Existence of solutions and stability analysis for a Darcy flow with extraction

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Abstract

In [8], a one-dimensional model describing the vertical movement of water and salt in a porous medium in which a continuous extraction of fresh water takes place was studied. Among other results, it was shown that for some range of parameter values, a heavier layer of water is formed above a lighter one in the transient state, having, however, a unique stable steady state. In this paper, we study the N-dimensional spatial model, for which Darcy's law must be introduced in the flow description. We prove the existence of weak solutions to the time evolution problem and perform a heuristic stability analysis in two ways: analytically, for a related problem, to find an approximation of the bifurcation curve in terms of the Raighley number, and numerically, to show the formation of instabilities in the original problem and their influence on the speed of convergence towards the stable steady state.

1 Introduction

Consider a water saturated bounded porous medium with horizontal upper and lower boundaries containing a solute, and suppose that a extracting mechanism within the upper part of the medium produces an upward flow of fresh water out through the upper boundary while keeping most of the solute content within the medium. If the fresh water extraction is strong

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enough then a solute high concentration layer is created in the extraction region, on the top of a lower concentration region and therefore gravitational driven instabilities are expected to arise. This is a well known phenomenon observed in the ecology of mangroves, see [15, 4]. In Section 4 we deduce the following mathematical model. Let $u \in [0,1]$ be the solute concentration, $\mathbf{q}$ the water flow discharge and p the pressure, and consider the domain $Q_T = \Omega \times (0,T)$ for $T > 0$ and $\Omega = B \times (0,1)$, with $B \subset \mathbb{R}^{N-1}$ bounded. Find $u, p : \bar{Q}_T \to \mathbb{R}$ and $\mathbf{q} : \bar{Q}_T \to \mathbb{R}^N$ such that

$$u_t + \operatorname{div}(Ru\mathbf{q} - \nabla u) = 0, \tag{1}$$

$$\operatorname{div} \mathbf{q} + mf(\cdot, u) = 0, \tag{2}$$

$$\mathbf{q} + \nabla p - u\mathbf{e}_z = 0, \tag{3}$$

in Q_T. Positive parameters R and m stand for the Rayleigh and the *extraction* numbers of the physical system, see (73)-(74). The vector $\mathbf{e}_z$ is the canonical vertical vector pointing downwards. In (2), the extraction function $f : \bar{B} \times [0,1] \times [0,1] \to \mathbb{R}_+$ is usually assumed to have the form

$$f(\mathbf{x}, z, \sigma) := s(z)(1-\sigma)_+^r, \tag{4}$$

with $r > 0$ and s describing the localization of the extraction region, given by (for $d \in (0,1)$)

$$s(z) := \begin{cases} 1 & \text{if } z \in [0,d], \\ 0 & \text{if } z \in (d,1]. \end{cases} \tag{5}$$

The spatial boundary is decomposed as $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D = B \times \{0\}$ and $\Gamma_N = \big(B \times \{1\}\big) \cup \big(\partial B \times (0,1)\big)$. The following boundary conditions are prescribed

$$u = u_D, \quad p = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{6}$$

$$\nabla u \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N \times (0,T). \tag{7}$$

A non-negative initial distribution, u_0, is considered to close the problem

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{8}$$

In this article we are interested in two questions: first is proving the existence of solutions of problem (1)- (3) and (6)-(8), which we shall refer to as *Problem P*. Second is a stability issue. In [8], in the context of one dimensional spatial variable (depth) it is proven that if the exponent r in function f, see (4), is smaller than one then the solute concentration may reach the threshold value $u = 1$ in finite time in some subset of $(0, d)$ while $u < 1$ below that layer for all $T < \infty$. This is clearly an instable situation and it is therefore expectable to observe gravitational instabilities when perturbations of the one-dimensional profile are considered in the N-dimensional setting. Our aim is to provide a range of values for the bifurcation parameter, R, for which these instabilities appear.

Existence of solutions for Darcy flows is already established for a large variety of physical situations which translate into different mathematical models, among which the most treated in the literature are the porous medium equation, the black oil system (two-phase filtration problem) and the dam problem see, for instance, [18, 2, 11, 7] and the references therein. Many of these problems neglect the gravity effects expressed by the term $u\mathbf{e}_z$ in the Darcy's equation (3) and set the problem only in terms of the concentration and the pressure. Similarly, system (1)-(3) may be reduced to equations

$$u_t - \operatorname{div}(Ru(\nabla p - u\mathbf{e}_z) + \nabla u) = 0, \tag{9}$$

$$-\Delta p + \frac{\partial u}{\partial z} + mf(\cdot, u) = 0. \tag{10}$$

Note that when gravity effects, expressed by the term $\partial u/\partial z$, may be neglected then the resulting problem lies in the general setting studied, for instance, in [1]. However, if gravity effects have to be taken into account then this reduction is not appropriate. The reason is that the key role of $\operatorname{div}\mathbf{q} \in L^\infty$ is hidden in formulation (9)-(10), and a usual fixed point technique for proving existence of solutions by compactness arguments lead to the consideration of Sobolev spaces with somehow rare dimension-dependent exponents. On the contrary, direct consideration of system (1)-(3) leads to a simpler proof of existence of solutions by uncoupling the original system in two sets of equations with independent physical meaning: concentration evolution with prescribed convection, in one hand, and flow-pressure balance with prescribed concentration, on the other. In addition, the numerical scheme corresponding to this approach is more efficient, see [5, 13].

The stability properties of equations (1)-(3) has also received attention for a variety of data, and phenomena like cellular convection or fingering have been proven to arise when the bifurcation parameter, R, is large enough. For instance, the steady state one-dimensional solution of the model problem $L = \infty$, $f = 0$, $\mathbf{q}\cdot\mathbf{n} = 0$ on $\partial\Omega\times(0, T)$ and $u = 1$ on $\Gamma_D\times(0, T)$ is known to be instable for values $R > 4\pi^2$, see for instance [16]. Other interesting models related to ours which also lead to gravitational instabilities are the salt lake formation by evaporation ($q \cdot \mathbf{n} = -const.$ on $\Gamma_D \times (0, T)$), see [9], or the peat moss formation ($f = 0$ and the temperature $u = u_D(t)$ on Γ_D), see [17].

The common feature of these models is the existence of an instable ground state, i.e., a steady one-dimensional solution which may be gravitationally instable. Analysis of the perturbation equations (linearized or not) and the study of a maximization problem for the bifurcation parameter is the usual approach for finding the threshold value of R above which instabilities occur.

For Problem P, due to the non-zero extraction term f and to the non-flow boundary condition $\mathbf{q} \cdot \mathbf{n} = 0$ on $\Gamma_N \times (0, T)$, the one-dimensional steady state solution is stable, see [8]. Therefore, what we mean by instabilities associated to solutions of Problem P is slightly different from the common use. Instabilities in solutions of Problem P appear, if they do, only in the transient state, when there exists the possibility of the formation of a layer of heavier fluid above a layer of lighter fluid. However, when $t \to \infty$ these instabilities diminish in size and do disappear in infinite time. A rigorous mathematical analysis of this phenomenon is out of our scope but its physical interest, which resides in the shortening of the time rate at which solutions to the evolution problem approach to the steady state, induce us to study certain approximations which may be treated rigorously, and to demonstrate by means of numeric simulations that the behavior of solutions to the approximated problems and to the original problem are similar, at least in the selected parameters range.

The outline of the paper is the following: in Section 2 we give the precise assumptions and definitions for the theorem on existence of solutions and we perform the stability analysis together with the numerical simulations. In Section 3 we give the proof of existence of solutions. Finally, we present the mathematical modelling of the physical problem in Section 4.

2 Main results

Although in the stability analysis and subsequent numerical experiments we assume $N = 2$, with B a bounded interval in $\mathbb{R}$, and we take f of the form (4), we may generalize these assumptions for the proof of existence of solutions. The main property of f, apart from regularity requirements, is *to switch off* the dynamics of the system when the solute concentration, u, takes the threshold value $u = 1$.

2.1 Hypothesis and definitions

We shall refer to problem (1)-(8) as to *Problem P*, for which we assume the following hypothesis:

H_1. The spatial domain $\Omega \subset \mathbb{R}^N$ is bounded with a Lipschitz continuous boundary, $\partial\Omega$, which is decomposed as $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$ and with Γ_D of positive $N-1$ dimensional measure.

