Non-invertible Symmetry

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Symmetry

- Symmetry has proven, from time and again, to be of fundamental importance for describing Nature. It leads to conservation laws and selection rules.
- In recent years, there has been a revolution in our understanding of symmetry. In particular, the notion of global symmetry has been generalized in different directions.
- Surprisingly, symmetry might **not** be invertible. What's done cannot be undone.
- These non-invertible symmetries appear in many familiar quantum systems, such as the Ising model, QED, QCD, axions.
- I will review some of these developments especially from the high energy physics community. I apologize in advance for the variety of fascinating papers, especially those in CMT and math, that are not discussed below.

Wigner's theorem

- Wigner [1931] proved that every symmetry in quantum mechanics is implemented by a unitary operator (or an anti-unitary operator).

- Unitary operator U has an inverse, i.e., $UU^{-1} = 1$.

- Ordinary symmetries are in particular invertible. A 90 degree rotation can be undone by a -90 degree rotation.

Symmetry in QFT

- For quantum field theory and lattice models in general spacetime dimensions, symmetry is subject to more constraints and has more structure.
- Locality is an important property of symmetry.
- A symmetry serves as two purposes:
  - It can be an operator that acts on the Hilbert space.
  - It can be a defect that changes the boundary condition in space.
- The consistency between these two pictures give highly nontrivial constraints on symmetry in QFT.

Noether current

$$\mu = t, x, y, z$$
$$i = x, y, z$$

- Consider a conserved Noether current

$$\partial^\mu j_\mu = -\partial_t j_t + \partial_i j_i = 0$$

- The charge is defined as

$$Q = \int d^3x \; j_t$$

- Thanks to the conservation equation, it is conserved

$$\partial_t Q = \int d^3x \; \partial_t j_t = \int d^3x \; \partial_i j_i = 0$$

- The $U(1)$ unitary symmetry operator (the exponentiated charge) is

$$U_\vartheta = \exp(i\vartheta Q) = \exp(i\vartheta \int d^3x \; j_t) \quad , \qquad \partial_t U_\vartheta = 0$$

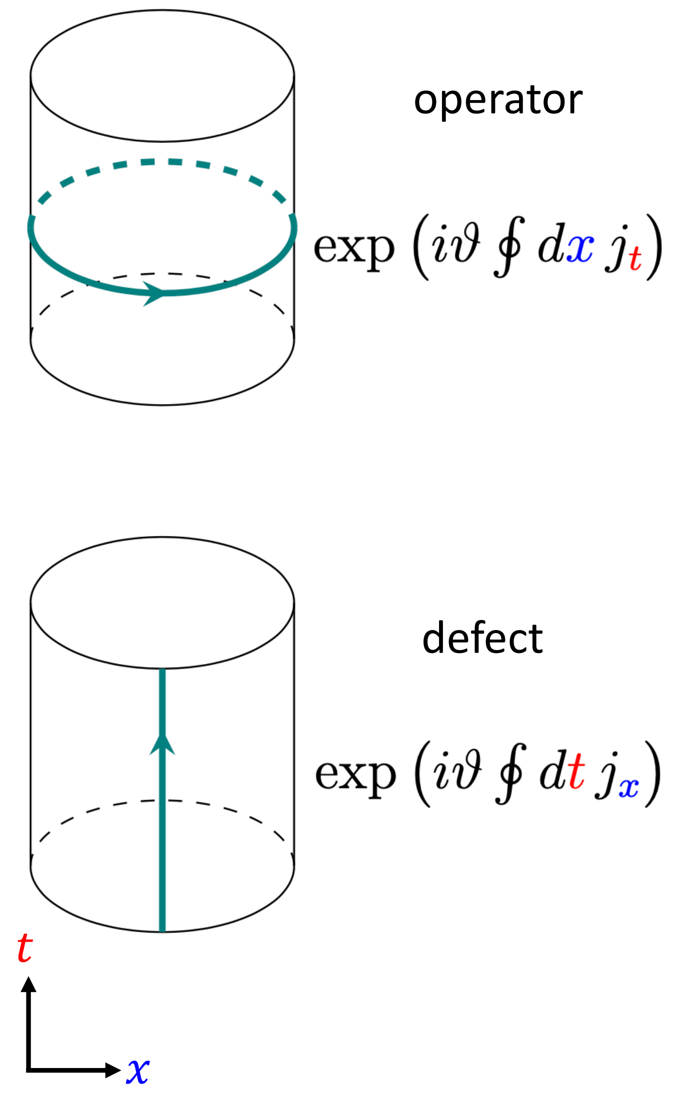
Symmetry and topology

- For relativistic systems in Euclidean signature, the time direction is on the same footing as any other spatial direction [Einstein 1905].
- We can therefore integrate the current on a general closed 3-manifold $M^{(3)}$ in 4-dimensional Euclidean spacetime [see Mark's talk]:

