Virasoro blocks and the reparametrization formalism

based on 2101.00880 and 2212.02527 + (2108.01095)

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Gijon

Brief history of the topic

Virasoro (identity) blocks

$$\langle V(1)V(2)W(3)W(4)\rangle = \frac{1}{(z_{12})^{2h_V}(z_{34})^{2h_W}} \sum_O C_{VVO}\, C_{WWO}\, \mathcal{V}_{h_O}(u)$$

- Recursion equations [Zamolodchikov 80s] + series repres. [Perlmutter '15]

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  [Action on coadjoint orbits of the Virasoro group [Alekseev-Shatashvili '89]]

  [Boundary action for AdS_3 gravity [Verlinde '90, Cotler-Jensen '18, KN '21b]]
- Derivation of this effective theory from first principles [KN '21a '22]

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Effective theory of Virasoro identity blocks

The bilocal vertex operator

A basic ingredient is the bilocal vertex operator

$$V(z_1)V(z_2) \quad \Longrightarrow \quad \mathcal{B}_{h_V}(1,2) \equiv \left(\frac{\partial f(z_1,\bar{z}_1)\, \partial f(z_2,\bar{z}_2)}{\left(f(z_1,\bar{z}_1) - f(z_2,\bar{z}_2)\right)^2} \right)^{h_V} \, ,$$

expanded in terms of the 'reparametrization mode' $\epsilon(z,\bar{z})$,

$$f = e^{\epsilon\, \partial_z} z = z + \epsilon + \frac{1}{2}\epsilon\partial\epsilon + O(\epsilon^3)\, .$$

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Objective

We want to understand the origin and meaning of $\mathcal{B}_h(1,2)$ in CFT$_2$.

Rules of the game

1. Dynamics of ϵ is modeled over that of the stress tensor

$$\langle \partial^3 \epsilon(z_1) \ldots \partial^3 \epsilon(z_n) \rangle = \left(\frac{12}{c} \right)^n \langle T(z_1) \ldots T(z_n) \rangle \, .$$

[Hence $\bar{\partial}\epsilon$ is the stress tensor shadow [Haehl-Reeves-Rozali '19]]

[The two-point function can be obtained from AS action [Cotler-Jensen '18]]

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2. Main conjecture:

$$\mathcal{V}_0(u) \simeq \langle \mathcal{B}_{h_V}(1,2)\mathcal{B}_{h_W}(3,4)\rangle_c\,.$$

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Large c formalism

expansion in ϵ $\Leftrightarrow$ expansion in $1/c$

Objective

Derive these rules!

Baby example: Virasoro identity block at $O(1/c)$

According to the above set of rules, we have

$$\mathcal{V}_0(u)\big|_{O(1/c)} = \langle \mathcal{B}^{(1)}_{h_V}(1,2)\mathcal{B}^{(1)}_{h_W}(3,4)\rangle = \frac{2h_V h_W}{c}\, u^2 \; {}_2F_1\left(2,2,4;u\right) \, ,$$

where

$$\mathcal{B}^{(1)}_h(1,2) = h\left(\partial\epsilon_1 + \partial\epsilon_2 - 2\frac{\epsilon_1-\epsilon_2}{z_{12}}\right) \, .$$

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Other examples: light-light exponentiation $[h_{V,W} = O(\sqrt{c})]$, heavy-light limit $[h_V = O(1), h_W = O(c)]$, six-point blocks in various regimes
[Cotler-Jensen '18, Haehl-Reeves-Rozali '19, Anous-Haehl '20]

Generating functionals for stress tensor insertions

Generating functionals

Let's consider a string of m primary operators,

$$\{O\} \equiv O_1(z_1)...O_m(z_m)\,.$$

We are interested in the generating functional for arbitrary number of additional stress tensor insertions,

$$Z\left[\mu\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^2w_1 ... \, d^2w_n \, \mu(w_1)...\mu(w_n) \langle \{O\} \, T(w_1)...T(w_n) \rangle \,,$$

where the source μ is naturally associated with a deformation of the conformal geometry,

$$ds^2 = dz\, d\bar{z} + \mu(z,\bar{z})\, d\bar{z}^2\,.$$

Stress tensor insertions are governed by the conf. Ward identity [BPZ '84], which turns into the diff. Ward identity [Verlinde '90]

$$\left(\bar{\partial} - \mu\partial - 2\partial\mu\right) \frac{\delta Z[\mu]}{\delta\mu(z)} = [-c\,\partial^3\mu + \sum_{j=1}^{m} \left(h_j \partial\delta(z - z_j) - \delta(z - z_j)\partial_{z_j}\right)] Z[\mu]$$

