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Strong coupling results in $\mathcal{N} = 2$ gauge theories

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This talk is mainly based on:

- M. Billo, M. Frau, **A. L.**, A. Pini, P. Vallarino, *“Strong coupling expansions in N=2 quiver gauge theories”*, JHEP 01 (2023) 119, arXiv:2211.11795

- M. Billo, M. Frau, **A. L.**, A. Pini, P. Vallarino, *“Localization vs holography in 4d N=2 quiver theories”*, JHEP 10 (2022) 020, arXiv:2207.08846

- M. Billo, M. Frau, **A. L.**, A. Pini, P. Vallarino, *“Structure Constants in N=2 Superconformal Quiver Theories at Strong Coupling and Holography”*, Phys. Rev. Lett. 129 (2022) 031602, arXiv:2206.13582

but it builds on a very vast literature …

Plan of the talk

1. Introduction / motivation
2. Localization and matrix model
3. Strong-coupling results
4. Conclusions

Introduction

The analysis of the strong-coupling regime in an interacting theory is a very difficult problem but, when there is a high amount of symmetry, significant progress can be made.

This is the case of $\mathcal{N}=4$ SYM where several exact results have been obtained over the years, especially in the planar limit:

$$N \to \infty \quad \text{with} \quad \lambda = N\, g_{\rm YM}^2 \quad \text{fixed}$$

They include:

- 2- and 3-point functions of protected scalar operators
- v.e.v. of BPS circular Wilson-loop
- cusp anomalous dimension
- Brehmsstrahlung function
- integrated 4-point functions of superconformal primaries
- "octagon" form factors in 4-point functions of very heavy scalar operators
- …

The main tools that are used are:

integrability, localization and holography.

$\mathcal{N} = 4$ SYM

- $\mathcal{N}=4$ SYM is the "simplest" gauge theory
- It is a superconformal theory and possess a holographic dual ($\mathrm{AdS}_5 \times S^5$)
- Field content in $\mathcal{N}=2$ language :

$$\begin{cases} 1 \text{ vector } A_\mu \\ 1 \text{ complex scalar } \phi \\ 2 \text{ chiral fermions } \psi^1_\alpha \ , \psi^2_\alpha \end{cases}$$

vector multiplet in the adjoint

$$\begin{cases} 2 \text{ complex scalars } \Phi^1 \ , \Phi^2 \\ 2 \text{ chiral fermions } \chi^1_\alpha \ , \chi^2_\alpha \end{cases}$$

hypermultiplet in the adjoint

$\mathcal{N}$ = 4 SYM

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- Field content in $\mathcal{N} = 2$ language : $\begin{cases} 1 \text{ vector } A_\mu \\ 1 \text{ complex scalar } \phi \\ 2 \text{ chiral fermions } \psi^1_\alpha \,, \psi^2_\alpha \end{cases}$ vector multiplet in the adjoint

  $\begin{cases} 2 \text{ complex scalars } \Phi^1 \,, \Phi^2 \\ 2 \text{ chiral fermions } \chi^1_\alpha \,, \chi^2_\alpha \end{cases}$ hypermultiplet in the adjoint

- Simplest observables:

  - Chiral operators $\mathcal{O}_n(x) = \mathrm{tr}\,\phi^n(x)$ primary operators with dimension n

  - Wilson loop $W = \dfrac{1}{N}\,\mathrm{tr}\,\mathcal{P}\exp\left[\oint_C d\tau\left(iA_\mu\,\dot{x}^\mu + \dfrac{1}{\sqrt{2}}(\phi+\bar{\phi})|\dot{x}|\right)\right]$

$\mathcal{N} = 4$ SYM

- 2-point functions $\langle \mathcal{O}_n(x)\, \overline{\mathcal{O}}_n(y) \rangle = \dfrac{G_n}{|x-y|^{2n}}$

- 3-point functions $\langle \mathcal{O}_{n_1}(x)\, \mathcal{O}_{n_2}(y)\, \overline{\mathcal{O}}_{n_3}(z) \rangle = \dfrac{G_{n_1,n_2,n_3}}{|x-z|^{2n_1}\,|y-z|^{2n_2}}$

- In the ‘t Hooft planar limit the structure constants

$$C_{n_1,n_2,n_3} = \frac{G_{n_1,n_2,n_3}}{\sqrt{G_{n_1}}\sqrt{G_{n_2}}\sqrt{G_{n_3}}} = \frac{\sqrt{n_1\, n_2\, n_3}}{N}$$

are independent of the coupling.

Lee, Minwalla, Rangamani, Seiberg, 1998

SYM perturbation theory (Feynman diagrams)

$$\boxed{C_{n_1,n_2,n_3} = \frac{\sqrt{n_1\, n_2\, n_3}}{N}}$$

AdS/CFT correspondence (Witten diagrams)

0 weak coupling — λ — ∞ strong coupling

$\mathcal{N}$ = 4 SYM

A less simple example is given by the v.e.v. of the circular Wilson loop for $N \to \infty$

$$W = \frac{1}{N} \operatorname{tr} \mathcal{P} \exp \left[\oint_C d\tau \left(i A_\mu \dot{x}^\mu + \frac{1}{\sqrt{2}} (\phi + \bar{\phi}) |\dot{x}| \right) \right]$$

weak coupling

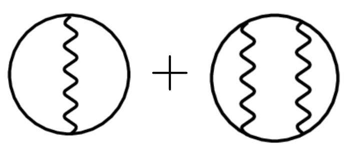
strong coupling

0 $\longrightarrow$ ∞

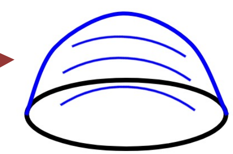
λ

SYM perturbation theory

AdS/CFT correspondence

$1 +$  $+ \cdots$

minimal area law in AdS_5



Maldacena, 1998

Erickson, Semenoff, Zarembo, 2000

$$\langle W \rangle = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \cdots$$

$$\langle W \rangle = e^{\sqrt{\lambda} - \frac{3}{4}\log\lambda + \frac{1}{2}\log\frac{2}{\pi} + \cdots}$$

$\lambda \to 0$

$\lambda \to \infty$

$$\boxed{\langle W \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})}$$

Erickson, Semenoff, Zarembo, 2000;
Drukker, Gross, 2000; …

As mentioned above, there are many other examples of exact results in $\mathcal{N}=4$ SYM

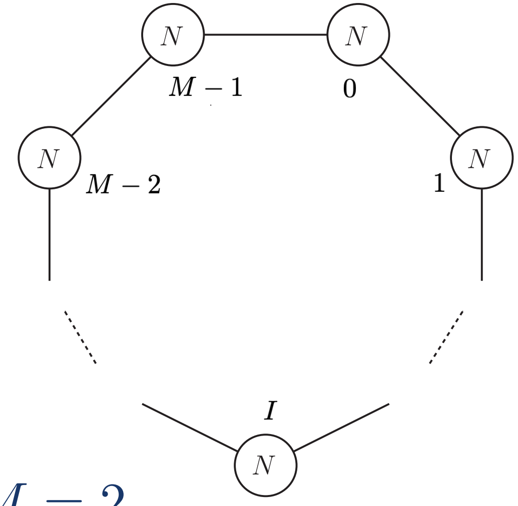
Finding exact results in non-maximally supersymmetric theories like $\mathcal{N}=2$ theories is more challenging!

