Lorentzian hyperthreads and nonlocal computation in holography

Juan F. Pedraza

IFT UAM/CSIC
Madrid, Spain

Work with: E. Cáceres (UT, Austin) & R. Carrasco (IFT, Madrid)

Main message

- It has been argued that holographic CFTs implement nonlocal computation protocols (first discussed in the field of quantum cryptography) [May; Cree & May; May, Sorce & Yoshida]
- We ask the extent to which current holographic proposals for computational complexity support this claim
- Although a microscopic interpretation of holographic complexity proposals is lacking, we focus on the recent reformulation of CV in terms of 'Lorentzian threads', which has an intuitive physical picture

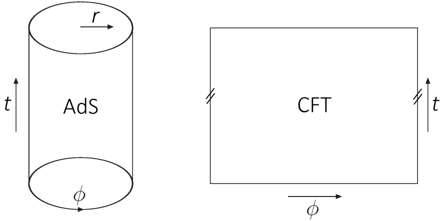
Upshot:

- The original prescription does not support this claim, but it is incomplete: it is insufficient to analyze the complexity of multipartite systems
- We correct the prescription and show it necessarily implies the existence of non-local operations (to account for holographic inequalities)

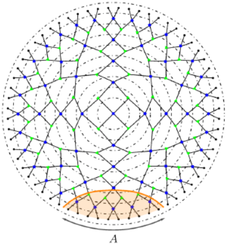
Table of Contents

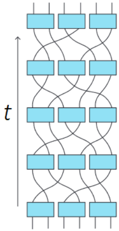
Motivation: holography and non-local computation

- Holography has uncovered deep connections between gravity and QI



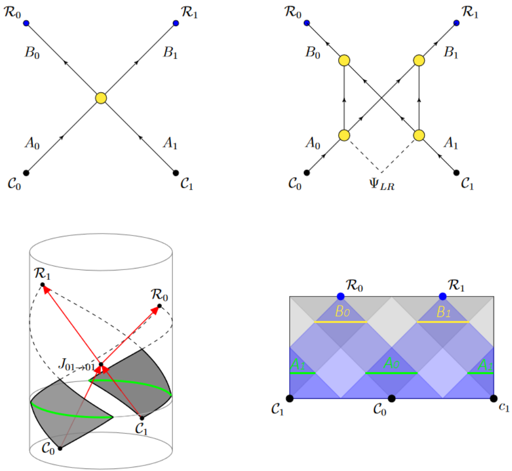
- Emergence of space is understood from entanglement. More importantly, QI concepts constrain holographic states and their time evolution





Motivation: holography and non-local computation

- Local tasks in the bulk are implemented nonlocally in the boundary [May]



- **Q:** Do current proposals for holographic complexity support this claim?

Holographic complexity: quick overview

- Entanglement not enough to describe late-time physics of BHs [Susskind]
- Computational complexity? Min number of gates to prepare a state:

$$|\psi_T\rangle = g_n \dots g_2\, g_1 |\psi_R\rangle = U_{TR}|\psi_R\rangle$$

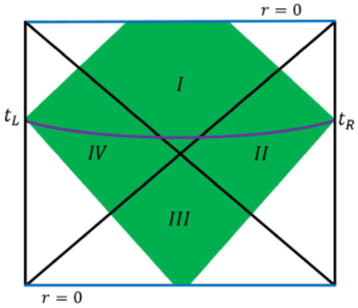
Two main proposals:

- CV [Susskind,...]:

$$C_V = \frac{1}{G_N \ell} \max \mathrm{Vol}(\Sigma)$$

- CA [Brown et al,...]:

$$C_A = \frac{I_{WDW}}{\pi\hbar}$$



- And generalizations: CV2.0 [Couch et al], 'complexity=anything´ [Belin et al]...

Holographic complexity: quick overview II

These proposals share qualitative properties:

- Capture late time growth of wormhole
- Lloyd's bound
- Switchback effect

However, physical interpretation remains largely unexplored:

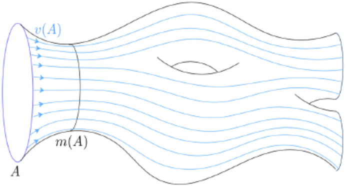
- Role of reference state?
- Set of gates?
- Tolerance ϵ?
- First principle derivation? / Match with quantum info definition?

