On a cross-diffusion population model deduced from mutation and splitting of a single species [1]

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Abstract

We deduce a particular case of the population cross-diffusion model introduced by Shigesada et al [31] by using the ideas of mutation and splitting from a single **species**, as described by Sánchez-Palencia for ODE's systems [30]. **The resulting equations of the PDE system only differ in the cross-diffusion terms, being the corresponding diffusion matrix self-diffusion dominated, which implies that the well known population segregation patterns of the Shigesada et al model do not appear in this case. We prove existence and uniqueness of solutions of the PDE system and use a finite element approximation to discuss, numerically, stability properties of solutions with respect to the parameters in comparison with related models.**

Key words: Cross-diffusion system, population dynamics, existence, uniqueness, finite element approximation, numerical examples, **stability**.

AMS Subject Classification: 35K55, 35D30, 92D25.

1 Introduction

In [31], Shigesada et al introduced the following time evolution drift-cross diffusion system of partial differential equations to model the interaction between two competitive species:

$$\partial_t u_i - \operatorname{div} J_i = f_i(u_1, u_2) \quad \text{in } Q_T = \Omega \times (0, T), \tag{1}$$

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for $i = 1,2$, in $Q_T =$, where $\Omega \subset \mathbb{R}^N$ is an open and bounded set with Lipschitz boundary, $\partial\Omega$, $T > 0$ is arbitrarily fixed, the unknowns u_i represent population densities, the flow is given by

$$J_i = \nabla(c_i u_i + a_{i1} u_i u_1 + a_{i2} u_i u_2) + d_i u_i \mathbf{q}, \tag{2}$$

f_i are competition Lotka-Volterra type functions,

$$f_i(u_1, u_2) = (\alpha_i - \beta_{i1} u_1 - \beta_{i2} u_2) u_i, \quad \alpha_i, \beta_{ij} \geq 0 \quad i,j = 1,2, \tag{3}$$

and the field $\mathbf{q}$ is usually given as $\mathbf{q} = \nabla\Phi$, with Φ an environmental potential, modeling areas where the environmental conditions are more or less favorable [31,27]. The above system of equations is completed with non-flux boundary conditions and non-negative initial data:

$$J_i \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{4}$$

$$u(\cdot, 0) = u_i^0 \geq 0 \quad \text{on} \quad \Omega, \tag{5}$$

for $i = 1,2$, where ν denotes the exterior unit normal to Ω.

This model has received much attention since its introduction due to the interesting spatial pattern formation of its solutions, referred to as *segregation*, **and in fact an intense effort has been devoted to the understanding of its mathematical properties, specially to the existence of solutions, see [34,22,11,14] for first results under restrictions on the coefficients, mainly condition (6) below, [15,7] for general existence results, [25,26] for the study of the stationary problem, and [33,3] for extensions to several populations and non-homogeneous Lotka-Volterra terms. The numerical approach to the problem has been treated in [2,18,28], and the search for exact solutions in [8]. Related models have been studied in [16,21,17,12] and others. However, it seems that the modeling itself has not been object of further study, and little more than the somehow ad hoc formulation given by Shigesada et al in their key work [31] is available. In this article, we propose a derivation of the model in terms of a well known mechanism of population differentiation, see Sánchez-Palencia [30]. Starting with a single species with density u, solution of certain evolution drift-diffusion PDE, we assume that mutation differentiates this single species into two sub-species with densities u_1 and u_2, which split in their behaviors such that we still have that $u_1 + u_2 = u$ satisfies the original problem, but u_1 and u_2 solve slightly different PDE's conforming a system which is a special case of the Shigesada et al model.**

As showed by Sánchez-Palencia [30], the strategy of differentiation and splitting in the ODE's model leads to a situation in which there

exists a full segment of steady state solutions which includes the cases of coexistence and extinction of one population. The interesting biological feature of the model is that, in general, small perturbations of the Lotka-Volterra terms involving advantages and disadvantages for both populations tend to induce coexistence. Therefore, differentiation-splitting strategies may be understood as mechanisms which promote diversity, rather than optimization of species. However, in general, this not seem to be the case when cross-diffusion spatial effects of the Shigesada et al type enter in the modeling, as we numerically demonstrate in Section 4. On the contrary, the effects of population pressure in the context of differentiation-splitting strategies seem to promote only the survival of the best fitted.

With respect to the segregation pattern formation of the Shigesada et al model, let us mention that they are not expected to arise in our differentiation-splitting model since this mechanism leads to a self-diffusion dominated diffusion matrix. As pointed out by Lou and Ni [25,26] in the context of the stationary problem corresponding to problem (1)-(5), while cross-diffusion helps to create segregation patterns, these patterns do not appear if the intensity of diffusion or self-diffusion is relatively large. Heuristically, we may have an idea of the relative size of diffusion parameters not leading to segregation patterns when considering the diffusion matrix of the system,

$$A(u_1,u_2) = \begin{pmatrix} c_1 + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & c_2 + 2a_{22}u_2 + a_{21}u_1 \end{pmatrix},$$

and observing that under the condition

$$8a_{11} \geq a_{12}, \quad 8a_{22} \geq a_{21}, \tag{6}$$

the diffusion matrix is positive definite

$$\xi^T A(u_1,u_2)\xi \geq \min\{c_1,c_2\} |\xi|^2 \quad \textbf{for all} \xi \in R^N,$$

hence yielding a uniform elliptic operator. Therefore, no segregation patterns are expected if condition (6) holds, as is the case for the differentiation-segregation model we shall deduce in the next section.