In the following I will discuss a class of $\mathcal{N}=2$ conformal theories in 4d

- where one can find exact results, that are valid for all values of the coupling constant
- where one can test the AdS/CFT holographic correspondence in a non-maximally supersymmetric context

$\mathcal{N}=2$ quiver gauge theories

$$\mathrm{SU}(N)\times\underbrace{\mathrm{SU}(N)\times\cdots}_{M\ \text{times}}\times\mathrm{SU}(N)$$



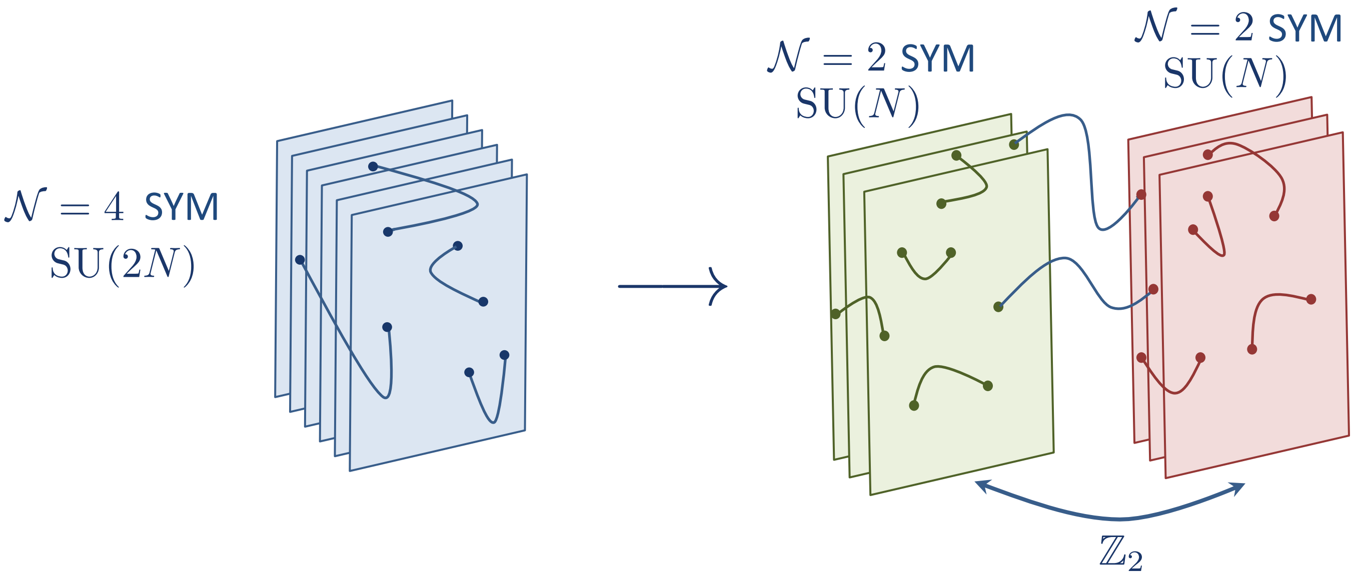
For simplicity in this talk I will consider the case $M=2$

$\mathcal{N}=2$ quiver theory

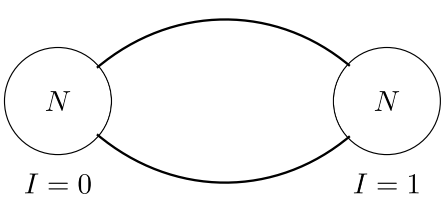
It is the "next-to-simplest" 4d gauge theory after $\mathcal{N}=4$ SYM

It arises as a $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM

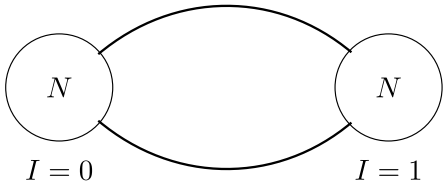
It admits a simple string theory realization in terms of fractional D3-branes



It is usually represented by the 2-node quiver diagram



$\mathcal{N}=2$ quiver theory



- gauge group: $\mathrm{SU}(N)\times\mathrm{SU}(N)$
- in each node: $\left\{\begin{array}{l} 1 \text{ vector } A_\mu^I \\ 1 \text{ complex scalar } \phi_I \\ + \text{ fermions}\end{array}\right.$ in the adjoint
- between nodes: hyper-multiplets in the bi-fundamental

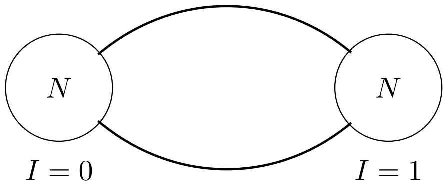
β – function = 0

- Local operators: $\boxed{\mathcal{O}_k^{\pm}(x)=\frac{1}{\sqrt{2}}\Big(\mathrm{tr}\,\phi_0(x)^k\pm\mathrm{tr}\,\phi_1(x)^k\Big)}$

$\mathcal{O}^+$ untwisted
($\mathbb{Z}_2$ symmetric)

$\mathcal{O}^-$ twisted
($\mathbb{Z}_2$ anti-symmetric)

$\mathcal{N} = 2$ quiver theory



- We are interested in studying the 2- and 3-point functions and the corresponding structure constants in the planar limit:

$$\langle \mathcal{O}_k^{\pm}(x)\, \overline{\mathcal{O}}_k^{\pm}(y) \rangle = \frac{G_k^{\pm}}{|x-y|^{2k}}$$

$$\langle \mathcal{O}_k^{+}(x)\, \mathcal{O}_\ell^{\pm}(y)\, \overline{\mathcal{O}}_p^{\pm}(z) \rangle = \frac{G_{k,\ell,p}^{\pm}}{|x-z|^{2k}|y-z|^{2\ell}} \qquad p = k+\ell$$

$$C_{k,\ell,p}^{\pm} = \frac{G_{k,\ell,p}^{\pm}}{\sqrt{G_k^{+}}\sqrt{G_\ell^{\pm}}\sqrt{G_p^{\pm}}}$$

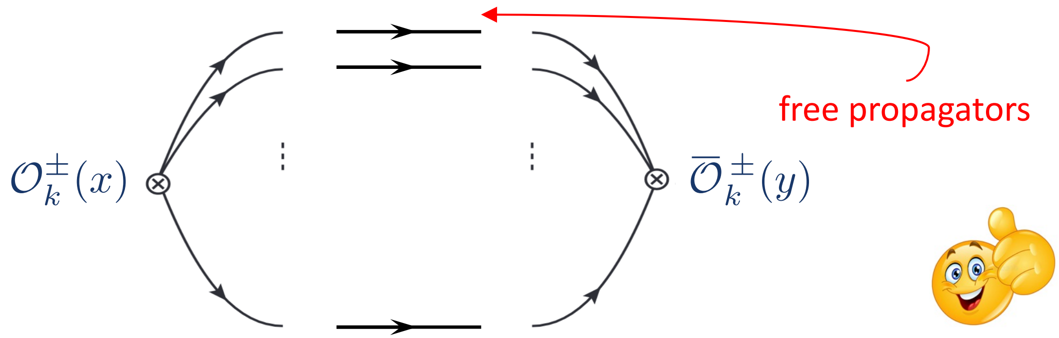
- The coefficients $G_k^{\pm}$, $G_{k,\ell,p}^{\pm}$, $C_{k,\ell,p}^{\pm}$ are non-trivial functions of N and λ

- How can we compute these functions?