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To solve it we introduce a Beltrami param. of the conformal deformations,

$$\mu = \frac{\bar{\partial} f}{\partial f}\,, \qquad (f \text{ quasi-conf. mapping})$$

such that

$$\partial f(z,\bar{z})\,\frac{\delta Z_0[\mu]}{\delta f(z,\bar{z})} = c\,\partial^3\mu(z,\bar{z})\,Z_0[\mu]\,,$$

$$\partial f(z,\bar{z})\,\frac{\delta Z_c[\mu]}{\delta f(z,\bar{z})} = -\sum_{i=1}^{m} \left(h_i\,\partial_z\delta(z - z_i) - \delta(z - z_i)\partial_{z_i}\right) Z_c[\mu]\,,$$

where Z_0, Z_c generate $\langle T ... T\rangle$ and $\langle \{O\} T ... T\rangle_c$, respectively.

Solutions of diff. Ward identity

1. No operator insertion $\{O\} = \emptyset$:

$$Z_0\,[\mu] = \exp\left[-\frac{c}{24\pi}\int d^2z\,\frac{\bar\partial f}{\partial f}\,\partial^2\ln\partial f\right] \equiv e^{-W_0[\mu]}$$

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2. Two identical operators $\{O\} = O(z_1)O(z_2)$:

$$Z_{2,c}[\mu] = \left(\frac{\partial f(z_1,\bar{z}_1)\,\partial f(z_2,\bar{z}_2)}{\left(f(z_1,\bar{z}_1) - f(z_2,\bar{z}_2)\right)^2}\right)^h \equiv \mathcal{B}_h(1,2)$$

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3. Three operators $\{O\} = O_1(z_1)O_2(z_2)O_3(z_3)$:

$$Z_{3,c}[\mu] = \frac{(\partial f_1)^{h_1}(\partial f_2)^{h_2}(\partial f_3)^{h_3}}{(f_1 - f_2)^{h_1+h_2-h_3}(f_1 - f_3)^{h_1+h_3-h_2}(f_2 - f_3)^{h_2+h_3-h_1}}$$

Solution to the Beltrami equation

We need the explicit relation between μ and f!

For a small conformal deformation of the form

$$\mu = \bar{\partial}\epsilon(z, \bar{z}) \, ,$$

the solution to the Beltrami equation takes the form [Donaldson's book]

$$f(z, \bar{z}) = z + \epsilon + \bar{\partial}^{-1} \left(\bar{\partial}\epsilon \, \partial\epsilon \right) + O(\epsilon^3) \, .$$

We have

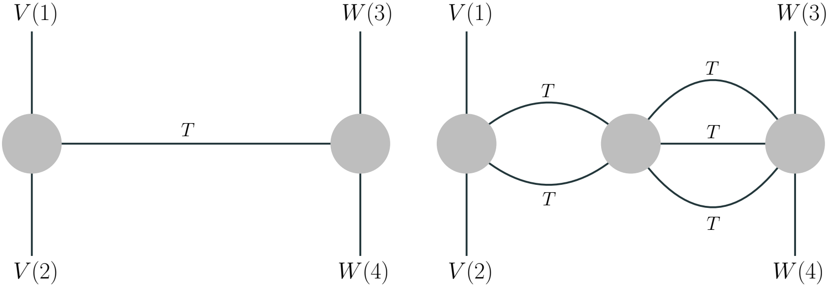
$$\langle \{O\} \, \bar{\partial}T(z_1) \ldots \bar{\partial}T(z_n) \rangle = \frac{\delta^n Z}{\delta\epsilon(z_1) \ldots \delta\epsilon(z_n)} \Big|_{\epsilon=0} \, .$$

Derivation of the effective theory

Position-space Feynman diagrams

Claim

The reparametrization formalism is equivalent to an expansion in terms of Feynman diagrams involving stress tensor exchanges



Exact vertices

The vertices are *partially amputated* $(2+n)$-point correlation functions

$$\langle V(1)V(2)\hat{T}(z_1)\ldots\hat{T}(z_n)\rangle$$

where $\hat{T}$ is the shadow of the stress tensor,

$$\hat{T}(z) \equiv -\frac{12}{\pi c}\,\partial^{-3}\bar{\partial}T(z)\,.$$

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This formula follows from

$$\langle\dots T(w_i)\dots\rangle = \int d^2y\,\langle\dots\hat{T}(y)\dots\rangle\langle T(y)T(w_i)\rangle$$

together with

$$\langle T(z)T(w)\rangle = \frac{c}{2(z-w)^4} = -\frac{c}{12}\,\partial_w^3\left(\frac{1}{z-w}\right) = -\frac{\pi c}{6}\,\partial_w^3\,(\partial_{\bar{w}})^{-1}\delta(z-w)\,.$$