The article is organized as follows. In Section 2 we introduce our model and comment on other related models. In Section 3, we state and prove the main analytical results of this article. Finally, in Section 4, we use a finite element approximation to compute several model examples and discuss on the stability of solutions with respect to the parameters of the model.

2 Mathematical model

We start considering the dynamics of one single **species** population satisfying

$$\begin{cases} \partial_t u - \operatorname{div} J(u) = F(u) & \text{in } \Omega \times (0,T), \\ J(u) \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 \geq 0 & \text{on } \Omega, \end{cases} \tag{7}$$

where the flow J is given by

$$J(u) = \nabla(cu + au^2) + du\mathbf{q},$$

with $a, c \geq 0$ and $d \in \mathbb{R}$, and where the Lotka-Volterra function is of competitive type

$$F(u) = u(\alpha - \beta u).$$

Here, $\alpha \geq 0$ is the intrinsic growth parameter and $\beta \geq 0$ is related to the carrying capacity of the ecosystem. In the homogeneous space case, i.e., when the PDE of problem (7) reduces to an ODE, the nonlinear term of the Lotka-Volterra function prevents the solution from unbounded increase. From the modeling point of view, observe that the flow J includes terms analogous to those of J_i given in (1). From the analytic point of view, the existence and uniqueness of solutions of problem (7) is a classical result under suitable assumptions on the regularity of functions $\mathbf{q}$ and u_0 see, for instance, [23]. For biological background and origins of the problem see, for instance, [29].

To deduce our final model, following Sánchez-Palencia [30] we suppose that, at some time, $t^* < T$, mutation differentiates population u (solution of problem (7)) into two populations, u_1 and u_2, with $u(\cdot,t^*) = u_1(\cdot,t^*) + u_2(\cdot,t^*)$ and that these new populations split in their behavior, satisfying

$$\begin{cases} \partial_t u_i - \operatorname{div} J_i(u_1,u_2) = F_i(u_1,u_2) & \text{in } \Omega \times (t^*,T), \\ J_i(u_1,u_2) \cdot \nu = 0 & \text{on } \partial\Omega \times (t^*,T), \\ u_i(\cdot,t^*) = u_i^0 & \text{on } \Omega, \end{cases} \tag{8}$$

for $i = 1,2$, with u_i^0 such that $u_1^0 + u_2^0 = u(\cdot,t^*)$, and with

$$J_i(u_1,u_2) = \nabla\Big(cu_i + au_i^2 + b_i u_1 u_2\Big) + du_i\mathbf{q}, \quad F_i(u_1,u_2) = u_i(\alpha - \beta u_1 - \beta u_2),$$

with $b_1 = a(1-\rho)$, $b_2 = a(1+\rho)$, for $-1 \leq \rho \leq 1$. Observe that if (u_1,u_2) is a solution of problem (8) then we still have that $u_1 + u_2$ is a solution of problem (7) in $\Omega \times (t^*,T)$ since $J_1(u_1,u_2) + J_2(u_1,u_2) = J(u_1+u_2)$ and $F_1(u_1,u_2) + F_2(u_1,u_2) = F(u_1+u_2)$. Observe also that the only difference between the equations satisfied by u_1 and u_2 lies in the cross-diffusion terms.

In this way we obtained, from the well known model in population dynamics (7), a particular case of the Shigesada et al problem given by the system (1)-(5). As mentioned in the previous section, of special interest are the situations in which the balance between the self-diffusion, Δu_i^2, and the cross-diffusion, $\Delta(u_1 u_2)$, is such that spatial segregation patterns arise. However, in the case of problem (8) segregation patterns are not expected to appear due to the small relative size of self and cross-diffusion coefficients. In particular, we have that condition (6) is always satisfied and therefore, if $c > 0$, the problem is uniformly parabolic.

Before stating our results, let us remark that there are other ways to perform the splitting of equations which leads to problem (8). **In [18], Gambino et al studied numerically a particle approximation or lagrangian version of the model generalizing the ideas of Degond et al [9], see also [10,24]. Simplifying their notation for clarity, and taking $F_i = 0$ and $\mathbf{q} = \nabla\Phi$ for simplicity, the ODE system for the particle positions x_n^i (position of particle n of population i) is given by, for $i, j = 1, 2$, $i \neq j$, $n = 1, \ldots, N$,**

$$\frac{d}{dt}x_n^i(t) = -\Big(\frac{c_i}{u_n^i(x_n^i,t)}\nabla u_N^i(x_n^i,t) + a_{ij}\nabla u_N^j(x_n^i,t) + d_i\nabla\Phi(x_n^i,t) + \Big(2a_{ii} + \frac{a_{ij}u_N^j(x_n^i,t)}{u_N^i(x_n^i,t)}\Big)\nabla u_N^i(x_n^i,t)\Big), \tag{9}$$

where $u_N^i(x,t) = \sum_{n=1}^N w_n^i \zeta_\varepsilon(x - x_n^i(t))$ is a particle approximation of u_i, with w_n^i the mass of particle n of population i and ζ_ε a regularizing kernel for approximating the Dirac δ distribution. The right hand side of equation (9) prescribes the particle velocity, which is modified by several factors which we describe in the same order as appearing in the equation:

- **random motion. This is a deterministic version of Fick's law, which takes the simpler form $\sqrt{c_i}dW_n^i(t)$ when written as a stochastic component of the equation, with W_n^i, $n = 1, \ldots, N$ a family of independent standard Wiener processes,**
- **inter-population pressure, with repulsive effects,**
- **environmental force, attracting towards the minima of Φ, and**
- **intra-population pressure, again with repulsive effects.**

As it can be seen, the intra-population pressure force has a formulation much more complicated than that of the inter-population pressure, due to the particular structure assumed for the cross-diffusion terms. In fact, in opposition to Fick's law, its interpretation is also unclear when considered as a part of a stochastic process [13].