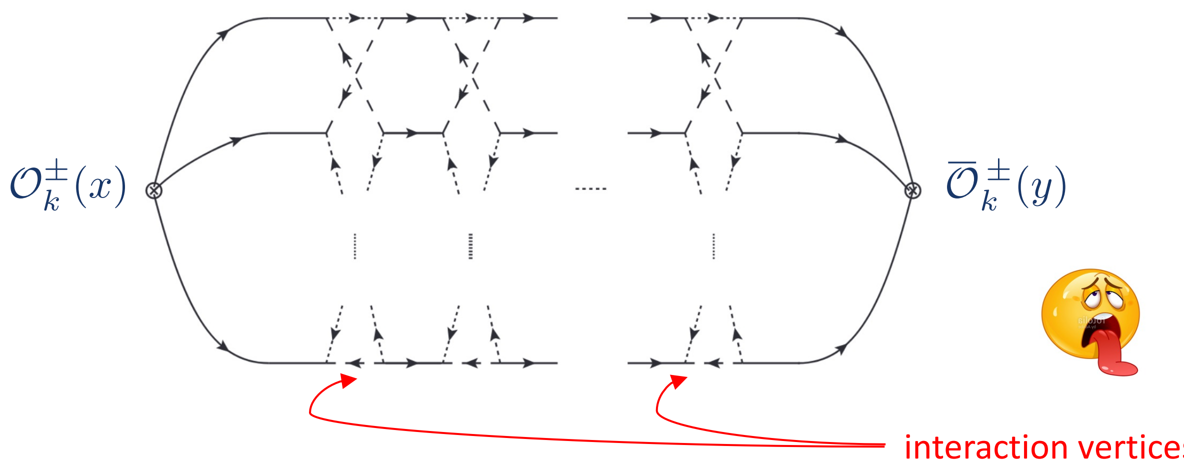
$\mathcal{N} = 2$ quiver theory

- At weak coupling one could use standard Feynman diagrams:

- at tree level ($\lambda = 0$)



- at loop level



- This is doable at the first orders, but with a lot of effort!

Localization

- A much more efficient way to compute these correlators is through

localization

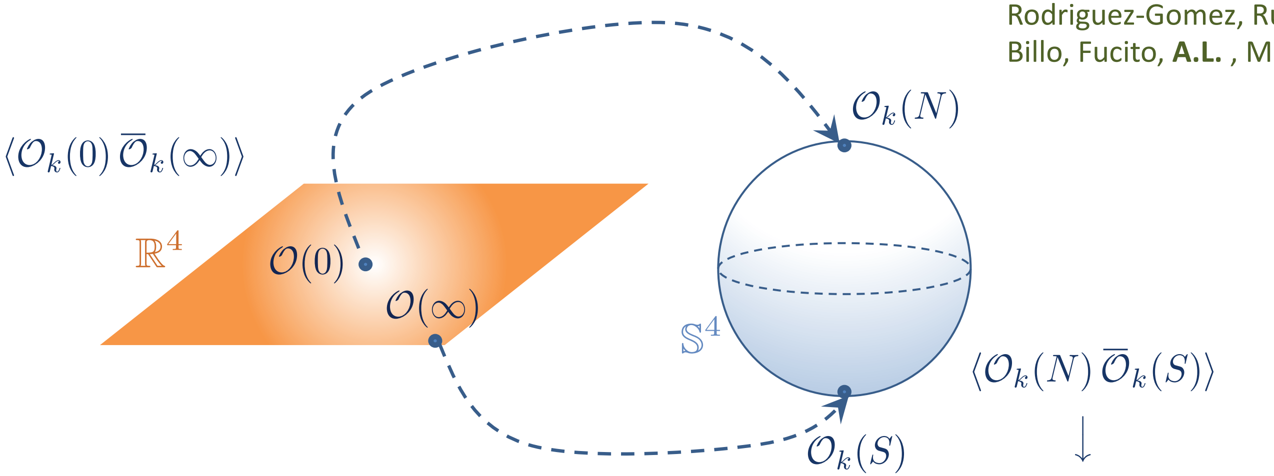
which for a theory on a compact manifold (like a 4-sphere) reduces path integrals to finite dimensional integrals in a

Pestun, 2007

matrix model

- This method applies to the partition function, the v.e.v. of circular Wilson loops, and the chiral/anti-chiral correlators

Baggio, Niarchos, Papadodimas, 2014, 2015;
Gerchkovitz, Gomis, Ishtiaque et al, 2016;
Rodriguez-Gomez, Russo, 2016; …
Billo, Fucito, **A.L.** , Morales, Stanev, Wen, 2017; …



Matrix model

Quiver theory on $\mathbb{R}^4$ $\xrightarrow{\text{conformal map}}$ Quiver theory on $\mathbb{S}^4$ $\xrightarrow{\text{localization}}$ Matrix model

- For our quiver theory the matrix model contains two $N \times N$ Hermitian matrices a_0 and a_1 corresponding to the v.e.v.'s of ϕ_0 and ϕ_1
- The partition function is

$$\mathcal{Z} = \int \Big(\prod_{I=0,1} da_I \; e^{-\operatorname{tr} a_I^2} \Big) \, Z_{1-\text{loop}} \, \cancel{Z_{\text{inst}}}$$

$= 1$ when $N \to \infty$

with

$$Z_{1-\text{loop}} = e^{-S_{\text{int}}}$$

odd Riemann ζ-values

$$S_{\text{int}} = 2 \sum_{m=2}^{\infty} \sum_{k=2}^{2m} (-1)^{m+k} \Big(\frac{\lambda}{8\pi^2 N} \Big)^m \binom{2m}{k} \frac{\zeta_{2m-1}}{2m} \big(\operatorname{tr} a_0^{2m-k} - \operatorname{tr} a_1^{2m-k} \big) \big(\operatorname{tr} a_0^k - \operatorname{tr} a_1^k \big)$$

Matrix model

$$\text{Quiver theory on } \mathbb{R}^4 \xrightarrow{\text{conformal map}} \text{Quiver theory on } \mathbb{S}^4 \xrightarrow{\text{localization}} \text{Matrix model}$$

Since $\phi_I(x) \longrightarrow a_I$, one may think that :

$$\mathcal{O}_k^{\pm}(x) = \frac{1}{\sqrt{2}}\Big(\mathrm{tr}\,\phi_0(x)^k \pm \mathrm{tr}\,\phi_1(x)^k\Big) \longrightarrow \frac{1}{\sqrt{2}}\Big(\mathrm{tr}\,a_0^k \pm \mathrm{tr}\,a_1^k\Big) \equiv A_k^{\pm}$$

However, $\mathcal{O}_k^{\pm}(x)$ do not have self-contractions, while $A_k^{\pm}$ do. So the right map is through normal-ordering

$$\mathcal{O}_k^{\pm}(x) \longrightarrow O_k^{\pm} = \,:\!A_k^{\pm}\!:\, = \sum_{\ell \leq k} \mathsf{M}_{k\ell}\, A_\ell^{\pm}$$

Thus

$$\boxed{\big\langle \mathcal{O}_k^{\pm}(x)\,\overline{\mathcal{O}}_k^{\pm}(y)\big\rangle = \frac{G_k}{|x-y|^2} \quad \longleftrightarrow \quad \big\langle O_k^{\pm}\, O_k^{\pm}\big\rangle = G_k}$$

and similarly for the 3-point functions. **Everything is reduced to a calculation of v.e.v's in the interacting matrix model.**