$$U_\vartheta = \exp(i\vartheta \int d^3x \, j_t)$$

$$\downarrow$$

$$U_\vartheta(M^{(3)}) = \exp(i\vartheta \oint_{M^{(3)}} j_\mu dn^\mu)$$



Conservation and topology

- The conservation equation $\partial_t U_\vartheta = 0$ is now **upgraded** to the fact that $U_\vartheta(M^{(3)})$ depends on $M^{(3)}$ only topologically because $\partial_\mu j^\mu = 0$ (divergence theorem).

Conserved → *Topological*

- The existence of the object $U_\vartheta\left(M^{(3)}\right)$, which can be an operator or a defect depending on how we choose $M^{(3)}$, is the manifestation of the locality of a global symmetry.
- $\frac{1}{7} U_\vartheta\left(M^{(3)}\right)$ or $U_\vartheta\left(M^{(3)}\right) - U_\vartheta{}'\left(M^{(3)}\right)$ are generally not valid topological defects because there are no twisted Hilbert spaces associated with them.

Topological operators

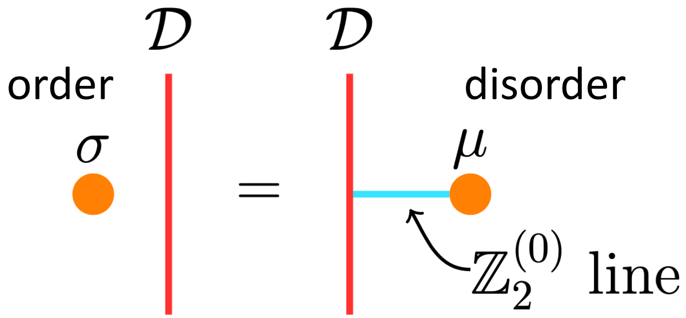
- Ordinary symmetries are implemented by codimension-1 topological operators, which are in particular conserved.
- Are all topological operators associated with an ordinary symmetry?
- **No**!
- Already in 1+1d, there are topological line operators that do not abide the group multiplication law:

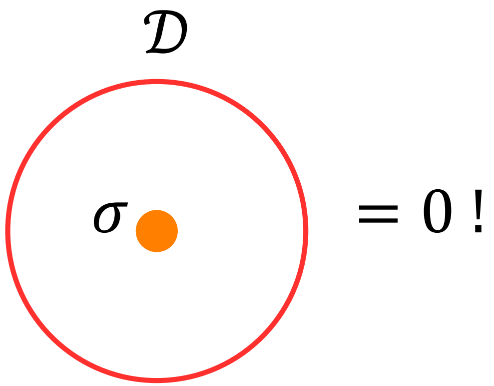
$$L_a \times L_b = \sum_c N_{ab}^c L_c$$

← More than one term on RHS

Ising CFT

- The simplest example of a non-invertible topological line operator is the Kramers-Wannier duality defect $\mathcal{D}$ in 1+1d Ising CFT [Frohlich-Fuchs-Runkel-Schweigert 2004].
- One way to understand it is to start from the Majorana fermion CFT, a spin QFT with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ global symmetry:
$$(-1)^{F_L}: \psi_L \to -\psi_L\,, \qquad (-1)^{F_R}: \psi_R \to -\psi_R$$
- Gauging $(-1)^F = (-1)^{F_L+F_R}$ of the Majorana gives the Ising CFT, i.e., bosonization. The quantum symmetry [Vafa 1989] of $(-1)^F$ is the $\mathbb{Z}_2$ symmetry of the Ising CFT.
- $(-1)^{F_L}$ turns into a non-invertible symmetry $\mathcal{D}$ [Thorngren 2018, Ji-SHS-Wen 2019]. Indeed, $\mathcal{D}$ flips the sign of $\varepsilon = \psi_L \psi_R$.
- However, it maps the local, order operator σ into the non-local, disorder operator μ.
- When you encircle σ by $\mathcal{D}$, it gives 0! Non-invertible symmetry.