To obtain an alternative model from the differentation-splitting strategy let us observe that the nonlinear diffusive term, $a\Delta u^2$, of problem (7) is better understood when written as $2a\,\mathrm{div}(u\nabla u)$ since it emphasizes the original role of this term as a repelling term (population pressure). The mutation into two populations then takes the form $2a\,\mathrm{div}\Big((u_1+u_2)\nabla(u_1+u_2)\Big)$, and a possible and natural splitting strategy is to consider new terms which are a sum of repelling forces

$$2a\,\mathrm{div}\Big(u_1\nabla(u_1+u_2)\Big) \quad \text{and} \quad 2a\,\mathrm{div}\Big(u_2\nabla(u_1+u_2)\Big),$$

for the u_1 and u_2 equations, respectively. In this way, the nonlinear diffusion terms of equations of problem (8) change and a new model problem may be considered:

$$\begin{cases} \partial_t u_i - \mathrm{div}\,\tilde{J}_i(u_1,u_2) = F_i(u_1,u_2) & \text{in } \Omega\times(T,\infty), \\ \tilde{J}_i(u_1,u_2)\cdot\nu = 0 & \text{on } \partial\Omega\times(T,\infty), \\ u_i(\cdot,t^*) = u_i^0 & \text{on } \Omega, \end{cases} \tag{10}$$

for $i=1,2$, with the initial data, u_i^0, and Lotka-Volterra functions, F_i, defined as those of problem (8), and with the new flow functions defined by

$$\tilde{J}_i(u_1,u_2) = (c+2au_i)\nabla u_i + 2au_i\nabla u_j + du_i\mathbf{q},$$

for $i,j=1,2$, $i\neq j$. Observe that $\tilde{J}_i$ is written as the addition of a conservative flow plus a term yielding linear diffusion. From the particle approximation point of view $\tilde{J}_i$ leads to the formulation

$$\begin{aligned} \frac{d}{dt}x_n^i(t) = -\Big(&\frac{c}{u_n^i(x_n^i,t)}\nabla u_N^i(x_n^i,t) + 2a\nabla u_N^i(x_n^i,t) + 2a\nabla u_N^j(x_n^i,t) \\ &+ d\nabla\Phi(x_n^i,t)\Big), \end{aligned} \tag{11}$$

which is a much simpler expression than that obtained for the Shigesada et al model (8), see (9) with the substitution $a_{ii}=a$ and $a_{ij}=b_i$. In fact, expression (12) resembles that obtained for the single **species** problem

$$\frac{d}{dt}x_n(t) = -\Big(\frac{c}{u_n(x_n,t)}\nabla u_N(x_n,t) + d\nabla\Phi(x_n,t) + 2a\nabla u_N(x_n,t)\Big), \tag{12}$$

with a similar definition of u_N than that of u_N^i given in (9). **Problem (10) is an alternative approach to cross-diffusion problems in Biology which was introduced by Gurtin et al [20], see also Busenber et al [6] in the context of epidemic models. Mathematical analysis of simplified versions of problem (10) have been carried out in [4,5].**

3 Main results

By using the translation $\tilde{t} = t - t^*$ we may consider problem (8) holding in the time domain $(0, \tilde{T}) = (0, T - t^*)$, with $\tilde{\mathbf{q}}(\cdot, \tilde{t}) = \mathbf{q}(\cdot, t^* + \tilde{t})$. In the following, we omit the tildes for clarity. The first result is on the existence of solutions of problem (8), which is a particular case of the problem studied in [14,7].

Theorem 1 *Let $T > 0$ and assume that:*

(1) the parameters satisfy $c + a > 0$,
(2) $\mathbf{q} \in L^2(Q_T)$,
(3) the initial data (of problem (7)) satisfy $u_0 \in L^2(\Omega)$, $u_0 \geq 0$.

Then there exists a weak solution (u_1, u_2) of problem (8) satisfying, for $i = 1, 2$, $u_i \geq 0$ in Q_T,

$$u_i \in L^2(0, T; H^1(\Omega)) \cap W^{1,r}(0, T; (W^{1,r'}(\Omega))'), \tag{13}$$

with $r = (2N + 2)/(2N + 1)$ if $c = 0$ and $r = 2$ if $c > 0$, and, for all $\varphi \in L^{r'}(0, T; W^{1,r'}(\Omega))$, and

$$\int_0^T \langle \partial_t u_i, \varphi \rangle + \int_{Q_T} J_i(u_1, u_2) \cdot \nabla \varphi = \int_{Q_T} F_i(u_1, u_2) \varphi, \tag{14}$$

with $< \cdot, \cdot >$ denoting the duality product between $W^{1,r'}(\Omega)$ and $(W^{1,r'}(\Omega))'$.

The second result is on the additional regularity and the uniqueness of solutions of problem (8), for which we are not aware of any previous results.

Theorem 2 *Let the assumptions of Theorem 1 hold. If, for $i = 1, 2$,*

$$u_i^0 \in L^\infty(\Omega) \quad \text{and} \quad \operatorname{div} \mathbf{q} \in L^1(0, T; L^\infty(\Omega)) \tag{15}$$

then any weak solution of problem (8) is such that $u_i \in L^\infty(Q_T)$ and r may be taken as $r = 2$ in (13). In addition, if

(i) $a = 0$ (and $c > 0$), or
(ii) $\operatorname{div} \mathbf{q} \in L^\infty(Q_T)$ and either $\beta = 0$ or $\nabla u_i \in L^\infty(Q_T)$,

then problem (8) admits a unique weak solution.