Preliminary step: the free matrix model $S_{\rm int} = 0$

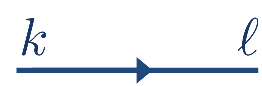
- In the free Gaussian model, in the planar limit, one finds

$$\big\langle O_k^{\pm}\, O_\ell^{\pm} \big\rangle_0 = k \Big(\frac{N}{2}\Big)^k \delta_{k,\ell} \;\equiv\; \mathcal{G}_k\, \delta_{k,\ell}$$

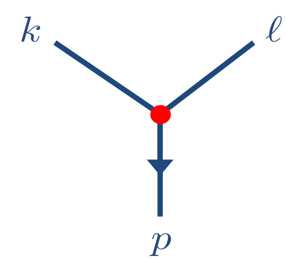
$$\big\langle O_k^{+}\, O_\ell^{\pm}\, O_p^{\pm} \big\rangle_0 = \frac{k\,\ell\,p}{2\sqrt{2}} \Big(\frac{N}{2}\Big)^{\frac{k+\ell+p}{2}-1} \delta_{k+\ell,p} \;\equiv\; \mathcal{G}_{k,\ell,p}\, \delta_{k+\ell,p}$$

- Defining the normalized operators $P_k^{\pm} = \dfrac{1}{\sqrt{\mathcal{G}_k}}\, O_k^{\pm}\big|_0$, one has

$$\big\langle P_k^{\pm}\, P_\ell^{\pm} \big\rangle_0 = \delta_{k,\ell} \qquad \longleftrightarrow$$



$$\big\langle P_k^{+}\, P_\ell^{\pm}\, P_p^{\pm} \big\rangle_0 = \frac{\sqrt{k\,\ell\,p}}{\sqrt{2}\,N}\, \delta_{k+\ell,p} \qquad \longleftrightarrow$$



like in $\mathcal{N}=4$ SYM (up to the $\sqrt{2}$ due to the orbifold)

The interacting matrix model

The interaction action of the quiver matrix model

$$S_{\rm int} = 2\sum_{m=2}^{\infty}\sum_{k=2}^{2m}(-1)^{m+k}\Big(\frac{\lambda}{8\pi^2 N}\Big)^{m}\binom{2m}{k}\frac{\zeta_{2m-1}}{2m}\big(\mathrm{tr}\, a_0^{2m-k} - \mathrm{tr}\, a_1^{2m-k}\big)\big(\mathrm{tr}\, a_0^{k} - \mathrm{tr}\, a_1^{k}\big)$$

can be rewritten as

$$S_{\rm int} = -\frac{1}{2}\sum_{k,\ell} P_k^- \,\mathsf{X}_{k,\ell}\, P_\ell^-$$

where

$$\mathsf{X}_{k,\ell} = -8\sqrt{k\,\ell}\sum_{p=0}^{\infty}(-1)^p\frac{(k+\ell+2p)!^{\,2}}{p!(k+p)!(\ell+p)!(k+\ell+p)!}\,\frac{\zeta_{k+\ell+2p-1}}{k+\ell+2p}\Big(\frac{\lambda}{16\pi^2}\Big)^{\frac{k+\ell+2p}{2}}$$

or

$$\mathsf{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}}\sqrt{k\,\ell}\int_0^\infty\frac{dt}{t}\,\frac{e^t}{(e^t-1)^2}\,J_k\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)\,J_\ell\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

Beccaria, Billo, Galvagno, Hasan, **A.L.**, 2020.

This convolution of Bessel functions contains the exact dependence on the coupling constant !

The X matrix

- The structure of the X matrix is

$$\mathsf{X} = \begin{pmatrix} \mathsf{X}_{2,2} & 0 & \mathsf{X}_{2,4} & 0 & \mathsf{X}_{2,6} & 0 & \cdots \\ 0 & \mathsf{X}_{3,3} & 0 & \mathsf{X}_{3,5} & 0 & \mathsf{X}_{3,7} & \cdots \\ \mathsf{X}_{4,2} & 0 & \mathsf{X}_{4,4} & 0 & \mathsf{X}_{4,6} & 0 & \cdots \\ 0 & \mathsf{X}_{5,3} & 0 & \mathsf{X}_{5,5} & 0 & \mathsf{X}_{5,7} & \cdots \\ \mathsf{X}_{6,2} & 0 & \mathsf{X}_{6,4} & 0 & \mathsf{X}_{6,6} & 0 & \cdots \\ 0 & \mathsf{X}_{7,3} & 0 & \mathsf{X}_{7,5} & 0 & \mathsf{X}_{7,7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Thus it is convenient to define

$$\mathsf{X}^{\mathrm{even}} = \begin{pmatrix} \mathsf{X}_{2,2} & \mathsf{X}_{2,4} & \mathsf{X}_{2,6} & \cdots \\ \mathsf{X}_{4,2} & \mathsf{X}_{4,4} & \mathsf{X}_{4,6} & \cdots \\ \mathsf{X}_{6,2} & \mathsf{X}_{6,4} & \mathsf{X}_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathsf{X}^{\mathrm{odd}} = \begin{pmatrix} \mathsf{X}_{3,3} & \mathsf{X}_{3,5} & \mathsf{X}_{3,7} & \cdots \\ \mathsf{X}_{5,3} & \mathsf{X}_{5,5} & \mathsf{X}_{5,7} & \cdots \\ \mathsf{X}_{7,3} & \mathsf{X}_{7,5} & \mathsf{X}_{7,7} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The 2-point functions

Now we can compute the 2-point functions.

- For the untwisted operators we have

$$\left\langle P_k^+ \, P_\ell^+ \right\rangle \underset{N\to\infty}{\sim} \frac{\left\langle P_k^+ \, P_\ell^+ \, e^{-\frac{1}{2}P^- \mathsf{X} \, P^-} \right\rangle_0}{\left\langle e^{-\frac{1}{2}P^- \mathsf{X} \, P^-} \right\rangle_0} = \frac{\left\langle P_k^+ \, P_\ell^+ \right\rangle_0 \cancel{\left\langle e^{-\frac{1}{2}P^- \mathsf{X} \, P^-} \right\rangle_0}}{\cancel{\left\langle e^{-\frac{1}{2}P^- \mathsf{X} \, P^-} \right\rangle_0}} = \left\langle P_k^+ \, P_\ell^+ \right\rangle_0 = \delta_{k,\ell}$$

- For the twisted operators we have

$$\left\langle P_k^- \, P_\ell^- \right\rangle \underset{N\to\infty}{\sim} \frac{\left\langle P_k^- \, P_\ell^- \, e^{\frac{1}{2}P^- \mathsf{X} \, P^-} \right\rangle_0}{\left\langle e^{\frac{1}{2}P^- \mathsf{X} \, P^-} \right\rangle_0} = \left\langle P_k^- \, P_\ell^- \right\rangle_0 + \left\langle P_k^- \, P_\ell^- \left(\frac{1}{2}P^- \mathsf{X} \, P^-\right) \right\rangle_0^{(\mathrm{c})}$$

$$+ \frac{1}{2}\left\langle P_k^- \, P_\ell^- \left(\frac{1}{2}P^- \mathsf{X} \, P^-\right)^2 \right\rangle_0^{(\mathrm{c})} + \cdots$$

Doing the contractions we find

$$\boxed{\left\langle P_k^- \, P_\ell^- \right\rangle = \delta_{k,\ell} + \mathsf{X}_{k,\ell} + \mathsf{X}_{k,\ell}^2 + \cdots = \left(\frac{1}{1-\mathsf{X}}\right)_{k,\ell}}$$

