) = 1/R_n > 0$ in $\overline{D_T}$, where $D\gamma_n$ is the Jacobian matrix of γ_n. Note that, in the limit $n \to \infty$, $\partial y_n/\partial t$ does not converge, in general, to a continuous function.

Since $\partial y_n/\partial X > 0$, we have that for every $t \in (0,T)$, there exists a function $Y_n(\cdot,t) : [0,L] \to [0,1]$ which is the inverse of $y_n(\cdot,t)$. We define

$\Gamma_n : \overline{Q_T} \to \overline{D_T}$ by $\Gamma_n(x,t) := (Y_n(x,t),t)$. Observe that Y_n is the unique solution of the problem

$$\frac{\partial Y}{\partial x} = R_n(Y,\cdot) \quad \text{in } Q_T, \quad Y(0,\cdot) = 0 \quad \text{in } (0,T).$$

Now, $0 = Y_n(y_n(0,t),t) = Y_n(0,t)$ and $1 = Y_n(y_n(1,t),t) = Y_n(L,t)$ imply $\Gamma_n(\overline{Q_T}) = \overline{D_T}$. Moreover, from $\frac{\partial}{\partial t}(y_n(Y_n(x,t),t)) = 0$, we deduce

$$\frac{\partial Y_n}{\partial t}(x,t) = -R_n(Y_n(x,t),t)U_n(Y_n(x,t),t). \tag{4.5}$$

Hence, $D\Gamma_n(Y_n,\cdot) \in C^\infty(Q_T)$ and $\det(D\Gamma_n) = R_n(Y_n,\cdot) > 0$ in $\overline{Q_T}$. Therefore, γ_n and Γ_n define smooth changes of variables.

Finally, observe that both γ_n and Γ_n converge uniformly, up to a subsequence, to some continuous functions γ and Γ. For the first, it is an easy consequence of the convergence (4.3). For the second, we have for all $(x_1,t_1),(x_2,t_2) \in Q_T$,

$$\begin{aligned}
&||\Gamma_n(x_2,t_2) - \Gamma_n(x_1,t_1)|| \\
&\leq |Y_n(x_2,t_2) - Y_n(x_1,t_2)| + |Y_n(x_1,t_2) - Y_n(x_1,t_1)| + |t_2 - t_1| \\
&\leq \|R_n\|_{L^\infty} \left(|x_2 - x_1| + \|U_n\|_{L^2(L^\infty)} |t_2 - t_1|^{1/2}\right) + |t_2 - t_1|,
\end{aligned}$$

which implies that the sequence Γ_n is uniformly continuous, and the assertion follows. We used the continuous imbedding $H^1 \subset L^\infty$.

Step 3. Convergence in Eulerian coordinates. We define

$$\rho_n(x,t) := R_n(Y_n(x,t),t), \quad u_n := U_n(Y_n(x,t),t) \quad \text{for} \quad (x,t) \in Q_T.$$

Using the convergence of (R_n,U_n) to (R,U), see (4.3) and (4.4), the bounds (4.2), the identity (4.5) and the continuous imbedding $H^1 \subset L^\infty$, we obtain

$$\begin{aligned}
\int_{Q_T} |\rho_{nt}|^2 &= \int_{Q_T} \Big| \frac{\partial R_n}{\partial t}(Y_n,\cdot) - \frac{\partial R_n}{\partial X}(Y_n,\cdot)R_n(Y_n,\cdot)U_n(Y_n,\cdot)\Big|^2 \\
&= \int_{D_T} \Big| \frac{\partial R_n}{\partial t} - \frac{\partial R_n}{\partial X} R_n U_n \Big|^2 \frac{1}{R_n} \\
&\leq c \left(\|R_{nt}\|_{L^2} + \|R_n\|_{L^\infty(H^1)} \|U_n\|_{L^2(L^\infty)} \right)^2 \leq c.
\end{aligned}$$

In a similar way, we obtain uniform bounds for $\|\rho_{nx}\|_{L^2}$, $\|u_n\|_{L^\infty(L^2)}$ and $\|u_{nx}\|_{L^2}$. These bounds imply the existence of a subsequence (ρ_n,u_n) converging weakly to (ρ,u), with