H_2. The function $f : \bar{\Omega} \times [0,1] \to \mathbb{R}$ satisfies

$$
\begin{aligned}
&f(x,\cdot) \in C([0,1]) \text{ for a.e. } x \in \Omega,\\
&f(\cdot,s) \in L^\infty(\Omega) \text{ for all } s \in [0,1],\\
&f(x,\cdot) \text{ is non-increasing in } [0,1] \text{ and } \ f(x,1) = 0 \text{ for a.e. } x \in \Omega.
\end{aligned}
$$

Note that, in particular, $f \geq 0$ in $\bar{\Omega} \times [0,1]$.

H_3. The initial and boundary data posses the regularity

$$
\begin{aligned}
&u_0 \in L^\infty(\Omega) \quad \text{and } 0 \leq u_0 \leq 1 \quad \text{a.e. in } \Omega,\\
&u_D \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \quad \text{and } 0 \leq u_D \leq 1 \quad \text{a.e. in } Q_T.
\end{aligned}
$$

H_4. The numbers R and m are positive.

There are several reasons because we can not expect to find classical solutions of Problem P. Among them, the regularity of the domain (corners), the mixed boundary conditions, and the possible degeneration of the problem through the semilinear term f. Thus, we introduce the following notion of solution.

Definition 1. We say that $(u, \mathbf{q}, p)$ is a *weak solution* of Problem P if $u : \bar{Q}_T \to (0,1]$, $\mathbf{q} : \bar{Q}_T \to \mathbb{R}^N$ and $p : \bar{Q}_T \to \mathbb{R}$ satisfy the following properties:

1. Regularity of solutions:

$$
\begin{aligned}
u \in& u_D + L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}') \cap L^\infty(Q_T),\\
\mathbf{q} \in& L^2(0,T;H_{0,N}(\mathrm{div},\Omega)) \cap \mathcal{W}_T,\\
p \in& L^2(0,T;\mathcal{V}),
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{V} :=& \left\{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_D\right\},\\
H_{0,N}(\mathrm{div},\Omega) :=& \left\{\boldsymbol{\phi} \in L^2(\Omega)^N : \mathrm{div}\,\boldsymbol{\phi} \in L^2(\Omega), \boldsymbol{\phi}\cdot\mathbf{n} = 0 \text{ on } \Gamma_N\right\},\\
\mathcal{W}_T :=& \left\{\boldsymbol{\phi} \in L^2(Q_T)^N : \mathrm{div}\,\boldsymbol{\phi} \in L^\infty(Q_T)\right\}.
\end{aligned}
$$

2. For all test function $\varphi \in \mathcal{V}$, $\xi \in L^2(\Omega)$, $\boldsymbol{\phi} \in H_{0,N}(\mathrm{div}, \Omega)$ and for a.e. $t \in (0, T)$, we have

$$< u_t, \varphi > - \int_\Omega (Ru\mathbf{q} - \nabla u) \cdot \nabla \varphi = 0, \tag{11}$$

$$\int_\Omega \mathbf{q} \cdot \boldsymbol{\phi} - \int_\Omega p \,\mathrm{div}\, \boldsymbol{\phi} - \int_\Omega u\mathbf{e}_z \cdot \boldsymbol{\phi} = 0, \tag{12}$$

$$\int_\Omega (\mathrm{div}\, \mathbf{q} + mf(\cdot, u))\xi = 0, \tag{13}$$

with $< \cdot, \cdot >$ denoting de duality product $\mathcal{V}' \times \mathcal{V}$.

3. The initial distribution is satisfied in the sense

$$\lim_{t \to 0} \|u(\cdot, t) - u_0\|_{L^2(\Omega)} = 0.$$

2.2 Existence of solutions

We prove the following theorem in Section 3.

Theorem 1. *Assume H_1-H_4. Then there exists a weak solution of Problem P.*

2.3 Stability analysis

As mentioned in the Introduction, in [8] the one dimensional setting of problem (1)-(8) is studied neglecting the horizontal dependence of all unknowns of the problem. It is proven that if the exponent in function f is $r < 1$ then the concentration may reach the threshold value, $u = 1$, in finite time in some subset of the extraction region $(0, d)$, while $u < 1$ below, in $(d, 1)$, for all $T < \infty$. Therefore, when we consider the one-dimensional profile as a solution of the N-dimensional problem (with data independent of the horizontal variables) then it is possible for perturbations of this solution to grow. However, this situation only may take place in the transient state since the steady state for the one dimensional problem is given by a concentration function which is increasing with depth, i.e., stable. The mathematical explanation for such a stable solution after a possibly instable transient state is given by the non-flow boundary condition at the bottom of the domain, condition which allows the solute *produced* in the extraction zone to fill up the region below this zone till the bottom boundary in infinite time.

Therefore, the question of stability for Problem P is not whether the steady state one-dimensional profile is stable or not under perturbations (which it is) but if the solutions of Problem P may develop transient instabilities which, in fact, will be attenuated when $t \to \infty$.

To give some clues to this question, we shall consider a related problem which is simpler to deal with but which keeps relevant information about solutions of Problem P. In this section and for the numerics, we fix $N = 2$, although it is not an essential assumption for the analysis that follows.

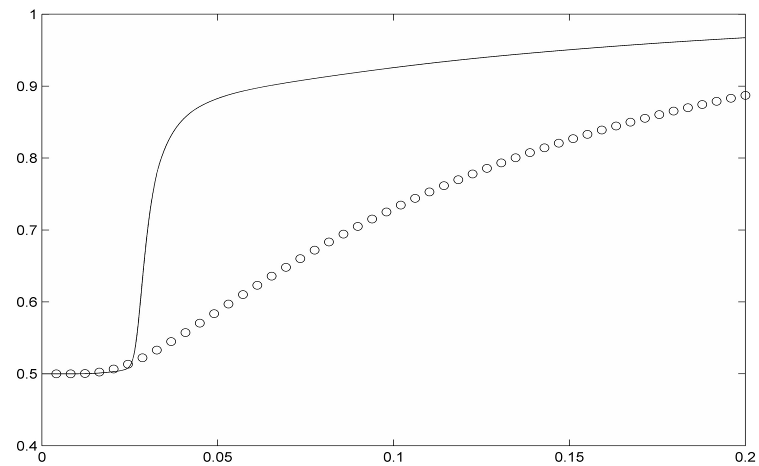


Figure 1: Time evolution of concentrations of the one (circles) and two-dimensional (continuous line) models at the bottom boundary, $z = 1$. Transient instabilities are reflected in the steep increase of the concentration of the 2D model. However, the concentration of the 1D model grows slowly. $R = 500$.

We consider the situation in which, after some time $T^* < T$, the solution to the one-dimensional problem has developed a dead core, i.e., an interval $(a, b) \subset (0, d)$ in which $u = 1$ for all $t > T^*$. For simplicity, we assume $b = d$, i.e., the dead core reaches the boundary of the no extraction region. Following, we investigate the stability of this one-dimensional configuration in the two-dimensional setting, with a modified boundary condition on $z = 1$. Consider the domain $\Omega_d = (0, L) \times (d, 1)$, where $f \equiv 0$. The top boundary, $z = d$, corresponds to the boundary between the dead core and the no-extraction region so we prescribe $u = 1$ on this boundary. On the bottom boundary, $z = 1$, we take constant Dirichlet data $u^* < 1$ instead of the non-flow boundary data, assuming that the value of u in $z = 1$ for the one-dimensional problem does not vary too fast for the time scale of the transient instabilities we are studying, see Figure 1. Therefore, we set the following boundary conditions

$$u(x,d,t) = 1, \quad u(x,1,t) = u^* \quad \text{for } x \in (0,L) \tag{14}$$

$$p(x,d,t) = p_0, \quad q_2(x,1,t) = 0 \quad \text{for } x \in (0,L), \tag{15}$$

$$\frac{\partial u}{\partial x}(0,z,t) = \frac{\partial u}{\partial x}(L,z,t) = 0 \quad \text{for } z \in (d,1), \tag{16}$$

$$q_1(0,z,t) = q_1(L,z,t) = 0 \quad \text{for } z \in (d,1), \tag{17}$$

for $t > T^*$, $\mathbf{q} = (q_1, q_2)$ and a constant p_0. Since T^* will not play any important role in the analysis, we set $T^* = 0$. In the domain $\Omega_d \times (0,T)$, functions $(u, \mathbf{q}, p)$ satisfy