This formula is exact in λ

The 2-point functions

- Doing the normal ordering $P_k^- \longrightarrow O_k^- = \sqrt{\mathcal{G}_k} :P_k^-:$ we arrive at the final result

$$G_{2n}^- = \langle O_{2n}^- \, O_{2n}^- \rangle = \mathcal{G}_{2n} \boxed{\frac{\det\left(1 - \mathsf{X}_{[n+1]}^{\text{even}}\right)}{\det\left(1 - \mathsf{X}_{[n]}^{\text{even}}\right)}}$$

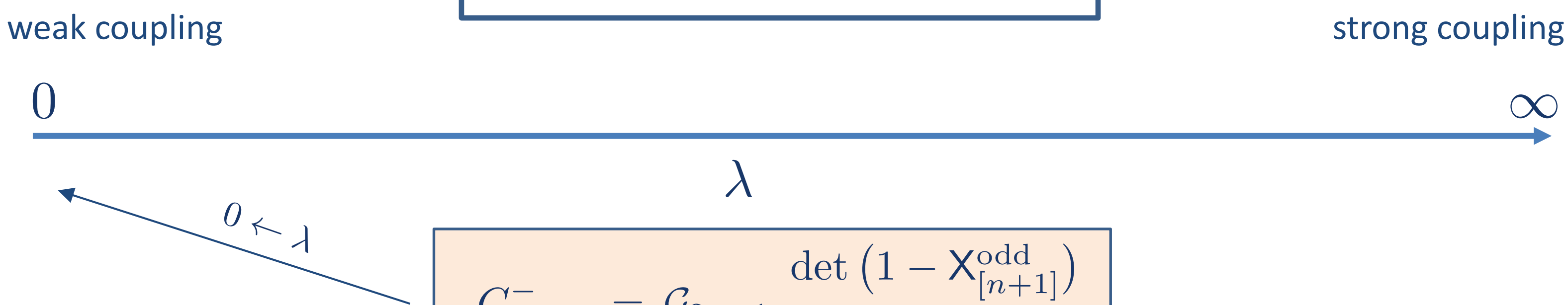
$$G_{2n+1}^- = \langle O_{2n+1}^- \, O_{2n+1}^- \rangle = \mathcal{G}_{2n+1} \boxed{\frac{\det\left(1 - \mathsf{X}_{[n+1]}^{\text{odd}}\right)}{\det\left(1 - \mathsf{X}_{[n]}^{\text{odd}}\right)}}$$

where $\mathcal{G}_k = k\left(\frac{N}{2}\right)^k$ and, for example,

$$\mathsf{X}_{[2]}^{\text{even}} = \begin{pmatrix} \mathsf{X}_{2,2} & \mathsf{X}_{2,4} & \mathsf{X}_{2,6} & \cdots \\ \mathsf{X}_{4,2} & \mathsf{X}_{4,4} & \mathsf{X}_{4,6} & \cdots \\ \mathsf{X}_{6,2} & \mathsf{X}_{6,4} & \mathsf{X}_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathsf{X}_{[3]}^{\text{odd}} = \begin{pmatrix} \mathsf{X}_{3,3} & \mathsf{X}_{3,5} & \mathsf{X}_{3,7} & \cdots \\ \mathsf{X}_{5,3} & \mathsf{X}_{5,5} & \mathsf{X}_{5,7} & \cdots \\ \mathsf{X}_{7,3} & \mathsf{X}_{7,5} & \mathsf{X}_{7,7} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

These formulas are valid for any value of λ

The 2-point functions



$$G^{-}_{2n+1} = \mathcal{G}_{2n+1}\,\frac{\det\left(1-\mathsf{X}^{\mathrm{odd}}_{[n+1]}\right)}{\det\left(1-\mathsf{X}^{\mathrm{odd}}_{[n]}\right)}$$

- Using the small λ expansion of the Bessel functions, it is quite straightforward to obtain the weak-coupling expansions. For example:

$$G^{-}_{3} = \frac{3\,N^3}{8}\Bigg[\,1-\frac{5\,\zeta_5}{256\pi^6}\,\lambda^3+\frac{105\,\zeta_7}{4096\pi^8}\,\lambda^4-\frac{1701\,\zeta_9}{65536\pi^{10}}\,\lambda^5$$

$$+\left(\frac{25\,\zeta_5^2}{65536\pi^{12}}+\frac{12705\,\zeta_{11}}{524288\pi^{12}}\right)\lambda^6+\cdots\;+\;O(\lambda^{160})\Bigg]$$

- The radius of convergence of these perturbative expansions is located at $\lambda \simeq \pi^2$. But they can be re-summed a la Padé and extended beyond that limit.

The 2-point functions

weak coupling — strong coupling

$0 \xrightarrow{\quad\lambda\quad} \infty$

$\lambda \to \infty$

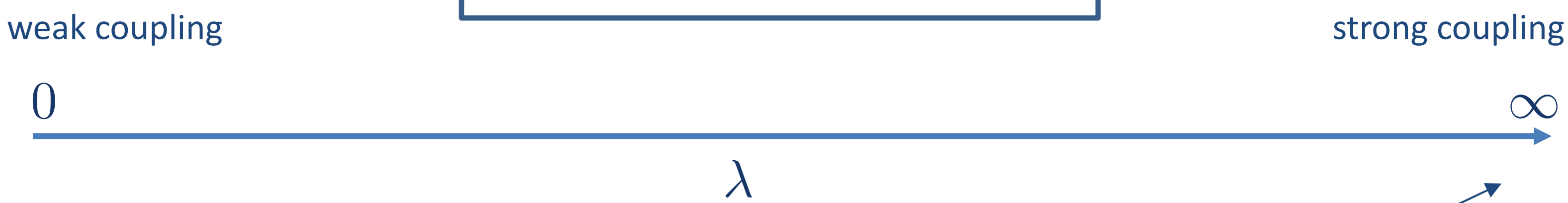
$$G^{-}_{2n+1} = \mathcal{G}_{2n+1}\,\frac{\det\left(1-\mathsf{X}^{\mathrm{odd}}_{[n+1]}\right)}{\det\left(1-\mathsf{X}^{\mathrm{odd}}_{[n]}\right)}$$

- More interestingly, using the asymptotic behaviour of the Bessel functions for large λ we can derive analytically the strong-coupling expansions of the 2-point functions. In particular one finds

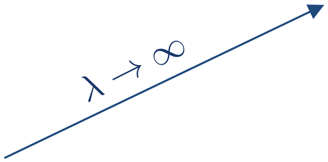
$$\mathsf{X}^{\mathrm{odd}} \underset{\lambda\to\infty}{\sim} -\frac{\lambda}{16\pi^2}\begin{pmatrix} 1 & -\frac{1}{\sqrt{15}} & 0 & 0 & 0 & 0 & \cdots \\ -\frac{1}{\sqrt{15}} & \frac{1}{3} & -\frac{2}{3\sqrt{35}} & 0 & 0 & 0 & \cdots \\ 0 & -\frac{2}{3\sqrt{35}} & \frac{1}{6} & -\frac{1}{6\sqrt{7}} & 0 & 0 & \cdots \\ 0 & 0 & -\frac{1}{6\sqrt{7}} & \frac{1}{10} & -\frac{2}{15\sqrt{11}} & 0 & \cdots \\ 0 & 0 & 0 & -\frac{2}{15\sqrt{11}} & \frac{1}{15} & -\frac{1}{3\sqrt{143}} & \cdots \\ 0 & 0 & 0 & 0 & -\frac{1}{3\sqrt{143}} & \frac{1}{21} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + O(\lambda^0)$$

Beccaria, Dunne, Tseytlin, 2021;
Beccaria, Billo, Frau, **A.L.** , Pini, 2021 + ...