Remark 1 *Another interesting result for solutions of problem (8) is obtained when the problem is of degenerate type. If $c = 0$ and (u_1^0, u_2^0) is an initial data for problem (8) with compact support in Ω then the support of $u_0 = u_1^0 + u_2^0$ is also compact in Ω and therefore, using well known results of the theory of degenerate parabolic equations, see for instance Antontsev et al [1], the support of $u(\cdot, t)$ remains compact for some time interval $(0, t_*)$, with $t_* > 0$. Therefore,* *since $u_i \geq 0$, then the supports of $u_i(\cdot, t)$ remain also compact for, at least, the same time interval.*

Proof of Theorem 1. Let us start recalling that the parameters a_{ij} of problem (1) corresponding to the particular case of problem (8) always satisfy condition (6). Therefore, if $c > 0$ then we may apply the results of Theorem 1 of [14] to deduce the existence of a weak solution of problem (8). If $c = 0$, and by condition (1) of Theorem 1, $a > 0$, then we may use Theorem 1.1 of [7] (see also its Remark 3.6 and [2] for a proof based on a finite element approximation) to obtain a weak solution of problem (8).

Proof of Theorem 2. If $a = 0$ and therefore $b_i = 0$ and $c > 0$, the diffusion matrix is constant and positive definite and the results of the theorem are classical. We start proving the additional regularity of weak solutions under assumptions (15) and $a > 0$. Let (u_1, u_2) be a weak solution of problem (8). By construction, $u = u_1 + u_2$ satisfies problem (7) with $u_0 = u_1^0 + u_2^0 \geq 0$. Under assumption (15), problem (7) admits a unique non-negative solution such that

$$u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))') \cap L^\infty(Q_T),$$

see for instance [32]. Therefore, since $u_i \geq 0$, we deduce that also $u_i \in L^\infty(Q_T)$. With this regularity of solutions, it is straightforward to show that $J_i(u_1, u_2) \in L^2(Q_T)$. A standard approximation argument allows us to obtain from (14) the estimate

$$\int_0^T \langle \partial_t u_i, \varphi \rangle \leq \|J_i(u_1,u_2)\|_{L^2(Q_T)} \|\nabla\varphi\|_{L^2(Q_T)} + \|F_i(u_1,u_2)\|_{L^2(Q_T)} \|\varphi\|_{L^2(Q_T)},$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$, implying that u_i is bounded in $H^1(0,T;(H^1(\Omega))')$, and therefore allowing us to take $r = 2$ in (13).

To prove the uniqueness of solutions of problem (8) under assumptions (15) and (ii), let us assume that there exist two solutions (u_1, u_2) and $(\hat{u}_1, \hat{u}_2)$ and let us define $u = u_1 + u_2$, $\hat{u} = \hat{u}_1 + \hat{u}_2$, $v = u_1 - u_2$, $\hat{v} = \hat{u}_1 - \hat{u}_2$. On one hand, since both u and $\hat{u}$ satisfy, by construction, problem (7), which only admits one weak solution, we deduce that $u = \hat{u}$. On the other, we have that v (resp. $\hat{v}$) satisfies, in a weak sense similar to that of (14), the problem

$$\begin{cases} \partial_t v - \operatorname{div} J_d(v) = F_d(v) & \text{in } Q_T, \\ J_d(v) \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ v(\cdot, 0) = u_{10} - u_{20} & \text{on } \Omega, \end{cases} \tag{16}$$

with $F_d(v) = v(\alpha - \beta u)$, and

$$J_d(v) = \nabla\Big(cv + \frac{a\rho}{2}(v^2 - u^2) + auv\Big) + d\mathbf{q}v,$$

(resp. with v replaced by $\hat{v}$). Our goal is to prove that $v = \hat{v}$ since, together with $u = \hat{u}$, this implies $u_1 = \hat{u}_1$ and $u_2 = \hat{u}_2$. Let us consider the problem satisfied by $w = v - \hat{v}$:

$$\begin{cases} \partial_t w - \operatorname{div} J_{dd}(w) = F_d(w) & \text{in } Q_T, \\ J_{dd}(w) \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ w(\cdot, 0) = 0 & \text{on } \Omega, \end{cases} \tag{17}$$

with

$$J_{dd}(w) = \nabla\Big(cw + \frac{a\rho}{2}(v + \hat{v})w + auw\Big) + d\mathbf{q}w.$$

Let φ be a smooth function defined in Q_T with $\nabla\varphi \cdot n = 0$ on $\partial\Omega \times (0,T)$ and $\varphi(\cdot, T) = 0$ on Ω. Using φ as a test function in the weak formulation of problem (17) and performing an additional integration by parts we obtain

$$\int_{Q_T} w\Big(- \partial_t\varphi - A\Delta\varphi + d\mathbf{q} \cdot \nabla\varphi - (\alpha - \beta u)\varphi\Big) = 0, \tag{18}$$

with $A = c + a(\frac{\rho}{2}(v + \hat{v}) + u)$. Observe that, since $\rho \in [-1, 1]$ and $u = \hat{u}$, we deduce

$$\frac{\rho}{2}(v + \hat{v}) + u = \frac{1}{2}\Big((1 + \rho)(u_1 + \hat{u}_1) + (1 - \rho)(u_2 + \hat{u}_2)\Big) \geq 0,$$

and therefore $A \geq 0$. We perform the change of variable $\tau = T - t$ and set the following problem to choose function φ (writing again $t \equiv \tau$ for clarity):