$$u_t + R\mathbf{q} \cdot \nabla u - \Delta u = 0, \tag{18}$$

$$\operatorname{div} \mathbf{q} = 0, \tag{19}$$

$$\mathbf{q} + \nabla p - u\mathbf{e}_z = 0. \tag{20}$$

Stability conditions for equations (18)-(20) is a well known issue and has been established for different types of boundary conditions, see for instance [9, 16]. In fact, problem (14)-(20) only differs from the one treated in [16] in two points: the domain is bounded in the horizontal direction ($L < \infty$) and the non-flow boundary condition on the top domain, $q_2(d,x,t) = 0$ of [16] is replaced by our condition on the pressure $p(d,x,t) = p_0$. The stability analysis is based on the expansion

$$\tilde{u} = U_0 + u, \quad \tilde{\mathbf{q}} = Q_0 + \mathbf{q} \quad \text{and} \quad \tilde{p} = P_0 + p, \tag{21}$$

with $\tilde{u}$, $\tilde{\mathbf{q}}$ and $\tilde{p}$ satisfying problem (14)-(20) and with U_0, Q_0 and P_0 the solution to the corresponding one dimensional steady state problem, given by

$$U_0(z) = 1 - \gamma(z-d), \quad Q_0(z) = 0, \quad P_0(z) = p_0 + \int_d^z U_0(s)ds,$$

for $z \in (d,1)$ and $\gamma = (1-u^*)/(1-d)$. Substituting (21) into equations (18)-(20) and omitting tildes, yields the following system for the perturbations

$$u_t + R\mathbf{q} \cdot \nabla u - \Delta u = \gamma R q_2, \tag{22}$$

$$\operatorname{div} \mathbf{q} = 0, \tag{23}$$

$$\mathbf{q} + \nabla p - u\mathbf{e}_z = 0, \tag{24}$$

in $\Omega_d \times (0,T)$, satisfying the homogeneous boundary conditions corresponding to (14)-(17). Conditions for nonlinear stability are deduced in the usual way. Multiplying (22) by u, integrating by parts and using (23) and the boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_d} u^2 + \int_{\Omega_d} \nabla u^2 = \gamma R \int_{\Omega_d} q_2 u. \tag{25}$$

Multiplying (24) by $\alpha\mathbf{q}$, $\alpha > 0$ and adding the result to (25) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_d} u^2 + \int_{\Omega_d} \nabla u^2 + \alpha \int_{\Omega_d} |\mathbf{q}|^2 = (\gamma R + \alpha) \int_{\Omega_d} q_2 u. \tag{26}$$

Then, if

$$\frac{1}{\gamma R + \alpha} \geq \frac{1}{\gamma R^*(\alpha) + \alpha} = \sup \frac{\int_{\Omega_d} q_2 u}{\int_{\Omega_d} \nabla u^2 + \alpha \int_{\Omega_d} |\mathbf{q}|^2}, \tag{27}$$

where the supremo is taken among the admissible functions, i.e., satisfying (i) the regularity requirements of a weak solution of Problem P, (ii) the homogeneous boundary conditions corresponding to (14)-(17), and (iii) $\operatorname{div}\mathbf{q} = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_d} u^2 \leq 0. \tag{28}$$

Therefore, the stability criterium for solutions of problem (14)-(20) is reduced to solving the maximization problem of the right hand side of (27). We observe that, as in [16], the Euler-Lagrange equations associated to the maximization problem are just the time independent linearized version of (22)-(24), implying that the linear and nonlinear estimates for the bifurcation parameter coincide. We note that the above mentioned differences between the problem treated in [16] and problem (14)-(20) affect to the maximization problem via the set of admissible functions. Indeed, our condition $p = 0$ on $\Gamma_D \times (0, T)$ implies that the linear problem to be solved is coupled for the three unknowns and not only for u and $\mathbf{q}$, as in [16]. Nevertheless, the similarities between both problems are evident.

Assuming periodic behavior in the horizontal variable for the steady state solution of the perturbation problem (22)-(24), namely $u(x,z) = e^{iax}U(z)$, $p(x,z) = e^{iax}P(z)$ and $q_2(x,z) = e^{iax}Q(z)$, we are led to solve the following problem: Find the minimum R^* such that there exist a non-trivial solution $U, P, Q : (d, 1) \to \mathbb{R}$ of

$$-U'' + a^2 U = \gamma R^* Q, \quad U(d) = 0, \quad U(1) = 0, \tag{29}$$
$$-P'' + a^2 P = -U', \quad P(d) = 0, \quad P'(1) = 0, \tag{30}$$
$$-Q'' + a^2 Q = a^2 U, \quad Q(d) + P'(d) = 0, \quad P'(1) = 0. \tag{31}$$

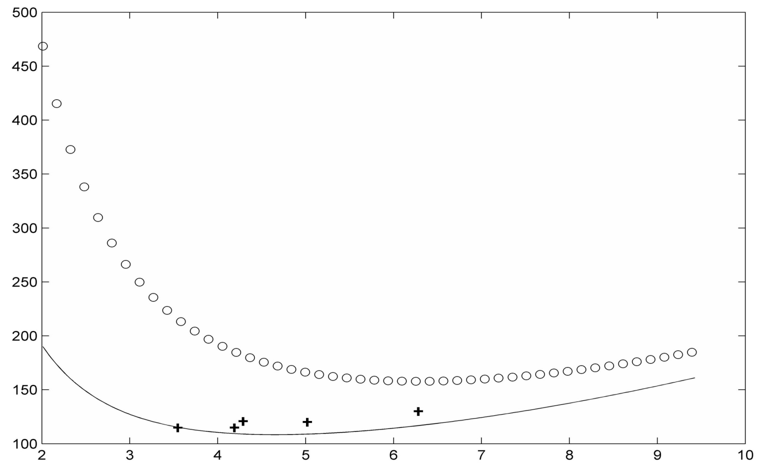


Figure 2: Bifurcation parameter, R^*, as a function of wave number, a. Circles line, for boundary condition $q_2 = 0$ on $z = d$, as in [16]. Continuous line, for $p = 0$ on $z = d$ (Problem (14)-(20)). Crosses: some values of R^* for which instabilities arise in the problem with extraction.

Eigenvalue problem (29)-(31) was solved numerically using a standard routine of [19]. In Figure 2 we summarize the results concerning to the size of the bifurcation parameter, R^*:

- We considered Problem P with the non-flow bottom boundary condition $\nabla u \cdot \mathbf{n} = 0$ replaced by the fixed Dirichlet boundary condition $u = u^*$, as in (14). In other words, we investigate the importance of the extraction region and the dead core formation for the development of transient instabilities. We computed numerical solutions for parameters values which imply no dead core formation ($u < 1$ on $z = d$), and even in this case we checked the formation of instabilities for values of R which are very close to those of problem (14)-(20). Therefore, the actual formation of a dead core in the extraction region seems not to be relevant for instabilities occurrence as long as the concentration on the bottom boundary keeps lower than that on the extraction region.

- We compare the bifurcation curves corresponding to our problem (14)-(20) and to the problem studied in [16], i.e., with our condition $p = 0$ on $z = d$ replaced by $q_2 = 0$ on $z = d$. The non-flow boundary condition seems to give more stability to the system, possibly as a consequence of the shortening of the region in which they may develope.

2.4 Numerical simulations

We used a stabilized mixed finite element method in space and implicit finite differences scheme in time to approximate solutions of an equivalent formulation of Problem P, consisting on combining equations (1) and (2) to replace (1) by

$$u_t + R\mathbf{q}\cdot\nabla u - \Delta u = Rmuf(\cdot,u). \tag{32}$$

The main advantage on using a mixed formulation instead of a simpler formulation based only in concentration-pressure, see (9)-(10), is that the flow, $\mathbf{q}$, is obtained directly from the discrete solution and there is no need of deducing it by numerical differentiation of the pressure, with the loss of accuracy it implies. Since our investigation is centered on stability, keeping a good grade of accuracy in the flow approximation is important. Moreover, as we already mentioned in the Introduction, the study of the existence of solutions indicates that the decoupling of the system inherent to the mixed formulation is more natural than that of the concentration-pressure formulation in the sense that it produces energy estimates which allow to obtain a simpler proof of the existence of solutions.