The 2-point functions



$$G^{-}_{2n+1} = \mathcal{G}_{2n+1}\, \frac{\det\left(1 - \mathsf{X}^{\mathrm{odd}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\mathrm{odd}}_{[n]}\right)}$$



- More interestingly, using the asymptotic behaviour of the Bessel functions for large λ we can derive analytically the strong-coupling expansions of the 2-point functions. In particular one finds

$$\mathsf{X}^{\mathrm{odd}} \underset{\lambda\to\infty}{\sim} -\lambda\, \mathsf{S} + O(\lambda^0)$$

- Heuristically

$$\det\left(\cancel{1} - \mathsf{X}^{\mathrm{odd}}\right) \underset{\lambda\to\infty}{\sim} \det(\lambda\, \mathsf{S}) \quad \Longrightarrow \quad \frac{\det\left(\cancel{1} - \mathsf{X}^{\mathrm{odd}}_{[n+1]}\right)}{\det\left(\cancel{1} - \mathsf{X}^{\mathrm{odd}}_{[n]}\right)} \underset{\lambda\to\infty}{\sim} \frac{1}{\lambda}$$

The 2-point functions

weak coupling

strong coupling

0 $\longrightarrow$ ∞

λ

$$G^{-}_{2n+1} \;=\; \mathcal{G}_{2n+1}\,\frac{\det\left(1-\mathsf{X}^{\rm odd}_{[n+1]}\right)}{\det\left(1-\mathsf{X}^{\rm odd}_{[n]}\right)}$$

$\lambda \to \infty$

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$$\mathsf{X}^{\rm odd} \underset{\lambda\to\infty}{\sim} -\lambda\,\mathsf{S} \;+\; O(\lambda^{0})$$

- More rigorously,

$$G^{-}_{2n+1} \underset{\lambda\to\infty}{\sim} \mathcal{G}_{2n+1}\,\frac{8\pi^2\, n\,(2n+1)}{\lambda} \;+\; O(\lambda^{-\frac{3}{2}})$$

$$G^{-}_{2n} \underset{\lambda\to\infty}{\sim} \mathcal{G}_{2n}\,\frac{8\pi^2\, n\,(2n-1)}{\lambda} \;+\; O(\lambda^{-\frac{3}{2}})$$

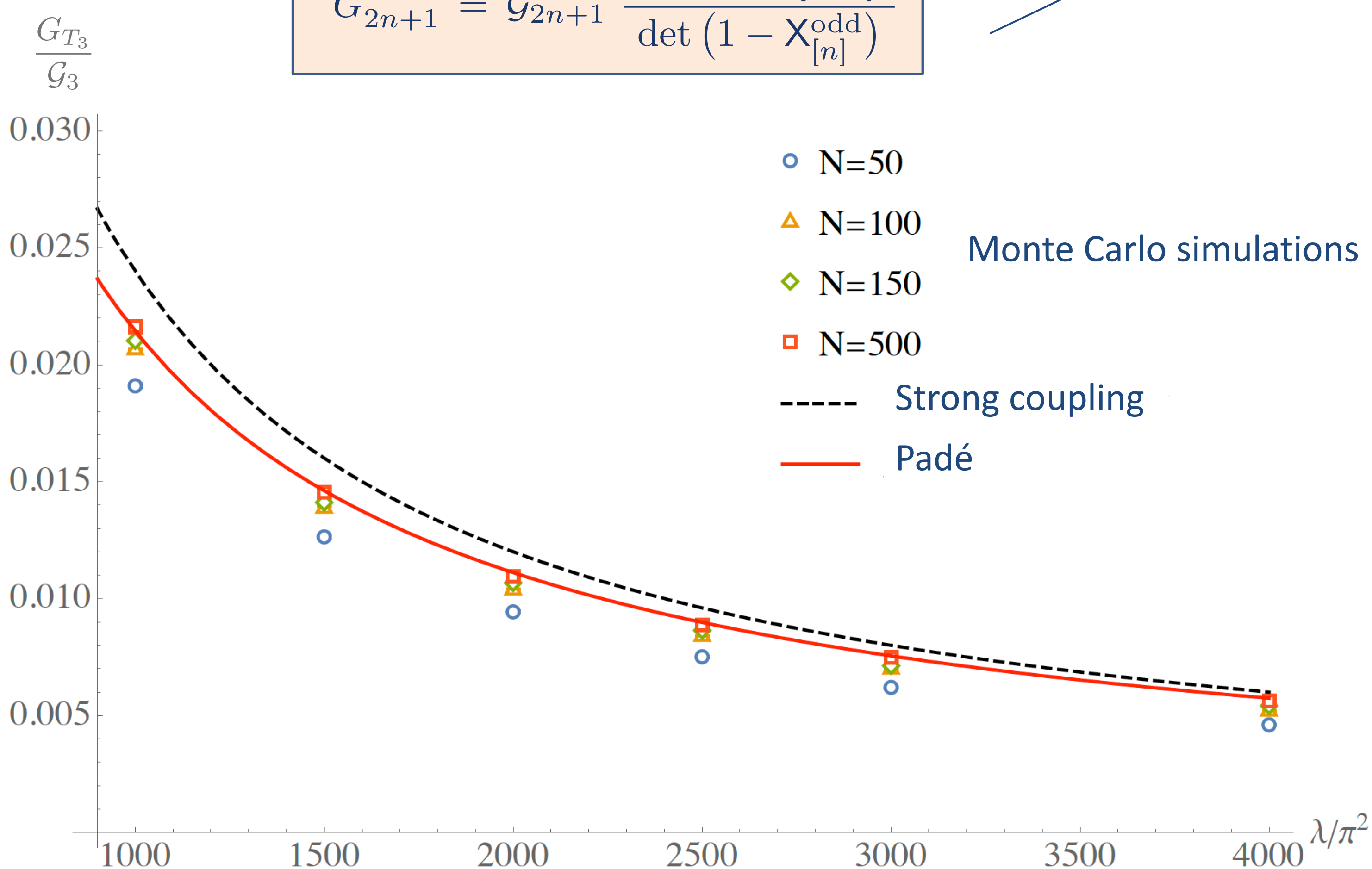
The 2-point functions

weak coupling — strong coupling

$0 \xrightarrow{\quad\lambda\quad} \infty$

$$G^-_{2n+1} = \mathcal{G}_{2n+1}\,\frac{\det\left(1-\mathsf{X}^{\rm odd}_{[n+1]}\right)}{\det\left(1-\mathsf{X}^{\rm odd}_{[n]}\right)}$$

$\lambda \to \infty$



Monte Carlo simulations

The 2-point functions

- Actually, one can do more and derive the full strong-coupling expansion:

$$\log\left[\det\left(1-\mathsf{X}^{\text{even}}_{[n]}\right)\right]=\frac{\sqrt{\lambda}}{4}-\left(2n-\frac{3}{2}\right)\log\left(\frac{\sqrt{\lambda}}{4\pi}\right)+B_{2n-1}+f_{2n-1}$$

$$\log\left[\det\left(1-\mathsf{X}^{\text{odd}}_{[n]}\right)\right]=\frac{\sqrt{\lambda}}{4}-\left(2n-\frac{1}{2}\right)\log\left(\frac{\sqrt{\lambda}}{4\pi}\right)+B_{2n}+f_{2n}$$

where

Beccaria, Korchemsky, Tseytlin, 2022

$$B_k=-6\log\mathsf{A}+\frac{1}{2}+\frac{1}{6}\log 2-k\,\log 2+\log\Gamma(k)$$

Gleisher constant

$$f_k=\frac{1}{16}(2k-3)(2k-1)\log\left(\frac{\lambda'}{\lambda}\right)+(2k-5)(2k-3)(4k^2-1)\frac{\zeta_3}{32\lambda'^{\,3/2}}$$
$$-(2k-7)(2k-5)(4k^2-9)(4k^2-1)\frac{3\,\zeta_5}{256\lambda'^{\,5/2}}$$
$$-(2k-5)(2k-3)(4k^2-1)(4k^2-8k-17)\frac{3\,\zeta_3^2}{64\lambda'^{\,3}}+O\left(\frac{1}{\lambda'^{\,7/2}}\right)$$