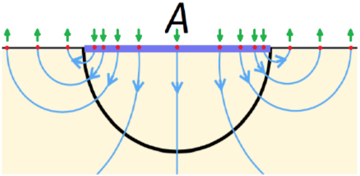
Some progress was recently achieved for CV, by recasting the proposal in terms of 'Lorentzian threads'[JP, Russo, Svesko, Weller-Davies]

Entanglement and Riemannian 'bit threads'

The RT prescription [Ryu & Takayanagi] can be reformulated in terms of flows v^μ [Freedman & Headrick]

$$S(A) = \min_{m \sim A} \frac{\mathrm{area}(m(A))}{4G_N} = \max_{v \in \mathcal{F}} \int_A v\,, \quad \mathcal{F} = \left\{ v \,|\, \nabla \cdot v = 0,\ |v| \leq \frac{1}{G_N \ell} \right\}$$

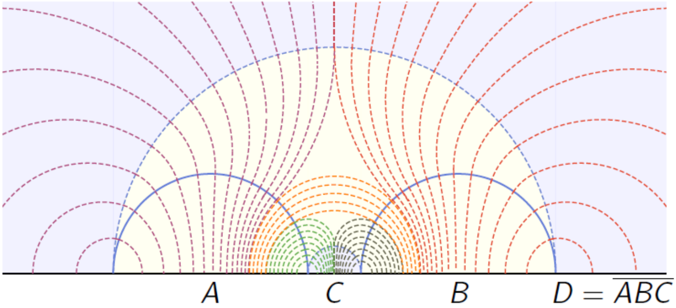




which can be derived from the min cut-max flow theorem. More rigorously, one can use Lagrange duality and convex optimization [Headrick & Hubeny]

$$L[v, \psi, \phi] = \int_A v + \int_\Sigma [-\psi(\nabla \cdot v) + \phi(1 - |v|)]$$

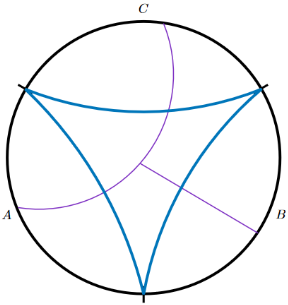
Solution to conceptual puzzles



- Insightful to think as a set of threads of finite thickness ($4G_N$)
- Entropy $\sim$ area: distillation of EPR pairs
- Properties of entropy are aligned with their QI meanings
- Threads and/or v^μ can be continuous across phase transitions

Generalization to k-threads or ‘hyperthreads´

- Bit threads lead to the ‘mostly bipartite’ property, conjectured by [Cui et al]
- However, it was subsequently shown that holographic entropy inequalities for multiple regions are inconsistent with this conjecture [Akers & Rath]
- Bit threads CAN be generalized to account for these multipartite contributions. The generalized program includes ‘hyperthreads´ [Harper]



Generalization to k-threads or 'hyperthreads´ II

- Concept of flow is no longer useful but we can reformulate the program in terms of measure theory
- Mathematically: define a measure function μ that maps an element p of $\mathcal{P}$ (the set of spacelike curves) into the reals $\mu : \mathcal{P} \to \{0, 1\}$
- Next, define the delta function $\Delta(x, p)$

$$\Delta(x, p) = \int_p ds \delta(x - y(s)),$$

  where $y(s)$ parametrize the curve of the thread p

- With these ingredients one can reformulate the program as

$$S(A) = \max \int_{\mathcal{P}} d\mu, \text{ s.t. } \rho(x) \leq 1 \quad \forall x \in \Sigma.$$

  where $\rho(x) = \int_{\mathcal{P}} d\mu \Delta(x, p)$ is the local density of threads

Generalization to k-threads or 'hyperthreads´ III

- We now introduce a Lagrange multiplier λ,

$$L(\mu,\lambda) = \int_{\mathcal{P}} d\mu + \int_{\Sigma} \lambda(x)\,(1-\rho(x))$$
$$= \int_{\Sigma} \lambda(x) + \int_{\mathcal{P}} d\mu \left(1 - \int_{p} ds\lambda(x)\right)$$

- Finally, it is easy to include hyperthreads in this language:

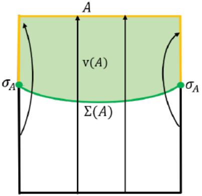
$$L = \sum_{i=2}^{N} k \int_{H_k} d\mu + \int_{\Sigma} \lambda(x)(1-\rho(x))$$
$$= \int_{\Sigma} \lambda + \sum_{k=2} \int_{H_k} \left(k - \int_{h} ds\lambda\right)$$

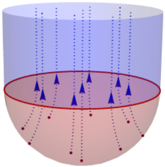
- The multicommodity theorem, implies that there is a solution to this program that computes simultaneously the entropy of multiple regions

Lorentzian threads and CV

The Lorentzian MinFlow-MaxCut theorem [Headrick & Hubeny] can be used to reformulate CV as [JP,Svesko,Russo,Weller-Davies]

$$C(\sigma_A) = \max_{\Sigma\sim A} \frac{\mathrm{Vol}(\Sigma(A))}{G_N \ell} = \min_{v\in\mathcal{F}} \int_A v \quad \mathcal{F} = \left\{ v \,|\, v^0 > 0,\ \nabla\cdot v = 0,\ |v| \geq \frac{1}{G_N \ell} \right\}$$





- Requires compact manifold: tightly connected with state preparation

$$|\psi_T\rangle = T e^{-\int_{\tau>0} d\tau d\vec{x} \sum_\alpha \lambda_\alpha \mathcal{O}_\alpha} |\psi_R\rangle = U_{TR} |\psi_R\rangle$$

- Makes evident the role of the reference state
- Threads interpreted as gates. Discrete version counts minimal # of gates
- Tolerance set by $1/G_N\ell$

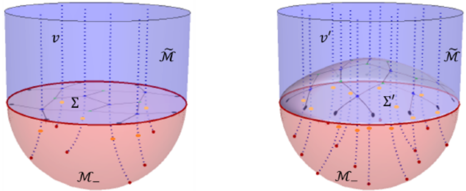
Physical interpretation

CV duality inspired by tensor networks. Tensors occupy finite volume

$$C \sim \#\ \text{tensors in the network}$$

Threads or 'gatelines' cut through physical tensors so that:

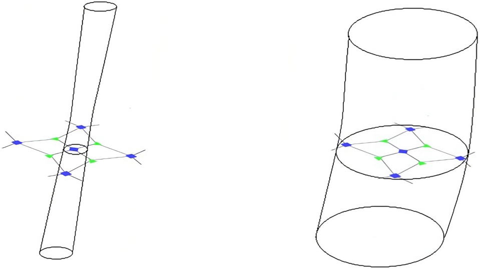
$$C \sim \#\ \text{gatelines} \sim \#\ \text{tensors in the network}$$



- Unitaries (gates) attached to each thread
- Suggests 'mostly commuting' gates

A refinement on the physical interpretation

- The 'mostly commuting' gates hypothesis seems problematic. A universal set of gates must include k-local gates in order to generate entanglement
- The above interpretation assumes $\ell \sim \ell_P$ so that the cross-section is of Planckian size. However, we can relax this condition

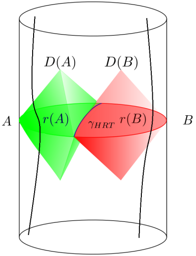


...such that $k \sim \ell/\ell_P$ (each gate acts simultaneously on k d.o.f.)

- This, however, does NOT allow for spatial non-locality

Shortcomings: multipartite systems & subregion complexity

- Let us consider a bipartite system:



- No threads connecting the two subsystems (as they are spacelike separated)
- Crucially, this implies the multicommodity theorem does NOT hold in the Lorentzian setting
- This would naively imply mutual complexity is always zero (superadditivity states $MC \leq 0$ [Agón, Headrick & Swingle]):

$$MC(A, B) = C(A) + C(B) - C(A \cup B) = N_A + N_B - (N_A + N_B) = 0$$

Multiple threads flavors

- We introduce two flavors of threads:

$$
\begin{aligned}
L = & \int_{\mathcal{P}_A} d\mu_A + \int_{\mathcal{P}_B} d\mu_B + \frac{1}{2}\int_{\mathcal{P}} (d\mu_A + d\mu_B) \\
& + \int_{\mathcal{M}} d^d x \left[\lambda_A \left(1 - \rho_A(x)\right) + \lambda_B \left(1 - \rho_B(x)\right)\right] \\
= & \int_{\mathcal{M}} d^d x \lambda_A + \int_{\mathcal{P}_A} d\mu_A \left[3/2 - \int_p ds \lambda_A\right] \\
& + \int_{\mathcal{P}_B} d\mu_A \left[1/2 - \int_p ds \lambda_A\right] + (A \leftrightarrow B).
\end{aligned}
$$