It is well known that classical mixed variational formulations need an adequate election of the discrete spaces for the flow and the pressure in order to satisfy the Babuska-Brezzi stability condition, see for instance [5]. Following [13], we consider a stabilized mixed finite element method for Darcy flows which allows to consider piecewise linear approximations and the same mesh for both pressure and flow. The only differences between our problem and the problem treated in [13] are the boundary condition for the pressure and the existence of a source term in (39). In any case, the adaptation to our problem is straightforward.

Let $t_n = n\delta t$, for $\delta t = T/N$, and $n = 0,...N$, and consider the discrete time approximation given by

$$u^n + \delta t(R\mathbf{q}^n\cdot\nabla u^n - \Delta u^n) = \delta t Rmu^n f(\cdot,u^n) + u^{n-1}, \tag{33}$$

$$\operatorname{div}\mathbf{q}^n = -mf(\cdot,u^n), \tag{34}$$

$$\mathbf{q}^n = -\nabla p^n + u^n\mathbf{e}_z, \tag{35}$$

in Ω, where the super-index n stands for the approximations in time t_n. The boundary conditions are

$$u^n = u_D, \quad p^n = 0 \quad \text{on } \Gamma_D, \tag{36}$$

$$\nabla u^n \cdot \mathbf{n} = \mathbf{q}^n \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \tag{37}$$

We solve this nonlinear system of equations by a fixed point method based on the proof of Theorem 1. We consider the map $S : L^2(\Omega) \to L^2(\Omega)$ given by $S(\hat{u}; u^{n-1}) = u$, where u is the solution of

$$u - \delta t \Delta u = -\delta t R \mathbf{q} \cdot \nabla \hat{u} + \delta t R m \hat{u} f(\cdot, \hat{u}) + u^{n-1}, \tag{38}$$

$$\operatorname{div} \mathbf{q} = -m f(\cdot, \hat{u}), \tag{39}$$

$$\mathbf{q} = -\nabla p + \hat{u} \mathbf{e}_z, \tag{40}$$

with the mentioned boundary conditions (36)-(37). A fixed point of $S(\cdot\,; u^{n-1})$ is denoted by u^n. The formulation of the stabilized mixed finite element method for problem (39)-(40) is: Find $\mathbf{q} \in H_{0,N}(\operatorname{div}, \Omega)$ and $p \in \mathcal{V}$ solutions of

$$\int_\Omega (\mathbf{q} + \nabla p) \cdot \boldsymbol{\phi} = \int_\Omega \hat{u} \mathbf{e}_z \cdot \boldsymbol{\phi} \quad \text{for all } \boldsymbol{\phi} \in H_{0,N}(\operatorname{div}, \Omega), \tag{41}$$

$$\int_\Omega (\nabla p - \mathbf{q}) \cdot \nabla \varphi = \int_\Omega (\hat{u} \mathbf{e}_z \cdot \nabla \varphi - 2 f \varphi) \quad \text{for all } \varphi \in \mathcal{V}. \tag{42}$$

Once that $\mathbf{q}$ and p are determined, we set the following problem for equation (38): Find $u \in u_D + \mathcal{V}$ solution of

$$\int_\Omega u\varphi + \delta t \int_\Omega \nabla u \cdot \nabla \varphi = \delta t R \int_\Omega (m \hat{u} f(\cdot, \hat{u}) - \mathbf{q} \cdot \nabla \hat{u}) \varphi + \int_\Omega u^{n-1} \varphi \quad \text{for all } \varphi \in \mathcal{V}. \tag{43}$$

We use the spatial discretization of (41)-(42) given in [13], and adapt it also for equation (43). It consists of finite triangular elements, continuous piecewise linear base functions and the same mesh for all the unknowns. For the practical implementation of the fixed point method, we consider that a discrete solution of (41)-(43) is a fixed point of $S(\cdot\,; u^{n-1})$ if a norm of $S(u^n_k; u^{n-1}) - u^n_k$, for $k = 0, 1, \ldots$, with $u^n_0 = u^{n-1}$, is smaller than a fixed tolerance.

Remark 1. When studying problem (14)-(20), we replace Ω by Ω_d and redefine the space $\mathcal{V}$ in (43) by $\mathcal{V}_D = \{u \in H^1(\Omega_d) : u = 0 \text{ on } (0, L) \times \{d, 1\}\}$.

For the numerical simulations we considered the spatial domains $\Omega = (0, 1) \times (0, 1)$ and $\Omega_d = (0, 1) \times (0.5, 1)$, i.e., the extraction zone is above $z = 0.5$, and the parameter values $R = 300$, $m = 1$ and $r = 1$. We used an uniform triangular mesh with 900 triangles and an initial time step of

$\delta t = 0.001$. The time step is adapted according to the size of $\|u^n - u^{n-1}\|$. Although instable regimes are found for values of R smaller than 300, see Fig. 2, we used this value to show more clearly in the pictures the phenomena we are interested in. We remark that even in the case $r = 1$ in which a dead core does not appear, the approximated problem studied in the stability analysis produces similar patterns than Problem P.

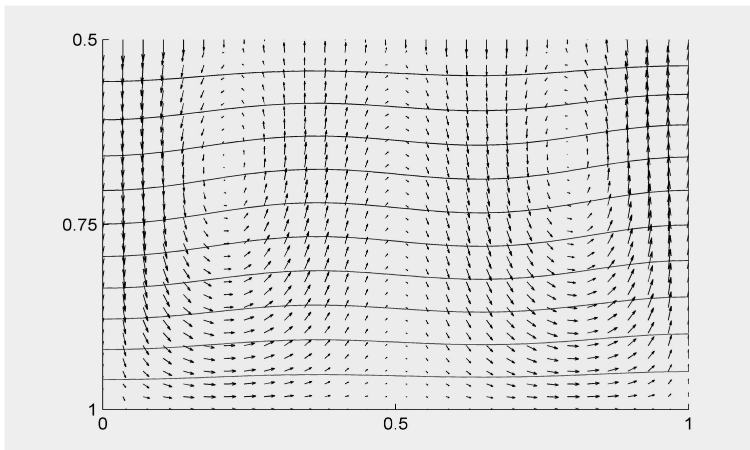


Figure 3: Flow and solute concentration for the approximated problem (14)-(20). $R = 300$, $t = 0.05$.

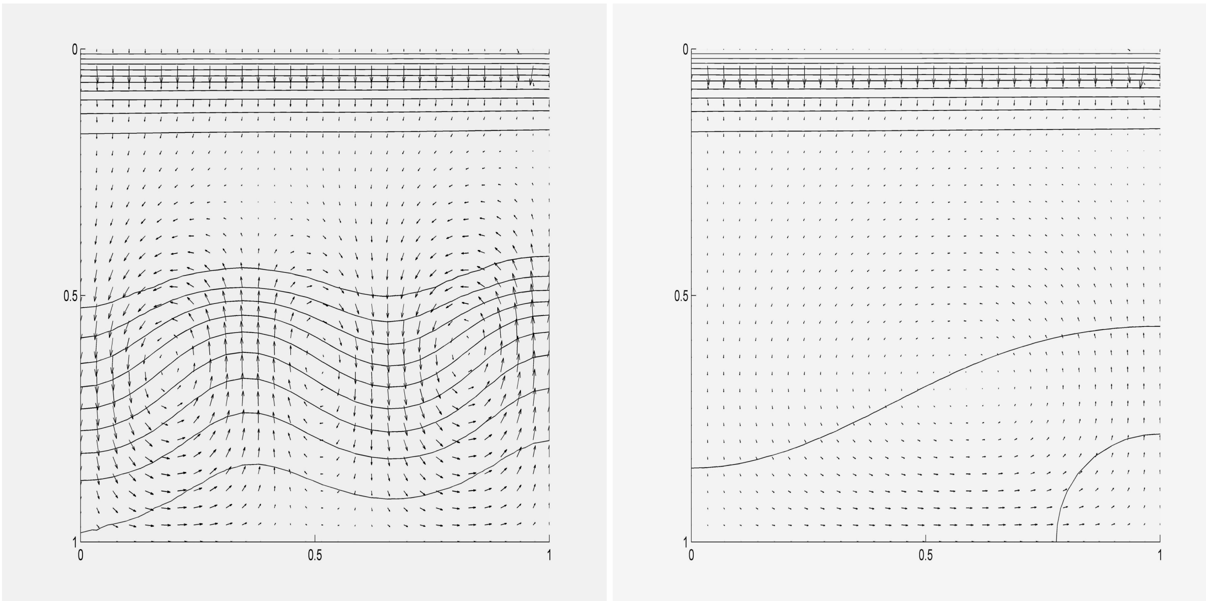
In Figures 3 and 4, flow and concentration contour lines are plotted.