Sketch of the proof

Thus the set of *connected* Feynman diagrams is given by

$$\begin{aligned}
\mathcal{V}_0 &= \sum_{n,m} \frac{1}{n!m!} \int \langle VV\hat{T}(z_1)...\hat{T}(z_n)\rangle_c \langle T(z_1)...T(w_m)\rangle \langle \hat{T}(w_1)...WW\rangle_c \\
&= \sum_{n,m} \frac{1}{n!m!} \int \langle VV\bar{\partial}T(z_1)\ldots\rangle_c \langle \epsilon(z_1)\ldots\epsilon(w_m)\rangle \langle \ldots\bar{\partial}T(w_m)WW\rangle_c \\
&= \sum_{n,m} \frac{1}{n!m!} \int \delta^{(n)}\mathcal{B}_{h_V}(1,2)\,\langle \epsilon(z_1)\ldots\epsilon(w_m)\rangle\,\delta^{(m)}\mathcal{B}_{h_W}(3,4) \\
&= \langle \mathcal{B}_{h_V}(1,2)\mathcal{B}_{h_W}(3,4)\rangle_c\,,
\end{aligned}$$

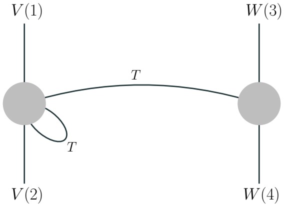
with the short-hand notation

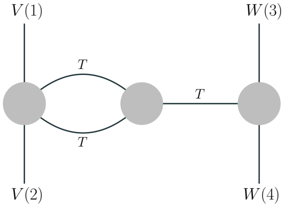
$$\delta^{(n)}\mathcal{B}_{h_V}(1,2) \equiv \left.\frac{\delta^n \mathcal{B}_{h_V}(1,2)}{\delta\epsilon(z_1)\ldots\delta\epsilon(z_n)}\right|_{\epsilon=0}.$$

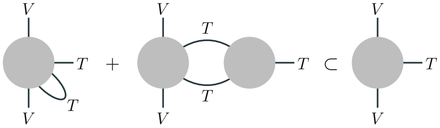
Open questions

UV divergences

- In the reparametrization formalism, we get coincident point divergences from $\langle \epsilon(z)\epsilon(z)... \rangle_c$ correlators
- Maps to UV divergences from loop diagrams
- Feature of EFT approach

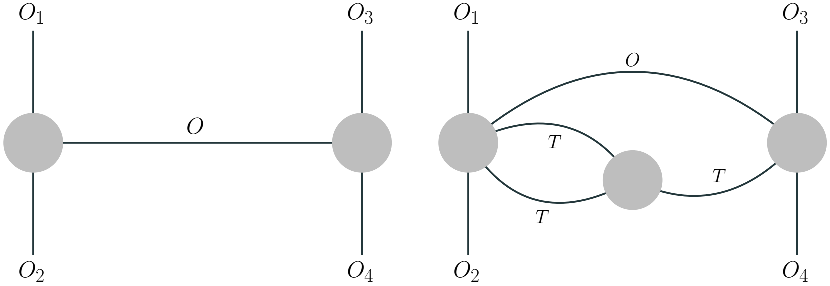






Generic Virasoro blocks

Exchange diagrams of a primary operator O and its descendants :



Proposed formula

$$\mathcal{F}_O = \int d^2y \, \langle Z_{3,c}(1,2,y) \, \mathsf{Sh}_y \, Z_{3,c}(y,3,4) \rangle_c$$

Relation to AdS_3 gravity

- Reparametrization formalism motivated by holography and maximal chaos [Cotler-Jensen '18, Haehl-Rozali '18]
- $Z_0[\mu]$ is the (sourced) vacuum onshell action [KN '21b]
- What is the gravitational realization of $Z_{2,c}[\mu] = \mathcal{B}_h(1,2)$?

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- What is the gravitational realization of $Z_{2,c}[\mu] = \mathcal{B}_h(1,2)$?

A safe conjecture

$\mathcal{B}_h(1,2)$ is a (sourced) gravitational Wilson line

Indeed, in a state with

$$\langle T \rangle = \frac{c}{12} S[f(z); z] \,,$$

the $SL(2,\mathbb{C})$ gravitational Wilson line becomes [D'Hoker-Kraus '19, ...]

$$\mathcal{W}_h(1,2)\big|_{T(f)} = \left(\frac{f'(z_1) f'(z_2)}{(f(z_1) - f(z_2))^2} \right)^h \quad \mathcal{W}_h(1,2) \equiv \langle h | P \left(\exp \int_{z_1}^{z_2} A \right) | h \rangle$$