$$\begin{cases} \partial_t\varphi - A_n\Delta\varphi + d\mathbf{q}_n \cdot \nabla\varphi - (\alpha - \beta u_n)\varphi = w_n & \text{in } Q_T, \\ \nabla\varphi \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(\cdot, 0) = 0 & \text{on } \Omega, \end{cases} \tag{19}$$

with A_n, $\mathbf{q}_n$, u_n, $w_n \in C^\infty(\bar{Q}_T)$, regularizations of A, $\mathbf{q}$, u and w, respectively, such that

$$A_n \geq 1/n \quad \text{a.e. in } \bar{Q}_T, \quad \|A_n - A\|_{L^2(Q_T)} \leq 1/n, \tag{20}$$

and

$$\begin{cases} \mathbf{q}_n \to \mathbf{q} \quad \text{strongly in } L^2(Q_T), \quad \text{with } \|\operatorname{div} \mathbf{q}_n\|_{L^\infty(Q_T)} \leq k, \\ u_n \to u \quad \text{strongly in } L^2(0,T; H^1(\Omega)), \quad \text{with } \|u_n\|_{L^\infty(Q_T)} \leq k, \\ w_n \to w \quad \text{strongly in } L^2(0,T; H^1(\Omega)), \end{cases} \tag{21}$$

with $k > 0$ independent of n. With these coefficients and data, we have that problem (19) has a solution $\varphi_n \in C^\infty(\bar{Q}_T)$, see [23]. Using this function we may rewrite (18) as

$$\int_{Q_T} ww_n + \int_{Q_T} w\Big((A_n - A)\Delta\varphi_n + d(\mathbf{q}-\mathbf{q}_n)\cdot\nabla\varphi_n + \beta(u-u_n)\varphi_n\Big) = 0. \quad (22)$$

We have the following estimates for φ_n. Multiplicating the equation of (19) by $-\Delta\varphi_n$, integrating by parts and using (20) we obtain

$$\int_\Omega |\nabla\varphi_n(T)|^2 + \frac{1}{n}\int_{Q_T} |\Delta\varphi_n|^2 \leq \Big(d\|\operatorname{div}\mathbf{q}_n\|_{L^\infty} + \|\alpha - \beta u_n\|_{L^\infty} + 1\Big)\int_{Q_T}|\nabla\varphi_n|^2$$
$$+\beta^2\|\nabla u_n\|^2_{L^\infty(Q_T)}\int_{Q_T}|\varphi_n|^2 + \int_{Q_T}|\nabla w_n|^2. \quad (23)$$

Using φ_n as test function in (19) we obtain

$$\frac{1}{2}\int_\Omega |\varphi_n(T)|^2 \leq \Big(\|d\operatorname{div}\mathbf{q}_n\|_{L^\infty} + \|\alpha-\beta u_n\|_{L^\infty} + \|w_n\|^2_{L^2}\Big)\int_{Q_T}|\varphi_n|^2$$
$$+\|\nabla A_n\|_{L^\infty}\int_{Q_T}|\nabla\varphi_n|^2. \quad (24)$$

We now distinguish the two cases in (ii) of Theorem 2 to pass to the limit $n\to\infty$ in identity (22). Let us start assuming $\beta = 0$. From (23), (20), (21), and Gronwall's lemma we deduce

$$\int_{Q_T}|\nabla\varphi_n|^2 + \frac{1}{n}\int_{Q_T}|\Delta\varphi_n|^2 \leq C(T) \quad (25)$$

with C independent of n. Observing that

$$\int_{Q_T} w(A_n - A)\Delta\varphi_n \leq \|w\|_{L^\infty}\|A-A_n\|_{L^2}\|\Delta\varphi_n\|_{L^2} \leq (\frac{C(T)}{n})^{1/2}\|w\|_{L^\infty} \quad (26)$$

we easily obtain, passing to the limit $n\to\infty$ in (22) that $w=0$ a.e. in Q_T.

If $\beta > 0$ then we need to use estimate (24) to control the L^2 norm of φ_n appearing in estimate (23). Therefore, we have to assume the additional regularity $\nabla u_i \in L^\infty(Q_T)$ in order to be able to define approximations u_n and A_n of u and A satisfying, in addition to (21),

$$\|\nabla u_n\|^2_{L^\infty(Q_T)} \leq k, \quad \text{and} \quad \|\nabla A_n\|_{L^\infty} \leq k, \quad (27)$$

with $k>0$ independent of n. In such a case, adding (23) and (24) and using (20), (21), (27) and Gronwall's lemma we obtain

$$\int_{Q_T}|\varphi_n|^2 + \int_{Q_T}|\nabla\varphi_n|^2 + \frac{1}{n}\int_{Q_T}|\Delta\varphi_n|^2 \leq C(T), \quad (28)$$

with C independent of n. Finally, we may pass to the limit $n \to \infty$ in (22) in a similar way than in the previous case to deduce $w = 0$ a.e. in Q_T.$\square$

4 Numerical examples

In this section we present numerical simulations in one space dimension illustrating differences between the behaviors of solutions corresponding to different mutation and splitting parameters in problem (8) and compare them to the corresponding solutions of problem (10). The main issue of the mutation and splitting procedure is, at least for the related ODE, the existence of a whole segment of equilibria, determined by $u_1 + u_2 = \alpha/\beta$. It is therefore interesting to see in which situations perturbations of the original system leads to stable equilibria. In the examples that follow and in others not shown, we have checked that problem (10) and problem (8) with $\rho = 0$ behaves in a similar way, but differently than problem (8) with $\rho \neq 0$. In our experiments, problem (8) with $\rho \neq 0$ always led to extinction of one population while this was not always the case for problem (10).