+ non-perturbative terms

$$\sqrt{\lambda'}=\sqrt{\lambda}-4\log 2$$

The 2-point functions

- Using these results, one can prove in full generality that

$$G_k^- = \mathcal{G}_k\,\frac{4\pi^2\,k\,(k-1)}{\lambda}\,\left(\frac{\lambda'}{\lambda}\right)^{k-1}\Big[1+(k-1)(2k-1)(2k-3)\,\frac{\zeta_3}{\lambda'^{3/2}}$$
$$-\,(k-1)(2k-3)(2k-5)(4k^2-1)\,\frac{9\,\zeta_5}{16\lambda'^{5/2}}$$
$$+\,(k-1)(2k-1)(2k-3)(2k-5)(4k^2-20k-3)\,\frac{\zeta_3^2}{4\lambda'^3}+O\Big(\frac{1}{\lambda'^{7/2}}\Big)\Big]$$

Leading Order term

Sub-leading corrections

(see also X. Zhang's poster)

\+ non-perturbative terms

weak coupling

strong coupling

$$0 \longrightarrow \infty$$

$$\lambda$$

$$G_k^-$$

$\lambda \to 0$

$\lambda \to \infty$

$$G_k^- \underset{\lambda\to 0}{\sim} \mathcal{G}_k + O(\lambda)$$

$$G_k^- \underset{\lambda\to\infty}{\sim} \mathcal{G}_k\,\frac{4\pi^2\,k\,(k-1)}{\lambda}+O(\lambda^{-\frac{3}{2}})$$

The 3-point functions and structure constants

- A similar analysis can be done for the 3-point functions.
- The key observation is that the 3-point functions are related to the 2-point functions by an exact Ward-like identity

$$G^{-}_{k,\ell,p} = \langle O^{+}_{k}\, O^{-}_{\ell}\, O^{-}_{p} \rangle$$

Billo, Frau, **A.L.** , Pini, Vallarino, 2022

$$\boxed{= \frac{1}{2\sqrt{N}}\sqrt{(k+\lambda\partial_\lambda)G^{+}_{k}}\,\sqrt{(\ell+\lambda\partial_\lambda)G^{-}_{\ell}}\,\sqrt{(p+\lambda\partial_\lambda)G^{-}_{p}}}$$

- Thus, knowing the 2-point functions, we know also the 3-point functions and the structure constants

$$C^{-}_{k,\ell,p} = \frac{G^{-}_{k,\ell,p}}{\sqrt{G^{+}_{k}\, G^{-}_{\ell}\, G^{-}_{p}}}$$

$$\boxed{= \frac{1}{\sqrt{2}\,N}\sqrt{k+\lambda\partial_\lambda\big(\log G^{+}_{k}\big)}\,\sqrt{\ell+\lambda\partial_\lambda\big(\log G^{-}_{\ell}\big)}\,\sqrt{p+\lambda\partial_\lambda\big(\log G^{-}_{p}\big)}}$$

Structure constants

weak coupling

strong coupling

$$0 \xrightarrow{\qquad\qquad \lambda \qquad\qquad} \infty$$

$$C^{-}_{k,\ell,p} = \frac{1}{\sqrt{2}\,N}\sqrt{k+\lambda\partial_\lambda\big(\log G^{+}_{k}\big)}\;\sqrt{\ell+\lambda\partial_\lambda\big(\log G^{-}_{\ell}\big)}\;\sqrt{p+\lambda\partial_\lambda\big(\log G^{-}_{p}\big)}$$

$\lambda \to 0$

$\lambda \to \infty$

$$C^{-}_{k,\ell,p} \underset{\lambda\to 0}{\sim} \frac{\sqrt{k\,\ell\,p}}{\sqrt{2}\,N}$$

$$C^{-}_{k,\ell,p} \underset{\lambda\to \infty}{\sim} \frac{\sqrt{k\,(\ell-1)\,(p-1)}}{\sqrt{2}\,N}$$

Easy to prove, as in $\mathcal{N}=4$ SYM

It should follow from the AdS/CFT correspondence. And indeed it does!!

Billo, Frau, **A.L.** , Pini, Vallarino, 2022

Holographic description: untwisted sector

- The untwisted operators $\mathcal{O}_k^+$ are dual to

Maldacena, 1997;
Lee, Minwalla, Rangamani, Seiberg, 1998; …

$s_k =$ K.K. modes of the metric and the R-R 4-form fluctuations (as in $\mathcal{N}=4$)

- Their effective action (derived from Type II B sugra in d=10) is

$$S = \frac{1}{2\kappa_{10}^2}\int_{\mathrm{AdS}_5} d^5z\,\sqrt{g}\,\Big[\sum_{k\geq 2} A_k\Big(\nabla_\mu s_k^*\,\nabla^\mu s_k + k(k-4)s_k^*\,s_k\Big) + \sum_{k,\ell,p}\Big(V_{k,\ell,p}\,s_k^*\,s_\ell^*\,s_p + \mathrm{c.c.}\Big)\Big]\,\frac{\pi^3}{2}$$

Coefficients computed by
Lee, Minwalla, Rangamani, Seiberg, 1998

Volume of $S^5/\mathbb{Z}_2$

- Note that

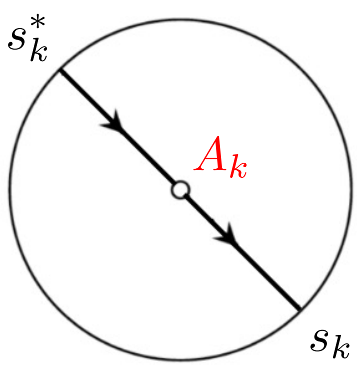
$$\frac{1}{2\kappa_{10}^2} = \frac{1}{(2\pi)^7 g_s^2\,\alpha'^{\,4}} = \cdots = \frac{4(2N)^2}{(2\pi)^5}\,\cancel{\frac{1}{R^8}}$$

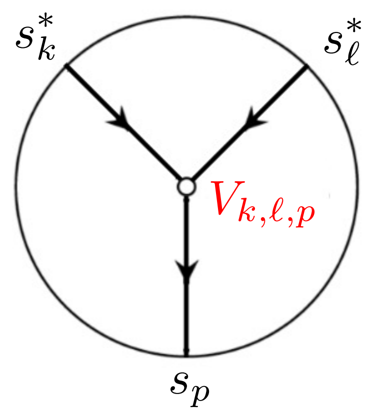
$8\pi N g_s = \lambda$ $\qquad$ $\alpha'\sqrt{\lambda} = R^2$

Holographic description: untwisted sector

From $$S = \frac{1}{2\kappa_{10}^2}\int_{\mathrm{AdS}_5} d^5z\,\sqrt{g}\,\Big[\sum_{k\geq 2} A_k\Big(\nabla_\mu s_k^*\,\nabla^\mu s_k + k(k-4)s_k^*\,s_k\Big) + \sum_{k,\ell,p}\Big(V_{k,\ell,p}\,s_k^*\,s_\ell^*\,s_p + \text{c.c.}\Big)\Big]\frac{\pi^3}{2}$$

using the holographic dictionary, one computes the 2- and 3- point functions from Witten diagrams





and finds

$$\boxed{C^+_{k,\ell,p} \underset{\lambda\to 0}{\sim} \frac{\sqrt{k\,\ell\,p}}{\sqrt{2}\,N} = C^+_{k,\ell,p}\Big|_{\lambda=0}}$$

(like in $\mathcal{N}=4$)

Holographic description: twisted sector

- The twisted operators $\mathcal{O}_k^-$ are dual to

Gukov, 1998;
Billo, Frau, Galvagno, **A.L.**, Pini, 2021.