- Figure 3. Solution of approximated problem (14)-(20) for time $t = 0.05$, showing the typical Benard instability cells corresponding to the selected data.

- Figure 4 (a). Solution of problem P for same time than in Figure 3, $t = 0.05$. Similar instability patterns than in the approximated problem do appear.

- Figure 4 (b). Long time behavior of solution of problem P, for time $t = 0.2$. Transient instabilities tend to disappear.

- Figure 5. We compare the speed of approach to the steady state between the one and the two-dimensional solutions of Problem P when instabilities in the transient state are present. We plot, against time, the ratio
$$\frac{\min_{x\in(0,L)} u_2(x,1,t)}{u_1(1,t)},$$



(a) $t = 0.05$

(b) $t = 0.2$

Figure 4: Flow and solute concentration for Problem P. (a) Transient state, $t = 0.05$. (b) *Long time* profile, $t = 0.2$. $R = 300$.

where u_1 and u_2 are the one and two-dimensional solutions, respectively, of Problem P. Different curves correspond to different values of R. For large values of R, we observe three time intervals with different behaviors of this ratio. Initially, when instabilities did not develop yet, both solutions are practically equal. Afterwards, when instabilities appear, the mixing in the two-dimensional model is accelerated and the increase of the solute concentration, u_2, at the bottom boundary is up to the fifty per cent greater than the corresponding one-dimensional solution, u_1. Finally, for later times, this ratio decreases and slowly approaches to one, since the stationary states are the same for both models.

For smaller values of R, the behavior is more complicated. For $R = 200$, the minimum value of u_2 on $z = 1$ is smaller than $u_1(1, \cdot)$ for all plotted times. However, we checked that the maximum value of u_2 on this boundary is always greater than $u(1, \cdot)$ implying, perhaps,

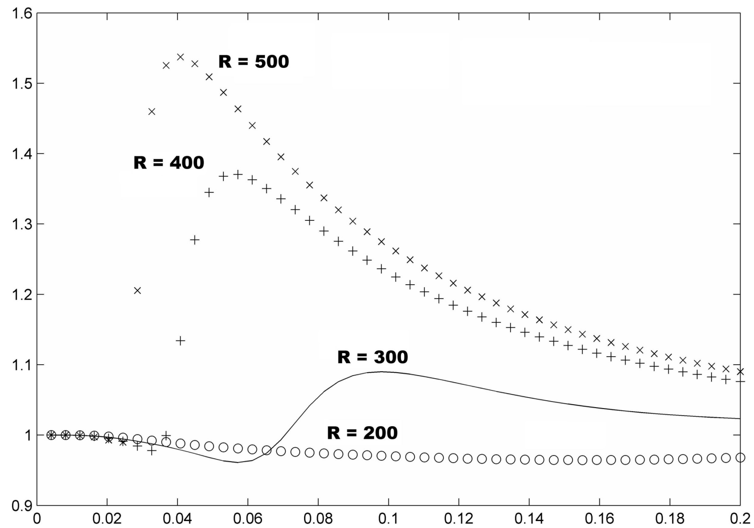


Figure 5: Two to one dimensional solutions ratio at $z = 1$.

that while the mixing accelerates the increase of solute concentration in some parts of the bottom boundary, it may not reach to the whole boundary, and regions such as corners may be free of this influence.

3 Proof of Theorem 1

In this section, for clarity, we set $R = 1$ and we replace the term mf by just f. Observe that defining $\hat{f} := mf$, we have that $\hat{f}$ satisfies Hypothesys H_2. The proof of Theorem 1 is based in the Schauder's fixed point theorem. We consider the map $S : L^2(Q_T) \to L^2(Q_T)$ given by $S(\hat{u}) = u$, where u is the solution of

$$\begin{aligned} & u \in u_D + L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}') \cap L^\infty(Q_T) \\ & < u_t, \varphi > + \int_\Omega \varphi \hat{\mathbf{q}} \cdot \nabla u + \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega u f(\cdot, u) \varphi, \\ & \lim_{t \to 0} \|u(\cdot, t) - u_0\|_{L^2(\Omega)} = 0, \end{aligned} \tag{44}$$

for all $\varphi \in \mathcal{V} \cap L^\infty(\Omega)$ and for a.e. $t \in (0,T)$, and where $\hat{\mathbf{q}}$ is one component of the solution, $(\hat{\mathbf{q}}, \hat{p})$, of

$$\mathbf{q} \in L^2(0,T; H_{0,N}(\mathrm{div},\Omega)) \cap \mathcal{W}_T, \quad p \in L^2(0,T;\mathcal{V}),$$

$$\int_\Omega \mathbf{q} \cdot \boldsymbol{\phi} - \int_\Omega p \,\mathrm{div}\, \boldsymbol{\phi} = \int_\Omega \hat{u}\mathbf{e}_z \cdot \boldsymbol{\phi}, \tag{45}$$

$$-\int_\Omega \xi \,\mathrm{div}\, \mathbf{q} = \int_\Omega f(\cdot,\hat{u})\xi, \tag{46}$$

for all $\xi \in L^2(\Omega)$, $\boldsymbol{\phi} \in H_{0,N}(\mathrm{div},\Omega)$ and for a.e. $t \in (0,T)$.

Observe that if u is a fixed point for S, i.e. $u = \hat{u}$, then $\xi = u(\cdot,t)\varphi \in L^\infty(\Omega) \subset L^2(\Omega)$ for a.e. $t \in (0,T)$ since $u, \varphi \in L^\infty(Q_T)$. Using ξ as a test function in (46), substituting in (44) and integrating by parts, we obtain a weak solution of our original problem in the sense of Definition 1. In particular, the regularity $\varphi \in L^\infty(\Omega)$ for the test functions of problem (44) can be removed. We start studying the uncoupled problems (44) and (45)-(46).

Lemma 1. *For any $\hat{u} \in L^2(Q_T)$, there exists a unique solution, $(\mathbf{q}, p)$, of problem (45)- (46) such that $\mathbf{q} \in \mathcal{W}_T$ and $p \in L^2(0,T;\mathcal{V})$. The norms*

$$\|\mathbf{q}\|_{L^2(0,T;H_{0,N}(\mathrm{div},\Omega))}, \qquad \|p\|_{L^2(0,T;\mathcal{V})}$$

are bounded by $c(\|\hat{u}\|_{L^2(Q_T)} + 1)$, with c independent of $\hat{u}$. In addition, $\|\mathbf{q}\|_{\mathcal{W}_T} \leq 1$.

Proof. We use Propositions 1.1 and 1.3 of [5]. Consider the bilinear forms $a : H_{0,N}(\mathrm{div},\Omega) \times H_{0,N}(\mathrm{div},\Omega) \to \mathbb{R}$ and $b : H_{0,N}(\mathrm{div},\Omega) \times L^2(\Omega) \to \mathbb{R}$ given by

$$a(\mathbf{q}, \boldsymbol{\phi}) = \int_\Omega \mathbf{q} \cdot \boldsymbol{\phi}, \qquad b(\boldsymbol{\phi}, p) = -\int_\Omega p \,\mathrm{div}\, \boldsymbol{\phi},$$

and observe that the linear operator corresponding to b is $B = -\mathrm{div} : H_{0,N}(\mathrm{div},\Omega) \to L^2(\Omega)$. Problem (45)-(46) may be stated as: find $\mathbf{q} \in L^2(0,T; H_{0,N}(\mathrm{div},\Omega))$ and $p \in L^2(0,T; L^2(\Omega))$ solutions of

$$a(\mathbf{q}, \boldsymbol{\phi}) + b(\boldsymbol{\phi}, p) = < g, \boldsymbol{\phi} >_{<V' \times V>}, \qquad \forall \boldsymbol{\phi} \in H_{0,N}(\mathrm{div},\Omega), \tag{47}$$

$$b(\mathbf{q}, \xi) = < h, \xi >_{<Q' \times Q>}, \qquad \forall \xi \in Q, \tag{48}$$

a.e. in $(0,T)$, where $g = \hat{u}(\cdot,t)\mathbf{e}_z \in L^2(\Omega) \subset (H_{0,N}(\mathrm{div},\Omega))'$ and $h = f(\cdot, \hat{u}(\cdot,t)) \in L^\infty(\Omega) \subset L^2(\Omega)$, for a.e. $t \in (0,T)$. To use the results in [5] we have to check that a is coercive on $\operatorname{Ker} B$ and that B is surjective. First assertion is straightforward since a is an equivalent norm of $H_{0,N}(\operatorname{div}, \Omega)$ on $\operatorname{Ker} B$. For the second assertion, B surjective, let $w \in L^2(\Omega)$ and consider the unique solution $\phi \in \mathcal{V}$ of

$$-\Delta\phi = w \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \Gamma_D, \quad \nabla\phi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N.$$