4.1 Finite element approximation

For simplicity, we take the spatial dimension to be $n = 1$, e.g. $\Omega \subset \mathbb{R}$ is chosen to be an open interval. For a more detailed description of the following finite element approximation, including the case of higher spatial dimensions, see [2]. On the interval Ω, we consider a family of quasi-uniform partitionings $\mathcal{T}^h$, $h > 0$, consisting of disjoint and open subintervals I with $h_I = |I|$ and $h = \max_{I \in \mathcal{T}^h} h_I$, so that $\bar{\Omega}_D = \cup_{I \in \mathcal{T}^h} \bar{I}$. Associated with $\mathcal{T}^h$ is the finite element space

$$S^h = \left\{ \varphi \in C(\bar{\Omega}) : \varphi|_I \quad \text{is linear for all} \quad I \in \mathcal{T}^h \right\} \subset H^1(\Omega).$$

Let J be the set of nodes of $\mathcal{T}^h$ and $\{p_j\}_{j \in J}$ the coordinates of these nodes. Let $\{\varphi_j\}_{j \in J}$ be the standard basis functions for S^h, that is $\varphi_j \in S^h$, $\varphi \geq 0$ in Ω, and $\varphi_j(p_i) = \delta_{ij}$ for all $i, j \in J$. The following functions were considered in [2], see also [19,35], to obtain a discrete analogue of the entropy inequality which allows to control the possible non-positivity of discrete approximate solutions. We define $\Lambda_\varepsilon : S^h \to L^\infty(\Omega)$ as

$$\Lambda_\varepsilon(z^h)|_\kappa = \begin{cases} \frac{z^h(p_k) - z^h(p_j)}{F'_\varepsilon(z^h(p_k)) - F'_\varepsilon(z^h(p_j))} & z^h(p_k) \neq z^h(p_j), \\ \lambda_\varepsilon(z^h(p_k)) & z^h(p_k) = z^h(p_j), \end{cases}$$

with $F_\varepsilon : \mathbb{R} \to [0, \infty)$ given by

$$F_\varepsilon(s) = \begin{cases} \frac{s^2-\varepsilon^2}{2\varepsilon} + (\ln \varepsilon - 1)s + 1 & s \leq \varepsilon, \\ (\ln s - 1)s + 1 & \varepsilon \leq s \leq \varepsilon^{-1}, \\ \frac{\varepsilon(s^2-\varepsilon^{-2})}{2} + (\ln \varepsilon^{-1} - 1)s + 1 & \varepsilon^{-1} \leq s, \end{cases}$$

and $\lambda_\varepsilon : \mathbb{R} \to \mathbb{R}$ as $\lambda_\varepsilon(s) = (F''_\varepsilon(s))^{-1}$, for some $\varepsilon > 0$. Observe that we always have $\Lambda_\varepsilon(z^h)|_\kappa = \lambda_\varepsilon(z^h(\xi))$ for some $\xi \in \kappa$ and that $\lambda_\varepsilon(s) \to s1_{[0,\infty)}(s)$ as $\varepsilon \to 0$.

For the time discretization, we consider a partitioning $0 = t_0 < t_1 < \ldots < t_{N-1} < T_N = T$ of $[0, T]$ into possibly variable time steps $\tau_n = t_n - t_{n-1}$, $n = 1, \ldots, N$. We set $\tau = \max_n \tau_n$. For any given $\varepsilon \in (0, 1)$, we then consider the following finite element approximation of problem (8): For $n \geq 1$ find $(u^n_{\varepsilon,1}, u^n_{\varepsilon,2}) \in (S^h)^2$ such that for $i, j = 1, 2$, with $j \neq i$, and for all $\varphi \in S^h$

$$\begin{aligned} \frac{1}{\tau_n} \left\langle u^n_{\varepsilon,i} - u^{n-1}_{\varepsilon,i}, \varphi \right\rangle + \left\langle \Big[c + 2a\Lambda_\varepsilon(u^n_{\varepsilon,i}) + b_i\Lambda_\varepsilon(u^n_{\varepsilon,j})\Big] \nabla u^n_{\varepsilon,i}, \nabla\varphi \right\rangle \\ + \left\langle \Lambda_\varepsilon(u^n_{\varepsilon,i}) \Big[b_i \nabla u^n_{\varepsilon,j} + d\nabla(\pi^h\Phi)\Big], \nabla\varphi \right\rangle \\ = \left\langle u^n_{\varepsilon,i} \Big[\alpha - \beta\lambda_\varepsilon(u^{n-1}_{\varepsilon,1}) - \beta\lambda_\varepsilon(u^{n-1}_{\varepsilon,2}))\Big], \varphi \right\rangle, \end{aligned} \tag{29}$$

where $\pi^h : C(\bar{\Omega}_D) \to S^h$ is the usual interpolation operator, with $(\pi^h\eta)(p_j) = \eta(p_j)$ for all $j \in J$, and $u^0_{\varepsilon,i} \in S^h$ is an approximation of u^0_i, for instance its L^2 projection on S^h. As shown in Theorem 2.1 of [2], problem (29) admits a solution. More concretely, if $(u^{n-1}_{\varepsilon,1}, u^{n-1}_{\varepsilon,2}) \in (S^h)^2$, $\varepsilon \in (0, e^{-2})$ and $2(\alpha+\beta)\tau_n < 1$, then there exists a solution $(u^n_{\varepsilon,1}, u^n_{\varepsilon,2}) \in (S^h)^2$ to the n-th step of problem (29). In addition, they prove (Theorem 3.1 of [2]) that if $\tau \to 0$ with either $\tau_1 \leq Ch^2$ or $u^0_i \in H^1(\Omega)$, and if $\varepsilon h^{-1/2} \to 0$ then a subsequence (not relabeled) of

$$u_{\varepsilon,i}(t) = \frac{t - t_{n-1}}{\tau_n} u^n_{\varepsilon,i} + \frac{t_n - t}{\tau_n} u^{n-1}_{\varepsilon,i} \quad t \in [t_{n-1}, t_n], \quad n \geq 1,$$

may be extracted such that $(u_{\varepsilon,1}, u_{\varepsilon,2}) \to (u_1, u_2)$ in a suitable sense, being (u_1, u_2) a weak solution of problem (8).