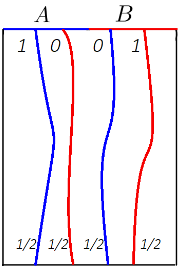
- Factors of $1/2$ $(3/2)$ appear because we want threads type A to contribute to $C(A)$ and $C(A \cup B)$ and likewise for threads type B.
- It can be shown that the dual program indeed computes simultaneously the 3 volumes, associated with all the relevant complexities

Multiple threads flavors

- A solution to this program will spit four numbers N_X^Y, the number of threads of type X that cross $D(Y)$
- In terms of these 4 numbers we have $C(A) = N_A^A$, $C(B) = N_B^B$, $C(A \cup B) = \frac{1}{2}(N_A^A + N_A^B + N_B^A + N_B^B)$, so:

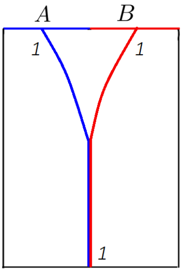
$$MC(A,B) = \frac{1}{2}(N_A^A + N_A^B - N_B^A - N_B^B) \leq 0$$

- This yields 4 fundamental 'gates', however, the crossed ones are difficult to interpret on their own:



Lorentzian hyperthreads are part of the program

- To understand the role of 'crossed' threads, we first notice that standard hyperthreads ARE part of this program:



- Mathematically:

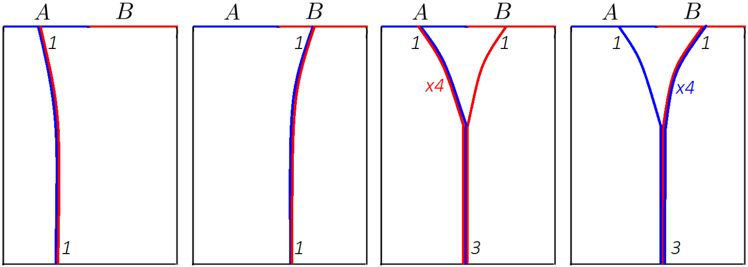
$$\min \int_{H_A} (d\mu_A + d\mu_{AB}) + \int_{H_B} (d\mu_B + d\mu_{AB}) + \frac{1}{2} \int_{H} (d\mu_A + d\mu_B + 2d\mu_{AB})$$

$$\text{s.t. } \rho_A + \rho_{AB} \geq 1, \ \rho_B + \rho_{AB} \geq 1 \quad \forall x \in \mathcal{M}.$$

- Redefining $\tilde{\mu}_A = \mu_A + \mu_{AB}$ and $\tilde{\mu}_B = \mu_B + \mu_{AB}$, one can show we recover the original 2-flavor program

Change of basis & generalized hyperthreads

- This motivates us to combine the 4 'elementary gates´ and define a new physical basis
- For the new basis, we require some conditions: must be invertible and must yield $N_i \geq 0$ for $i = 1, 2, 3, 4$. This rules out standard Lorentzian hyerthreads, because they contribute positively to the mutual complexity.
- One possible basis (iff $C(A \cup B) \leq \min\{2C(A) + C(B), C(A) + 2C(B)\}$):



Summary and outlook

- Lorentzian flows are useful to compute the complexity of the full state, but fail to capture subregion complexities (because there is no Lorentzian analog of the multicomodity theorem)
- We can overcome this problem by introducing multiple flavors of Lorentzian threads (one per subregion). These new objects appear instead of spacelike threads connecting the various subregions
- Upon a change of basis, we observe that Lorentzian hyperthreads (and generalized hyperthreads) ARE included in this new program
- Thus, CV (and derived inequalities) imply the boundary theory implements nonlocal tasks, through shared entanglement
- We are currently investigating nonlocal constraints on the time evolution of holographic states (e.g. black hole collapse)
- Riemanian threads with multiple flavors are also under investigation (they may help prove higher-order entropy inequalities)