Taking $\mathbf{q} = \nabla\phi$ we get $\mathbf{q} \in L^2(\Omega)^N$, $-\operatorname{div} \mathbf{q} = w \in L^2(\Omega)$ and $\mathbf{q} \cdot \mathbf{n} = 0$ on Γ_N, i.e., $\mathbf{q} \in H_{0,N}(\operatorname{div}, \Omega)$ and $B\mathbf{q} = w$. Propositions 1.1 and 1.3 of [5] ensure the existence of a unique solution $(\mathbf{q}(t), p(t)) \in H_{0,N}(\operatorname{div}, \Omega) \times L^2(\Omega)$ for a.e. $t \in (0, T)$ of problem (45)- (46) such that the norms $\|\mathbf{q}(t)\|_{H_{0,N}(\operatorname{div},\Omega)}$ and $\|p(t)\|_{L^2(\Omega)}$ are bounded by a constant times $\|\hat{u}(t)\|_{L^2(\Omega)} + \|f(\cdot, \hat{u}(t))\|_{L^2(\Omega)}$. Finally, from identity (46) we obtain $\|\mathbf{q}\|_{\mathcal{W}_T} \leq 1$ and from identity (45) we obtain, for a.e. $t \in (0, T)$

$$-\int_\Omega p \operatorname{div} \boldsymbol{\phi} = \int_O (\hat{u}\mathbf{e}_z - \mathbf{q}) \cdot \boldsymbol{\phi}, \quad \text{for all } \boldsymbol{\phi} \in H_{0,N}(\operatorname{div}, \Omega).$$

Since $\hat{u}\mathbf{e}_z, \mathbf{q} \in L^2(Q_T)^N$, we deduce $\nabla p \in L^2(Q_T)^N$ and

$$\|p\|_{L^2(0,T;\mathcal{V})} \leq c\big(\|\hat{u}\|_{L^2(Q_T)} + \|\mathbf{q}\|_{L^2(Q_T)^N}\big) \leq c(\|\hat{u}\|_{L^2(Q_T)} + 1).$$

□

Lemma 2. *For any $\hat{\mathbf{q}} \in L^2(0, T; H_{0,N}(\operatorname{div}, \Omega)) \cap \mathcal{W}_T$ there exists a unique solution, u, of problem (44) such that the norms*

$$\|u\|_{L^\infty(Q_T)}, \quad \|u\|_{L^2(0,T;\mathcal{V})}, \quad \|u_t\|_{L^2(0,T;\mathcal{V}')},$$

are bounded in terms of the norms $\|u_0\|_{L^\infty(\Omega)}$, $\|\hat{\mathbf{q}}\|_{L^2(0,T;H_{0,N}(\operatorname{div},\Omega))\cap\mathcal{W}_T}$, and

$$\|u_D\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap L^\infty(Q_T)}.$$

Proof. We consider a sequence $\{\hat{\mathbf{q}}_n\} \subset L^\infty(Q_T)^N \cap L^2(0, T; H_{0,N}(\operatorname{div}, \Omega)) \cap \mathcal{W}_T$ such that

$$\hat{\mathbf{q}}_n \to \hat{\mathbf{q}} \quad \text{strongly in } L^2(0, T; H_{0,N}(\operatorname{div}, \Omega)),$$

and apply Theorem 1.7 of [1] to problem (44), with $\hat{\mathbf{q}}$ replaced by $\hat{\mathbf{q}}_n$, which ensures the existence of a unique weak solution, u_n. We now obtain some uniform estimates for norms of u_n.

$L^\infty(Q_T)$ estimate. We prove that $\min\{u_D, u_0\} \leq u_n \leq 1$ a.e. in Q_T. Using in (44) (with $\hat{\mathbf{q}}$ replaced by $\hat{\mathbf{q}}_n$) the admissible test function $T(u_n)$, where T

is the Stampaccia truncature function $T(s) = s - 1$ for $s > 1$ and $T(s) = 0$ for $s \leq 1$, we obtain, after integration by parts

$$\frac{d}{dt}\int_\Omega \mathcal{T}(u_n) \leq \|\operatorname{div}(\hat{\mathbf{q}}_n)\|_{L^\infty(Q_T)} \int_\Omega \mathcal{T}(u_n).$$

where $\mathcal{T}$ is the primitive of T with $\mathcal{T}(0) = 0$. Gronwall's Lemma implies $\mathcal{T}(u_n) = 0$ in Q_T, and then $u_n \leq 1$ in Q_T. Since $\hat{\mathbf{q}}_n \to \hat{\mathbf{q}}$ strongly in $L^2(0,T;H_{0,N}(\operatorname{div},\Omega))$ and $\operatorname{div}\hat{\mathbf{q}} \in L^\infty(Q_T)$, the estimate $u_n \leq 1$ is valid for all n. To prove $u_n \geq \min\{u_D, u_0\}$ one first prove that $u_n \geq 0$ using a Stampaccia tructaure function, as above. Then, once we know that $u_n f(\cdot, u_n) \geq 0$, we apply the maximum principle to conclude.

Energy estimate. We use $\varphi = u_n - u_D \in L^2(0,T;\mathcal{V}) \cap L^\infty(Q_T)$ as test function. Standard inequalities give us, after integration in $(0,T)$,

$$\frac{1}{4}\int_\Omega {u_n}^2(T) + \frac{1}{2}\int_{Q_T} |\nabla u_n|^2 \leq \frac{1}{2}\int_\Omega u_0^2 + 2\int_{Q_T} {u_n}^2 + 4\int_\Omega u_D^2(T) \\ + 4\int_{Q_T} |\hat{\mathbf{q}}_n|^2 + 4\int_{Q_T} (u_{Dt}^2 + |\nabla u_D|^2 + u_D^2), \quad (49)$$

and Gronwall's Lemma implies

$$\|u_n\|_{L^\infty(L^2)} + \|u_n\|_{L^2(\mathcal{V})} \leq C,$$

with C depending only on norms of u_D, u_0 and on the $L^2(Q_T)$ norm of $\hat{\mathbf{q}}_n$. Observe that since $\hat{\mathbf{q}}_n \to \hat{\mathbf{q}}$ strongly in $L^2(Q_T)$, C may be taken independent of n.

Time derivative estimate. Integrating by parts in the convective term of (44) we obtain

$$< u_{nt}, \varphi > = \int_\Omega u_n \operatorname{div}(\varphi \hat{\mathbf{q}}_n) - \int_\Omega \nabla u_n \cdot \nabla\varphi + \int_\Omega u_n f(\cdot, u_n)\varphi,$$

from where

$$< u_{nt}, \varphi > \leq c_1 \|\nabla\varphi\|_{L^2(\Omega)} + c_2\|\varphi\|_{L^2(\Omega)},$$

with $c_1 = (\|\hat{\mathbf{q}}_n\|_{L^2(\Omega)} + \|\nabla u_n\|_{L^2(\Omega)})$ and $c_2 = (\|\operatorname{div}\hat{\mathbf{q}}_n\|_{L^\infty(\Omega)} + \|u_n\|_{L^2(\Omega)})$. Therefore $\|u_{nt}\|_{\mathcal{V}'} \leq C$, with C independent of n for similar reasons than above.