$$\rho \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{4.6}$$

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \tag{4.7}$$

For the time derivative of u_n, we have that for any $\phi \in C^\infty(Q_T)$,

$$\begin{aligned}
\int_0^T \langle u_{nt}, \phi\rangle_{H^{-1}\times H_0^1} &= \int_{Q_T} \Big(\frac{\partial U_n}{\partial t}(Y_n,\cdot) - \frac{\partial U_n}{\partial X}(Y_n,\cdot)R_n(Y_n,\cdot)U_n(Y_n,\cdot)\Big)\phi \\
&= \int_{D_T} \Big(\frac{\partial U_n}{\partial t} - \frac{\partial U_n}{\partial X}R_n U_n\Big)\frac{\phi(y_n,\cdot)}{R_n} \\
&= \int_{D_T} \Big(\frac{\partial U_n}{\partial t}\frac{\phi(y_n,\cdot)}{R_n} + \frac{1}{2}U_n^2 \frac{\partial \phi}{\partial X}(y_n,\cdot)\Big) \\
&\le c\left(\|U_{nt}\|_{L^2} + \|U_n\|_{L^4}^2\right)\|\phi\|_{L^2(H_0^1)} \le c.
\end{aligned}$$

Using the uniform convergence of γ_n and the continuous embedding

$$L^2(0,T;H_0^1(I)) \cap L^\infty(0,T;L^2(I)) \subset L^4(D_T)$$

we obtain

$$\|u_{nt}\|_{L^2(H^{-1})} \le c(\|U_{nt}\|_{L^2} + \|U_n\|^2_{L^2(H_0^1)\cap L^\infty(L^2)}) \le c,$$

and therefore $u \in H^1(0,T;H^{-1}(\Omega))$ and

$$u_n \rightharpoonup u \quad \text{in } H^1(0,T;H^{-1}(\Omega)). \tag{4.8}$$

Step 4. Identification of the limit. To prove that (ρ, u) is a weak solution of Problem P_ε we first need to show that the test functions of problems PL_ε and P_ε are conveniently related to each other. Let $\phi \in C([0,T];C_0^1(\overline{\Omega})) \cap H^1(0,T;L^2(\Omega))$ and define $\Phi_n := \phi\circ\gamma_n$. We have

$$\Phi_{nt} = \frac{\partial \phi}{\partial t}(y_n,\cdot) + \frac{\partial \phi}{\partial x}(y_n,\cdot)U_n, \quad \text{and} \quad \Phi_{nX} = \frac{\partial \phi}{\partial x}(y_n,\cdot)\frac{1}{R_n},$$

in D_T, and then

$$\|\Phi_{nt}\|_{L^2} \le c(\|\phi_t\|_{L^2} + \|\phi_x\|_{L^\infty}\|U_n\|_{L^2}) \quad \text{and} \quad \sup_{D_T}|\Phi_{nX}| \le c \sup_{Q_T}|\phi_x|\,.$$

The convergence of γ_n and R_n imply that the sequence Φ_n is uniformly bounded in $C([0,T];C_0^1(\overline{I})) \cap H^1(0,T;L^2(I))$. Hence, there exists a function Φ such that $\Phi_{nX} \to \Phi_X$ uniformly in D_T and $\Phi_{nt} \rightharpoonup \Phi_t$ in $L^2(D_T)$. Furthermore, $\Phi = \phi\circ\gamma$, is a test function of Problem PL_ε.

To prove that (ρ, u) is a solution of Problem P_ε, we observe that (ρ_n, u_n) satisfy

$$\frac{\partial \rho_n}{\partial t} + \frac{\partial \rho_n}{\partial x}u_n + \rho_n\frac{\partial u_n}{\partial x} = \frac{\partial R_n}{\partial t}(Y_n,\cdot) - \frac{\partial R_n}{\partial X}(Y_n,\cdot)R_n(Y_n,\cdot)U_n(Y_n,\cdot)$$

$$+\frac{\partial R_n}{\partial X}(Y_n,\cdot)R_n(Y_n,\cdot)U_n(Y_n,\cdot)+R_n^2(Y_n,\cdot)\frac{\partial U_n}{\partial X}(Y_n,\cdot)=0.$$

Passing to the limit we obtain equation (2.10). On the other hand, for $\phi \in C([0,T];C_0^1(\overline{\Omega}))\cap H^1(0,T;L^2(\Omega))$, we have

$$\begin{aligned}&\int_{Q_T}(\rho_n u_n(\phi_t+u_n\phi_x)-(\rho_n+\varepsilon)u_{nx}\phi_x+p(\rho_n)\phi_x+\rho_n f(\rho_n,u_n,t)\phi)\\ &=\int_{D_T}(U_n\Phi_{nt}+(p(R_n)-(R_n+\varepsilon)R_nU_{nX})\Phi_{nX}+F(R_n,U_n,t)\Phi_n)\,. \qquad (4.9)\end{aligned}$$

The convergence of (R_n,U_n) to (R,U), see (4.3) and (4.4), and the convergence of Φ_n to a test function of Problem PL_ε imply that the right hand side of (4.9) tends to zero. The uniform convergence of Γ_n implies the uniform convergence of ρ_n, and the bound of u_n in $L^4(Q_T)$ implies the convergence of u_n^2 in $L^2(D_T)$. Then, since $(\rho_n,u_n)\to(\rho,u)$, see (4.6), (4.7) and (4.8), we may pass to the limit on the left hand of (4.9). □

5. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. Let $(\rho_\varepsilon,u_\varepsilon)$ be the solution of Problem P_ε given by Theorems 3 and 5. Then, using Theorem 5 and Corollary 4, we get

$$\begin{aligned}\int_0^L\rho_\varepsilon(\cdot,T)u_\varepsilon^2(\cdot,T)+\int_{Q_T}\big(\rho_\varepsilon u_\varepsilon^2+(\rho_\varepsilon+\varepsilon)|u_{\varepsilon x}|^2\big)+\int_0^L\rho_\varepsilon(\cdot,T)E^p(\frac{1}{\rho_\varepsilon}(\cdot,T))\\ =\int_0^1U_\varepsilon^2+\int_{D_T}\big(U_\varepsilon^2+\beta(U_\varepsilon)|U_{\varepsilon X}|^2\big)+\int_0^1E^p(W_\varepsilon)\\ \le C\left(T+\int_0^L\big(\rho_{0\varepsilon}u_0^2+\rho_{0\varepsilon}E^p(1/\rho_{0\varepsilon})\big)\right), \qquad (5.1)\end{aligned}$$

and

$$\begin{aligned}4||(\sqrt{\rho_\varepsilon})_x||^2_{L^\infty(L^2)}&=\sup_{[0,T]}\int_0^L\frac{\rho_\varepsilon^2\,|\rho_{\varepsilon x}|^2}{\rho_\varepsilon^3}\le\sup_{[0,T]}\int_0^L\frac{\beta^2(1/\rho_\varepsilon)}{\rho_\varepsilon}|(\frac{1}{\rho_\varepsilon})_x|^2 \qquad (5.2)\\ &=\sup_{[0,T]}\int_0^1\Big|E^\beta(W_\varepsilon)_X\Big|^2\le C\big(T+\int_0^L(\frac{\varepsilon+\rho_{0\varepsilon}}{\rho_{0\varepsilon}^{3/2}}\rho_{0\varepsilon x})^2\big).\end{aligned}$$

Using the assumptions $(\mathrm{H1})_\varepsilon$ and (H3) on the initial data we deduce that the right hand sides of (5.1)-(5.2) are uniformly bounded with respect to ε. In particular, from (5.1) we obtain uniform bounds for $\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty(L^2)}$ and

$\|\sqrt{\rho_\varepsilon+\varepsilon}u_{\varepsilon x}\|_{L^2}$, and from (5.2) we obtain a uniform estimate for $||\rho_{\varepsilon x}||_{L^\infty(L^2)}$. From equation (2.10), we also obtain the uniform estimate

$$\begin{aligned}||\rho_{\varepsilon t}||_{L^2(L^1)} = ||\rho_\varepsilon u_{\varepsilon x} + \rho_{\varepsilon x} u_\varepsilon||_{L^2(L^1)} &\leq C\|\sqrt{\rho_\varepsilon}\|_{L^\infty}\|\sqrt{\rho_\varepsilon}u_{\varepsilon x}\|_{L^2} \\ &+ C\|(\sqrt{\rho_\varepsilon})_x\|_{L^\infty(L^2)}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty(L^2)} \leq C.\end{aligned}$$

Finally, Theorem 3, gives $W_\varepsilon \geq W_- \equiv 1/\rho_+$ in D_T, with ρ^+ independent of ε. Then by (4.1) of Theorem 5 we deduce $\rho < \rho^+$ in Q_T, for the same ρ^+.

Proof of Theorem 2. Estimates of Theorem 1 and Corollary 4, page 85 of [26], imply (2.14) and first part of (2.15). To prove the first part of (2.17), we use the sign function defined, as usual, by sign(s) $= 1$ if $s > 0$, sign(s) $= -1$ if $s < 0$, and sign$(0) = 0$, and the characteristic function of the set Q^*, denoted by 1_{Q^*}. Estimate (2.13) implies

$$\int_{Q_T} m_\varepsilon \mathrm{sign(m)} 1_{\mathrm{Q}^*} \leq \|\sqrt{\rho_\varepsilon}\mathrm{u}_\varepsilon\|_{\mathrm{L}^\infty(\mathrm{L}^2)}\|\mathrm{sign(m)}\sqrt{\rho_\varepsilon}1_{\mathrm{Q}^*}\|_{\mathrm{L}^1(\mathrm{L}^2)} \to 0.$$

On the other hand

$$\int_{Q_T} m_\varepsilon \mathrm{sign(m)} 1_{\mathrm{Q}^*} \to \int_{\mathrm{Q}^*} |\mathrm{m}|,$$

and therefore $m = 0$ in Q^*.