4.2 Experiments

As in [2], we use the following fixed point algorithm for solving the system of nonlinear algebraic equations for $(u^n_{\varepsilon,1}, u^n_{\varepsilon,2})$ arising at each time level from the approximations (29). For $t = t_0 = 0$, set $u^0_i = u^0_{\varepsilon,i}$. For $t = t_n$, let $u^{n-1}_{\varepsilon,i}$ be given and set $u^{n,0}_{\varepsilon,i} = u^{n-1}_{\varepsilon,i}$. Then, for $k \geq 1$ find $u^{n,k}_{\varepsilon,i}$ such that for $i, j = 1, 2$, with $j \neq i$, and for all $\varphi \in S^h$

$$\frac{1}{\tau_n}\left\langle u_{\varepsilon,i}^{n,k}-u_{\varepsilon,i}^{n-1},\varphi\right\rangle+\left\langle\left[c+2a\Lambda_\varepsilon(u_{\varepsilon,i}^{n,k-1})+b_i\Lambda_\varepsilon(u_{\varepsilon,j}^{n,k-1})\right]\nabla u_{\varepsilon,i}^{n,k},\nabla\varphi\right\rangle$$
$$+\left\langle\Lambda_\varepsilon(u_{\varepsilon,i}^{n,k-1})\left[b_i\nabla u_{\varepsilon,j}^{n,k}+d\nabla(\pi^h\Phi)\right],\nabla\varphi\right\rangle$$
$$=\left\langle\alpha u_{\varepsilon,i}^{n,k}-\beta\left[\lambda_\varepsilon(u_{\varepsilon,i}^{n-1})-\lambda_\varepsilon(u_{\varepsilon,j}^{n-1}))\right]\lambda_\varepsilon(u_{\varepsilon,i}^{n,k-1}),\varphi\right\rangle.$$

We adopted the stopping criteria

$$\max_{i=1,2}\|u_{\varepsilon,i}^{n,k}-u_{\varepsilon,i}^{n,k-1}\|_\infty<\text{tol},$$

with $\text{tol}=10^{-7}$ in the experiments, and set $u_i^n=u_i^{n,k}$. Similar approaches are followed to approximate solutions of problems (7) and (10). In all experiments we integrated in time until a numerical stationary solution, u_i^S, was achieved. This was determined by

$$\max_{i=1,2}\|u_{\varepsilon,i}^{n,1}-u_{\varepsilon,i}^{n,0}\|_\infty<5\times10^{-12}.$$

4.3 Experiment 1

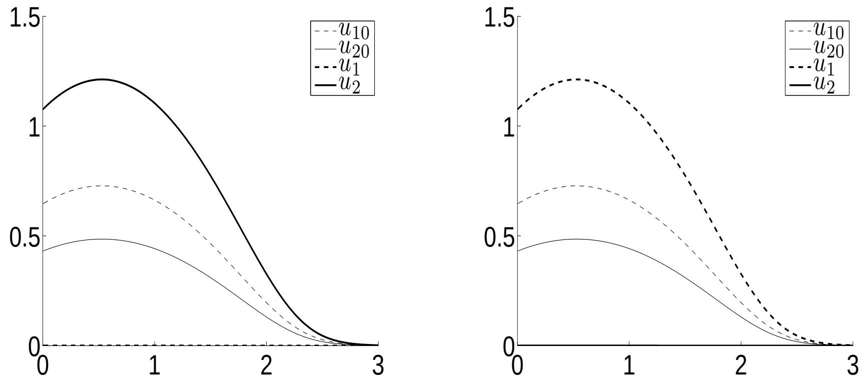
We take $\Omega=(0,3)$ with a spatial and time step sizes $h=0.01$ and $\tau=0.001$, respectively. The environmental potential is $\Phi(x)=-1.5(x-0.5)^2$. The flow and Lotka-Volterra parameters are given by $a=c=d=\alpha=\beta=1$, and the initial conditions are

$$u_{10}=0.6u^S,\quad u_{20}=0.4u^S,\tag{30}$$

with u^S the numerical stationary solution of problem (7) for the same parameters than above, and with the initial data $u_0=1$. In Fig. 1 we show the steady state solutions of problem (8) corresponding to $\rho=\pm0.75$, which lead to extinction of the population with lower cross-diffusion parameter b_i. Other non-zero values of ρ produce the same results. For any initial data of the form $u_{10}=\lambda u^S$, $u_{20}=(1-\lambda)u^S$, with $\lambda\in[0,1]$, the solution of problems (8) with $\rho=0$ and of problem (10) are just $u_1=u_{10}$, $u_2=u_{20}$, so both populations survive in this case. We observe that in all the cases solutions satisfy $u_1^S+u_2^S=u^S$.