We deduced that the sequence u_n is uniformly bounded with respect to n in the space $L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}') \cap L^\infty(Q_T)$. Therefore, there exists a subsequence u_n and a function $u \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}') \cap L^\infty(Q_T)$ such that $u_n \to u$ weakly in $L^2(0,T;\mathcal{V})\cap H^1(0,T;\mathcal{V}')$ and weakly star in $L^\infty(Q_T)$. In addition, applying Aubin's Lemma we deduce that $u_n \to u$ strongly in $L^2(Q_T)$ and that $u \in C((0,T],L^2(Q_T))$. Passing now to the limit $n\to\infty$ in the formulation (44) is straightforward. $\square$

End of Proof of Theorem 1. We now check the hypothesis of the fixed point theorem, which are: (i) $S: L^2(Q_T)\to L^2(Q_T)$ is continuous, (ii) S is compact, and (iii) the set

$$\Lambda := \{u \in L^2(Q_T) : u = \lambda S(u), \quad \text{for all } \lambda\in[0,1]\} \quad \text{is bounded.}$$

(i) S is continuous. Consider a sequence $\hat{u}_n$ such that $\hat{u}_n \to \hat{u}$ strongly in $L^2(Q_T)$. We have to prove that $S(\hat{u}_n)\to S(\hat{u})$ strongly in $L^2(Q_T)$. We have that $S(\hat{u}_n) = u_n$ with u_n the solution of problem (44) corresponding to $\hat{\mathbf{q}}_n$, where $\hat{\mathbf{q}}_n$ is the first component of the solution, $(\hat{\mathbf{q}}_n, \hat{p}_n)$, of problem (45)-(46) corresponding to $\hat{u}_n$. Lemma 1 implies $\hat{\mathbf{q}}_n \in L^2(0,T;H_{0,N}(\mathrm{div},\Omega))\cap\mathcal{W}_T$ and $\hat{p}_n\in L^2(0,T;\mathcal{V})$ with uniform bounds in these spaces. Therefore, Lemma 2 implies that the norms

$$\|u_n\|_{L^\infty(Q_T)}, \quad \|u_n\|_{L^2(0,T;\mathcal{V})}, \quad \|u_{nt}\|_{L^2(0,T;\mathcal{V}')},$$

are uniformly bounded with respect to n. Hence, there exist functions $u\in u_D + L^2(0,T;\mathcal{V})\cap H^1(0,T;\mathcal{V}')\cap L^\infty(Q_T)$, $\hat{\mathbf{q}}\in L^2(0,T;H_{0,N}(\mathrm{div},\Omega))\cap\mathcal{W}_T$ and $\hat{p}\in L^2(0,T;\mathcal{V})$, and subsequences $u_n, \hat{\mathbf{q}}_n, \hat{p}_n$ in these spaces such that

$$\hat{\mathbf{q}}_n \to \hat{\mathbf{q}} \qquad \text{weakly in } L^2(0,T;H_{0,N}(\mathrm{div},\Omega)), \tag{50}$$
$$\mathrm{div}\,\hat{\mathbf{q}}_n \to \mathrm{div}\,\hat{\mathbf{q}} \qquad \text{weakly star in } L^\infty(Q_T), \tag{51}$$
$$\hat{p}_n \to \hat{p} \qquad \text{weakly in } L^2(0,T;\mathcal{V}), \tag{52}$$
$$u_n \to u \qquad \text{weakly star in } L^\infty(Q_T), \tag{53}$$
$$u_n \to u \qquad \text{weakly in } L^2(0,T;\mathcal{V}), \tag{54}$$
$$u_{nt} \to u_t \qquad \text{weakly in } L^2(0,T;\mathcal{V}'). \tag{55}$$

From (54) and (55) and Aubin's theorem we deduce that

$$u_n \to u \quad \text{strongly in } L^2(Q_T), \quad u\in C([0,T];L^2(\Omega)).$$

From the formulations of problems (45)-(46) and (44) we have

$$\int_\Omega \hat{\mathbf{q}}_n\cdot\boldsymbol{\phi} - \int_\Omega \hat{p}_n\,\mathrm{div}\,\boldsymbol{\phi} = \int_\Omega \hat{u}_n\mathbf{e}_z\cdot\boldsymbol{\phi}, \tag{56}$$
$$-\int_\Omega \xi\,\mathrm{div}\,\hat{\mathbf{q}}_n = \int_\Omega f(\cdot,\hat{u}_n)\xi, \tag{57}$$

for all $\xi \in L^2(\Omega)$, $\boldsymbol{\phi} \in H_{0,N}(\mathrm{div}, \Omega)$ and for a.e. $t \in (0,T)$, and

$$< u_{nt}, \varphi > + \int_\Omega \varphi \hat{\mathbf{q}}_n \cdot \nabla u_n - \int_\Omega \nabla u_n \cdot \nabla \varphi = \int_\Omega u_n f(\cdot, u_n)\varphi, \tag{58}$$

for all $\varphi \in \mathcal{V} \cap L^\infty(\Omega)$ and for a.e. $t \in (0,T)$. Since, by assumption, $\hat{u}_n \to \hat{u}$ strongly in $L^2(Q_T)$ and $f(x,\cdot)$ is continuous for a.e. $x \in \Omega$, taking the limit $n \to \infty$ in (56)-(57) and using (50)-(52) we obtain

$$\int_\Omega \hat{\mathbf{q}} \cdot \boldsymbol{\phi} - \int_\Omega \hat{p} \,\mathrm{div}\, \boldsymbol{\phi} = \int_\Omega \hat{u} \mathbf{e}_z \cdot \boldsymbol{\phi}, \tag{59}$$

$$-\int_\Omega \xi \,\mathrm{div}\, \hat{\mathbf{q}} = \int_\Omega f(\cdot, \hat{u})\xi, \tag{60}$$

for all $\xi \in L^2(\Omega)$, $\boldsymbol{\phi} \in H_{0,N}(\mathrm{div}, \Omega)$ and for a.e. $t \in (0,T)$. Passing to the limit $n \to \infty$ in (58) is straightforward, with the exception of the convective term, since both sequences are only weakly convergent. Integrating by parts, we obtain

$$\int_\Omega \varphi \hat{\mathbf{q}}_n \cdot \nabla u_n = -\int_\Omega \varphi u_n \,\mathrm{div}\, \hat{\mathbf{q}}_n - \int_\Omega u_n \hat{\mathbf{q}}_n \cdot \nabla \varphi.$$

For the first term at the right hand side we use that $u_n \to u$ strongly in $L^2(Q_T)$ and that $\mathrm{div}\, \hat{\mathbf{q}}_n \to \mathrm{div}\, \hat{\mathbf{q}}$ weakly star in $L^\infty(Q_T)$. For the second, again that $u_n \to u$ strongly in $L^2(Q_T)$, that $\|u_n\|_{L^\infty(Q_T)}$ is uniformly bounded, and that $\hat{\mathbf{q}}_n \to \hat{\mathbf{q}}$ weakly in $L^2(Q_T)$.

Finally, observe that the uniqueness of solutions of problems (56)-(57) and (58) implies that not only a subsequence but the whole sequence converges.

(ii) S is compact. From the previous analysis, we know that for all $\hat{u} \in L^2(Q_T)$, $u = S(\hat{u}) \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$, which is compactly embedded in $L^2(Q_T)$, and therefore S is compact.

(iii) Λ is bounded. For $\lambda = 0$ is trivial. For $\lambda \in (0,1]$ it is straightforward too. Condition $\hat{u} = \lambda S(\hat{u})$ is equivalent to

$$< \hat{u}_t, \varphi > + \int_\Omega \varphi \hat{\mathbf{q}} \cdot \nabla \hat{u} + \int_\Omega \nabla \hat{u} \cdot \nabla \varphi = \int_\Omega \hat{u} f(\cdot, \hat{u}/\lambda)\varphi, \tag{61}$$

$$\int_\Omega \hat{\mathbf{q}} \cdot \boldsymbol{\phi} - \int_\Omega \hat{p} \,\mathrm{div}\, \boldsymbol{\phi} = \int_\Omega \hat{u} \mathbf{e}_z \cdot \boldsymbol{\phi}, \tag{62}$$

$$-\int_\Omega \xi \,\mathrm{div}\, \hat{\mathbf{q}} = \int_\Omega f(\cdot, \hat{u})\xi, \tag{63}$$

for all $\varphi \in \mathcal{V} \cap L^\infty(\Omega)$, $\xi \in L^2(\Omega)$, $\boldsymbol{\phi} \in H_{0,N}(\text{div}, \Omega)$ and for a.e. $t \in (0, T)$. Therefore, the only change estimating the $L^2(Q_T)$ norm of the solution of equation (61) is in the right hand side term. Since $\|f\|_{L^\infty} \leq 1$ we obtain the same estimates than in Lemma 2, and therefore, the boundedness of Λ.