$\eta_k =$ K.K. modes of the twisted scalars obtained by wrapping the NS-NS and R-R 2-forms on the exceptional 2-cycle:

$$\frac{1}{2\pi\alpha'}\int_e B_{(2)} \qquad \frac{1}{2\pi\alpha'}\int_e C_{(2)}$$

- Their effective action (derived from localizing Type II B sugra at the orbifold fixed point) is

$$S = \frac{(2\pi\alpha')^2}{4\kappa_{10}^2}\int_{\mathrm{AdS}_5} d^5z\,\sqrt{g}\,\Big[\sum_{k\geq 2}\frac{1}{2}\Big(\nabla_\mu \underbrace{\eta_k^*\,\nabla^\mu\eta_k + k(k-4)\eta_k^*}_{\text{Gukov, 1998}}\,\eta_k\Big) + \sum_{k,\ell,p}\Big(W_{k,\ell,p}\,s_k^*\,\eta_\ell^*\,\eta_p + \text{c.c.}\Big)\Big]\,\underbrace{2\pi}_{\text{Volume of } S^1}$$

where

$$W_{k,\ell,p} = -\frac{(k+\ell-p)(k+p-\ell)(k+l+p-2)(k+\ell+p-4)}{2^{\frac{k}{2}}(k+1)}$$

Billo, Frau, **A.L.**, Pini, Vallarino, 2022.

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- Note that

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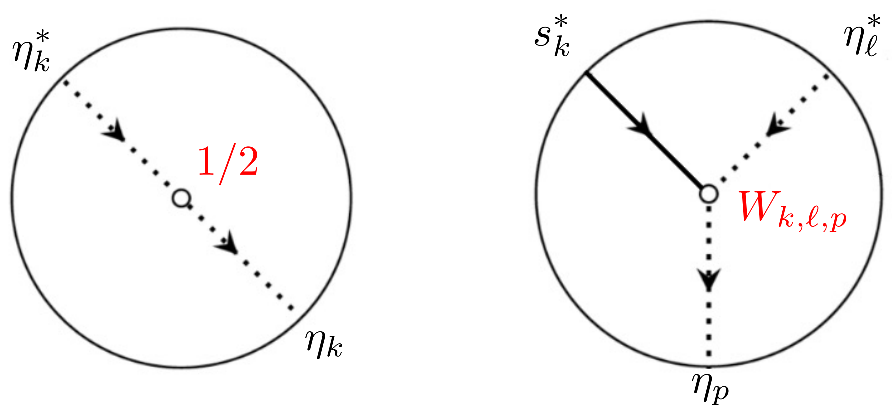
Holographic description: twisted sector

From
$$S = \frac{(2\pi\alpha')^2}{4\kappa_{10}^2}\int_{\mathrm{AdS}_5} d^5z\,\sqrt{g}\,\Big[\sum_{k\geq 2}\frac{1}{2}\Big(\nabla_\mu\eta_k^*\,\nabla^\mu\eta_k + k(k-4)\eta_k^*\,\eta_k\Big) + \sum_{k,\ell,p}\Big(W_{k,\ell,p}\,s_k^*\,\eta_\ell^*\,\eta_p + \text{c.c.}\Big)\Big]\,2\pi$$

using the holographic dictionary, one computes the 2- and 3- point functions from Witten diagrams



and finds

$$\boxed{C^-_{k,\ell,p} \underset{\lambda\to 0}{\sim} \frac{\sqrt{k\,(\ell-1)\,(p-1)}}{\sqrt{2}\,N}}$$

in perfect agreement with the localization results

Conclusions

- In this SCFT there exists a non-trivial relation between the 2- and 3-point functions of scalar operators, valid for all couplings in the planar limit

- The structure constants can be written in a very compact and exact way

$$\boxed{C^{-}_{k,\ell,p} = \frac{1}{\sqrt{2}N}\sqrt{k+\lambda\partial_\lambda \log G^{+}_{k}}\;\sqrt{\ell+\lambda\partial_\lambda \log G^{-}_{\ell}}\;\sqrt{p+\lambda\partial_\lambda \log G^{-}_{p}}}$$

- The 2-point functions (and hence the structure constants) are known for all couplings in terms of the matrix X (convolution of Bessel functions).

- At strong coupling

$$\boxed{C^{-}_{k,\ell,p} \underset{\lambda\to\infty}{\sim} \frac{\sqrt{k\,(\ell-1)\,(p-1)}}{\sqrt{2}\,N}}$$

in perfect agreement with the holographic results (Billò et al 2022).

This is the first explicit check of the AdS/CFT correspondence in a non-maximally supersymmetric set-up.

Conclusions

- In this SCFT there exists a non-trivial relation between the 2- and 3-point functions of scalar operators, valid for all couplings in the planar limit

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- The 2-point functions (and hence the structure constants) are known for all couplings in terms of the matrix X (convolution of Bessel functions).

- A systematic strong coupling expansion can be worked out

$$C^{-}_{k,\ell,p} = C^{-(\mathrm{LO})}_{k,\ell,p}\left(\frac{\lambda}{\lambda'}\right)^{\frac{1}{2}}\left[1 + c^{(1)}_{k,\ell,p}\frac{\zeta_3}{\lambda'^{3/2}} + c^{(2)}_{k,\ell,p}\frac{\zeta_5}{\lambda'^{5/2}} + c^{(3)}_{k,\ell,p}\frac{\zeta_3^2}{\lambda'^{3}} + \cdots\right]$$

- This expansion has the same form expected from closed string amplitudes and, since at the moment explicit calculations beyond the supergravity limit do not seem to be possible, this analysis provides a very strong prediction for the string corrections.

Conclusions

- The X matrix we have used for the structure constants in the $\mathcal{N}=2$ quiver theory

$$\mathsf{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}}\sqrt{k\,\ell}\int_0^\infty \frac{dt}{t}\,\frac{e^t}{(e^t-1)^2}\,J_k\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)\,J_\ell\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

is very similar to the X matrix that is used in the study of the cusp anomalous dimension in $\mathcal{N}=4$ SYM via the BES equation:

$$\mathsf{X}^{\mathrm{BES}}_{k,\ell} = -4(-1)^{\frac{k+\ell+2k\ell}{2}}\sqrt{k\,\ell}\int_0^\infty \frac{dt}{t}\,\frac{1}{e^t-1}\,J_k\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)\,J_\ell\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

- Like for the cusp anomalous dimensions, also for the structure constants one should find in a systematic way the non-perturbative exponentially small corrections.

- It seems that the formalism of the X matrix is much more general than it seems at first sight and it would be nice to uncover a general pattern.

Thanks a lot for listening!