4.4 Experiment 2

We take $\Omega=(-1,1)$ with a spatial and time step sizes $h=0.01$ and $\tau=0.001$, respectively. The environmental potential is $\Phi(x)=\exp(-10x^4)$. The flow and Lotka-Volterra parameters are given as in Experiment 1, except $d=5$, and



(a) $\rho = 0.75$ (b) $\rho = -0.75$

Fig. 1. Experiment 1. Steady state solutions of problem (8).

the *initial* conditions are

$$u_{10} = u^S 1_{(-0.25,0.25)}, \quad u_{20} = u^S - u_{10}, \tag{31}$$

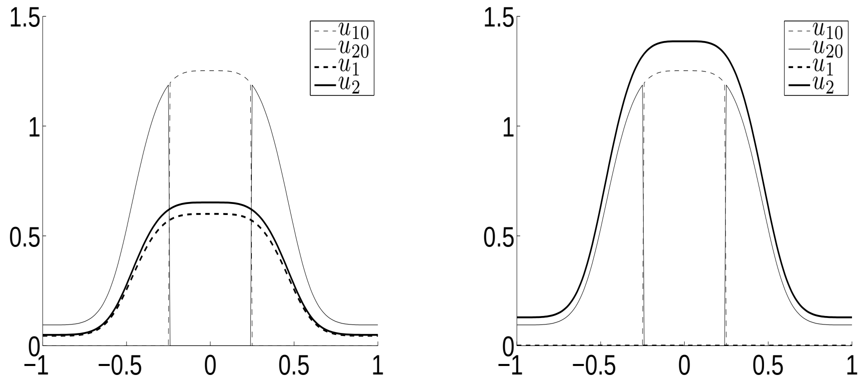
with u^S the numerical stationary solution of problem (7) for the same parameters than above, and with the initial data $u_0 = 1$.

The solutions of problem (8) behaves, qualitatively, as in Example 2: one population is extincted and the other coincides with u^S when $\rho \neq 0$. The solutions of problems (8) with $\rho = 0$ and of problem (10) are again different of those with $\rho \neq 0$, since both populations survive in these cases, see Fig. 2 (a). However, this seems to be an unstable situation as it is demonstrated when perturbing parameter α. Fig. 2 (b) shows the steady state solution of problems (8) (with $\rho = \pm 0.75$) and (10), all of them coinciding, for the same situation as before but with $\alpha = \alpha_1 = 0.9$ for the u_1-equation and $\alpha = \alpha_2 = 1.1$ for the u_2-equation. The result is that population 1 is extincted and population 2 satisfies $u_2^S = \tilde{u}^S$, with $\tilde{u}^S$ the steady state solution of problem (7) with the same data than above but $\alpha = \alpha_2 = 1.1$.

4.5 Experiment 3

We take the data as in Experiment 1 but modifying the Lotka-Volterra term corresponding to population 2 as Sánchez-Palencia in [30]: $\alpha_2 = 0.95$, $\beta_{21} = 0.9$. For the system of ODE's, this perturbation results in the passing from the existence of a segment of equilibria, $u_1 + u_2 = \alpha/\beta$, to a unique equilibrium (u_1, u_2) near $(0.5, 0.5)$, see [30]. We study in this example whether something similar happens to the PDE system or not. As in Experiment 2, we look for steady state solutions of problem (8) with $\rho = \pm 0.75$ and $\rho = 0$, and of problem (10). We take the following initial conditions:

$$u_{10} = u^S 1_{(0.25,0.50)}, \quad u_{20} = u^S - u_{10}, \tag{32}$$



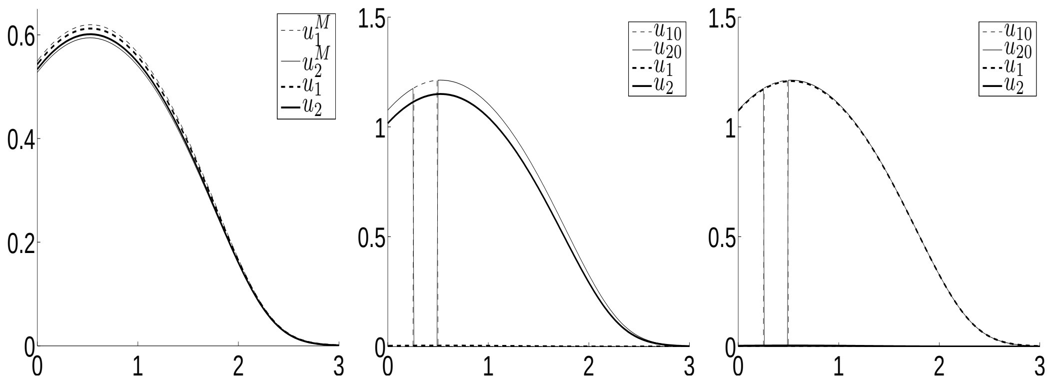
(a) $\alpha = 1$ (b) $\alpha_1 = 0.9,\ \alpha_2 = 1.1$

Fig. 2. Experiment 2. Steady state solutions; (a) problem (10), (b) problems (10) and (8) for $\rho = \pm 0.75$.

with u^S the numerical stationary solution of problem (7) for the same parameters than above, and with the initial data $u_0 = 1$. In Fig. 3 (a) we plot the steady state solutions corresponding to problem (10) and problem (8) with $\rho = 0$. Solutions to both problems are similar and satisfy, in coincidence with the solution of the ODE system,

$$\|u_1/u_2\|_\infty \approx 1.$$

However, solutions of problem (8) with $\rho = \pm 0.75$ again lead to extinction of one **species**, as shown in Fig. 3 (b) and (c).



(a) (b) $\rho = 0.75$ (c) $\rho = -0.75$

Fig. 3. Experiment 3. Steady state solutions: (a) problems (10) and (8) for $\rho = 0$, (b) problem (8) for $\rho = \pm 0.75$.

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