Therefore, the hypothesis of the fixed point theorem are verified, and we deduce the existence of a fixed point, u, which as already mentioned is a weak solution of problem (1)-(8).

4 Appendix: The physical model

Our main motivation to study the mathematical model described in the Introduction is found in the ecology of mangroves. Mangrove forests or swamps are located on low, muddy, tropical coastal areas around the world. Mangroves are woody plants that form the dominant vegetation of mangrove forests. They are characterized by their ability to tolerate regular inundation by tidal water with salt concentration c_w close to that of sea water see, for example, [12]. The mangrove roots take up fresh water from the saline soil and leave behind most of the salt, resulting in a net flow of water downward from the soil surface, which carries salt with it. As pointed out by Passioura *et al.* [15], in the absence of lateral flow, the steady state salinity profile in the root zone must be such that the salinity around the roots is higher than c_w, and that the concentration gradient is large enough so that the advective downward flow of salt is balanced by the diffusive flow of salt back up to the surface. In [15] the authors presented steady state equations governing the flow of salt and uptake of water in the root zone, assuming that there is an upper limit c_c to the salt concentration at which roots can take up water, and that the rate of uptake of water is proportional to the difference between the local concentration c and the assumed upper limit c_c. They also assumed that the root zone is unbounded, and that the constant of proportionality for root water uptake is independent of depth through the soil. In [8], the model was extended in two important ways. First, considering more general root water uptake functions and second, limiting the root zone to a bounded domain. The authors proved mathematical properties of the spatial one-dimensional model, such as the existence and uniqueness of solutions of the evolution and steady state problems, the conditions under which the threshold level of salt concentration is attained in finite time, and others.

In [8], as in the present work, it is assumed that tides, or other sources of fresh or not too saline water, renew the water on the soil-water interface allowing to prescribe the salt concentration at this boundary (Dirichlet boundary data). Although this is the usual environment in which mangroves live, other situations leading to different boundary conditions may also be considered. In [10], the authors focused in the situation in which the inflow of fresh or sea water is impeded due to some physical barrier, such as shore highways or flooding control dikes. Balance equations for salt and water content lead to a dynamical boundary condition at such interface, i.e., a boundary condition involving the time derivative of the solution. The main interest of that model is its ability to describe the complete salinization of the system, as reported in the biological literature, see for instance [4].

In this article we retake the model in [8] to study the problem in spatial dimension $N > 1$. We consider the case where the mangroves are present in the horizontal x, y plane, with an homogeneous porous medium located below this plane and a water flow above it. This is a significant change since in dimension $N = 1$ (only vertical coordinate) the fluid volume balance expressed by

$$\operatorname{div} \mathbf{q} + S = 0, \tag{64}$$

where $\mathbf{q}$ is the water discharge and S is the volume of water taken up by the roots per unit volume of porous material per unit time, is enough to determine the flow $\mathbf{q}$ through the equation $q_z + S = 0$, being z the vertical coordinate. However, if due to the loss of uniformity in the x–y plane more spatial variables have to be considered then equation (64) is not enough to determine the flow and an additional law must be taken into account. Since we are modelling a viscous flow through a porous medium, the soil in which mangroves roots grow, this law is Darcy's law,

$$\mathbf{q} + \frac{\kappa}{\mu}(\nabla p - \rho g \mathbf{e}_z) = 0, \tag{65}$$

where κ is the permeability, μ the viscosity and g, the gravitational acceleration constant. By $\mathbf{e}_z$ we denote the downwards vertical unitary vector, i.e. z denotes vertical depth: $z = 0$ is the soil surface and $z = H$ is the *bottom* of the spatial domain, under the soil surface. In (65), we assume the following equation of state for the density

$$\rho \equiv \rho(c) = \rho_0 + \alpha c, \tag{66}$$

with $\rho_0 \geq 0$ and $\alpha > 0$. We further assume S to have the form

$$S(\cdot, c) := \begin{cases} s(\cdot)\left(1 - \dfrac{c}{c_c}\right)^r & \text{for } 0 \leq c \leq c_c, \\ 0 & \text{for } c > c_c, \end{cases} \tag{67}$$

where $r > 0$, c_c is the upper limit of salt concentration at which mangroves may uptake water and s is the *root distribution*. This root distribution function is non-negative, and non-increasing with z. We keep in mind the following characteristic example

$$s(z) := \begin{cases} s_0/z_* & \text{for } 0 < z < z_*, \\ 0 & \text{for } z_* < z < H. \end{cases} \tag{68}$$

The quantity s_0 is the total amount of root water uptake with no salt present, in volume per unit surface per unit time, i.e. the *transpiration rate* of the mangrove plants in the absence of salinity.

In addition to equations (64)-(65) for the fluid discharge, we have the following equation for the evolution of salt concentration, see [3],

$$\theta c_t + \operatorname{div}(c\mathbf{q} - \theta \mathbf{D} \nabla c) = 0. \tag{69}$$

Here, we assume that the porous medium is characterized by a constant porosity $\theta \in (0,1)$, indicating that the mangroves roots are homogenized throughout the porous medium, without affecting its properties. We assume further that the hydrodynamic dispersion tensor, $\mathbf{D} = D\mathbf{I}$, with $\mathbf{I}$ the identity matrix, is constant and isotropic, i.e. the velocity dependence in the mechanical dispersion is neglected.

Equations (64), (65) and (69) stand on the domain $B \times (0,H) \times (0,\tau)$, with H denoting depth, τ the size of the time interval to be considered and $B \subset \mathbb{R}^{N-1}$, bounded, tipically $N = 2,3$. Concerning the boundary conditions, we prescribe the concentration and a reference pressure on the soil surface,

$$c = c_D, \quad p = p_{air} \quad \text{on } B \times \{0\} \times (0,\tau), \tag{70}$$

and consider no flow boundary condition in the rest of the boundary:

$$\mathbf{q} \cdot \mathbf{n} = \nabla c \cdot \mathbf{n} = 0 \quad \text{on } \big(\partial B \times (0,H) \times (0,\tau)\big) \cup \big(B \times \{H\} \times (0,\tau)\big), \tag{71}$$

with $\mathbf{n}$ denoting the unitary outwards normal vector to $B \times (0,H)$. We finally add to this formulation a given initial distribution of salt concentration

$$c(\cdot, 0) = c_0 \quad \text{in } B \times (0,H). \tag{72}$$

Finally, the system is rendered to dimensionless form by introducing the following variables, unknowns and parameters:

$$\tilde{t} := Dt/H^2, \quad \tilde{x} := x/H, \quad \tilde{z} := z/H,$$

$$u := c/c_c, \quad \tilde{\mathbf{q}} := \mathbf{q}/\hat{q}, \quad \hat{q} := \kappa\alpha c_c g/\mu \quad \tilde{p} := (p - p_{air} - \rho_0 gHz)/\alpha c_c gH,$$

$$\tilde{s}(\tilde{x}, \tilde{z}) := s(H\tilde{x}, H\tilde{z}), \quad d := z_*/H, \tilde{L} := L/H.$$

Let us mention that in the recasting of our model there appear two constants capturing the important physical parameters: the Rayleigh number

$$R := KH/\theta D, \tag{73}$$

and the *extraction number*

$$m := s_0 z_*/K, \tag{74}$$

with $K = \kappa\alpha c_c g/\mu$, the hydraulic conductivity.

Remark 2. Using [15] and [14] as references we find the following values for the physical constants: $D \approx 10^{-5}\,\mathrm{m}^2\mathrm{day}^{-1}$, $\theta \approx 0.5$, $s_0 \approx 1\ \ell\ \mathrm{m}^{-2}\mathrm{day}^{-1}$ and K in the range 10^{-4}–10^{-1} m day^{-1}. Taking z^* in the range 0.2–0.5 m and H in the range 0.5–1 m, this implies a time scale in the range 5–30 yr, R in the range 10–10^4 and m in 1–100.

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