To obtain the second part of (2.14) we observe that the estimate of $||\sqrt{\rho_\varepsilon + \varepsilon}(\frac{m_\varepsilon}{\rho_\varepsilon})_x||_{L^2}$ in (2.13) implies both the existence of a function $\eta \in L^2(Q_T)$ such that

$$\sqrt{\rho_\varepsilon}(\frac{m_\varepsilon}{\rho_\varepsilon})_x \rightharpoonup \eta \quad \text{in } L^2(Q_T),$$

and

$$\varepsilon(\frac{m_\varepsilon}{\rho_\varepsilon})_x = \sqrt{\varepsilon}\sqrt{\varepsilon}(\frac{m_\varepsilon}{\rho_\varepsilon})_x \rightharpoonup 0 \quad \text{in } L^2(Q_T).$$

Since (2.14) implies $\sqrt{\rho_\varepsilon} \to \sqrt{\rho}$ uniformly, there exists a function $\zeta \in L^2(Q_T)$ such that

$$\rho_\varepsilon(\frac{m_\varepsilon}{\rho_\varepsilon})_x \rightharpoonup \sqrt{\rho}\eta := \zeta \quad \text{in } L^2(Q_T),$$

and we thus obtain both second parts of (2.15) and (2.17).

The estimate of $||\frac{m_\varepsilon}{\sqrt{\rho_\varepsilon}}||_{L^\infty(L^2)}$ in (2.13) implies

$$\frac{m_\varepsilon}{\sqrt{\rho_\varepsilon}} \overset{*}{\rightharpoonup} \Gamma \quad \text{in } L^\infty(0,T;L^2(\Omega)).$$

To obtain (2.18) we consider the sets

$$Q_\varepsilon^\delta = \{(x,t) : \rho_\varepsilon(x,t) > \delta\} \quad \text{and} \quad Q^\delta = \{(x,t) : \rho(x,t) > \delta\}\,.$$

Due to the uniform convergence $\rho_\varepsilon \to \rho$, we have that for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that $Q^\delta \subset Q_\varepsilon^{\delta/2}$ for every $\varepsilon < \varepsilon_0$. Then

$$\begin{aligned} ||(\frac{m_\varepsilon}{\rho_\varepsilon})_x||_{L^2(Q^\delta)} = \|u_{\varepsilon x}\|_{L^2(Q^\delta)} \leq \|u_{\varepsilon x}\|_{L^2(Q_\varepsilon^{\delta/2})} \leq \sqrt{\frac{2}{\delta}}||\sqrt{\rho_\varepsilon}u_{\varepsilon_x}||_{L^2(Q_\varepsilon^{\delta/2})} \\ \leq \sqrt{\frac{2}{\delta}}||\sqrt{\rho_\varepsilon}u_{\varepsilon_x}||_{L^2(Q_T)}. \end{aligned}$$

On the other hand, the first two convergences in (2.14) imply

$$\rho_\varepsilon(\frac{m_\varepsilon}{\rho_\varepsilon})_x \rightharpoonup \rho(\frac{m}{\rho})_x \quad \text{in} \quad L^2(Q^\delta),$$

which prove (2.18).

The estimate of $m_\varepsilon^2/\rho_\varepsilon$ in (2.13) implies the convergence (2.16). We may identify the limit in the sets Q^δ and therefore obtain first part of (2.19). The second part of (2.19) is deduced by testing against functions with support in $Q \backslash Q^*$.

Finally, to obtain (2.20), we use (2.10), (2.11), (2.13), and Corollary 4, page 85 of [26]. □

Remark 4. The proofs of Theorems 1 and 2 in the general case $\alpha > 1/2$ are just a slight modification of the proof for the case $\alpha = 1$. The main difference arises when showing (2.14). In the general case we have, instead of (3.17), the following definition of the energy E^β

$$E^\beta(\sigma) := \int_{W'}^{\sigma} \beta(s) ds = \Big[\varepsilon \log s - \frac{1}{\alpha}\frac{1}{s^\alpha}\Big]_{W'}^{\sigma},$$

with which we may obtain inequality (3.30) again. The assumptions on the initial condition (2.6) allow us to bound the right hand side of inequality (3.30) uniformly in ε and h. Then in a similar way than in (5.2), we obtain a uniform bound of $\rho_x^{\alpha-1/2}$ in $L^\infty(0,T;L^2(\Omega))$. Obtaining the other estimates in (2.13) is straightforward. These estimates allow us to prove the uniform convergence of ρ_ε^α, and then we easily deduce (2.14).

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