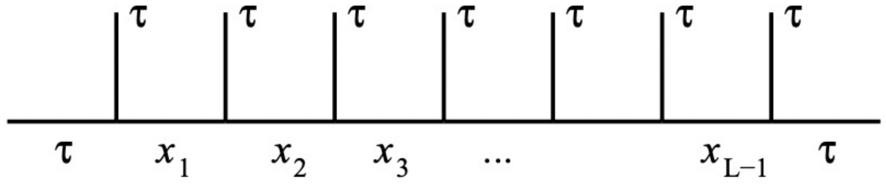




Non-invertible topological lines in 1+1d

- Non-invertible topological lines are everywhere in 1+1d:

1. They have a long history in RCFT -- Verlinde lines [Verlinde 1988, Moore-Seiberg 1988-1989, Petkova-Zuber 2000, (Frohlich)-Fuchs-Runkel-Schweigert 2002-2006,…].
2. Wilson lines $Rep(G)$ from gauging a non-abelian global symmetry G [..., Bhardwaj-Tachikawa 2017,…].
3. From anomalous global symmetries in a fermionic theory after bosonization [Thorngren 2018, Ji-SHS-Wen 2019, Lin-SHS 2019].
4. Lattice realization in condensed matter system: golden chain [Feiguin-Trebst-Ludwig-Troyer-Kitaev-Wang-Freedman 2006]. See [Aasen-Fendley-Mong 2016, 2020] for statistical models with noninvertible lines.



Why are they "symmetries"?

Why should we think of the non-invertible topological operators as generalized global symmetries?

- They lead to conservation laws and selection rules [..., Lin-Okada-Seifnashri-Tachikawa 2022,...].
- Some non-invertible symmetries can be gauged [Brunner-Carqueville-Plencner 2014].
- They can have generalized anomalies, which lead to generalized 't Hooft anomaly matching conditions. New constraints on renormalization group flows [Chang-Lin-SHS-Wang-Yin 2018, Komargodski et al. 2020].

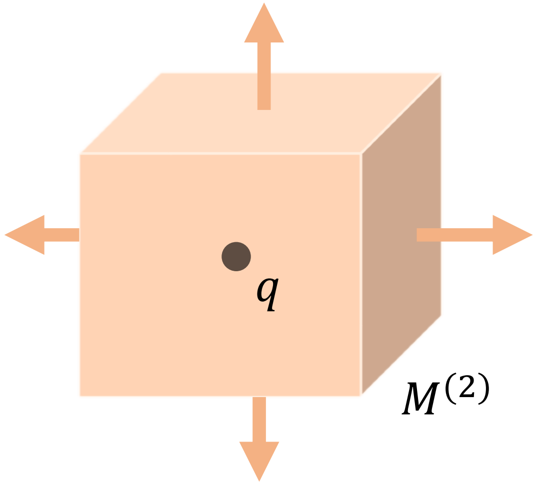
Higher-form global symmetries

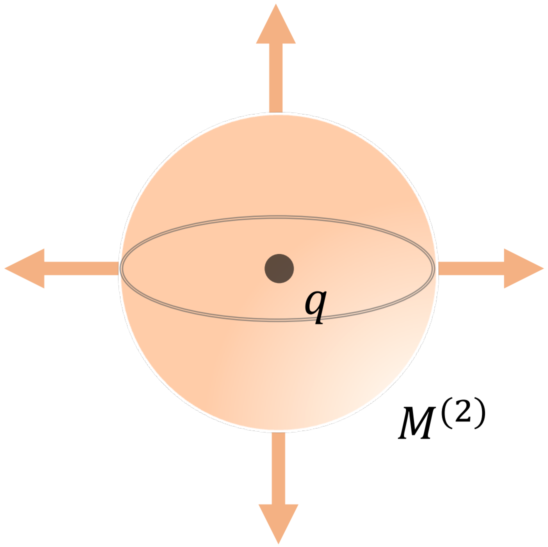
- While non-invertible (0-form) symmetries are ubiquitous in 1+1d, there were only few examples in higher spacetime dimensions before 2021.
- Meanwhile, higher-form global symmetries [Gaiotto-Kapustin-Seiberg-Willett 2014] have found a lot of interesting applications in higher spacetime dimensions.
- A q-form global symmetry operator has codimension $q+1$ in spacetime.
- For instance, the Gauss surface operator

$$\oint_{M^{(2)}} \vec{E} \cdot d\vec{n}$$

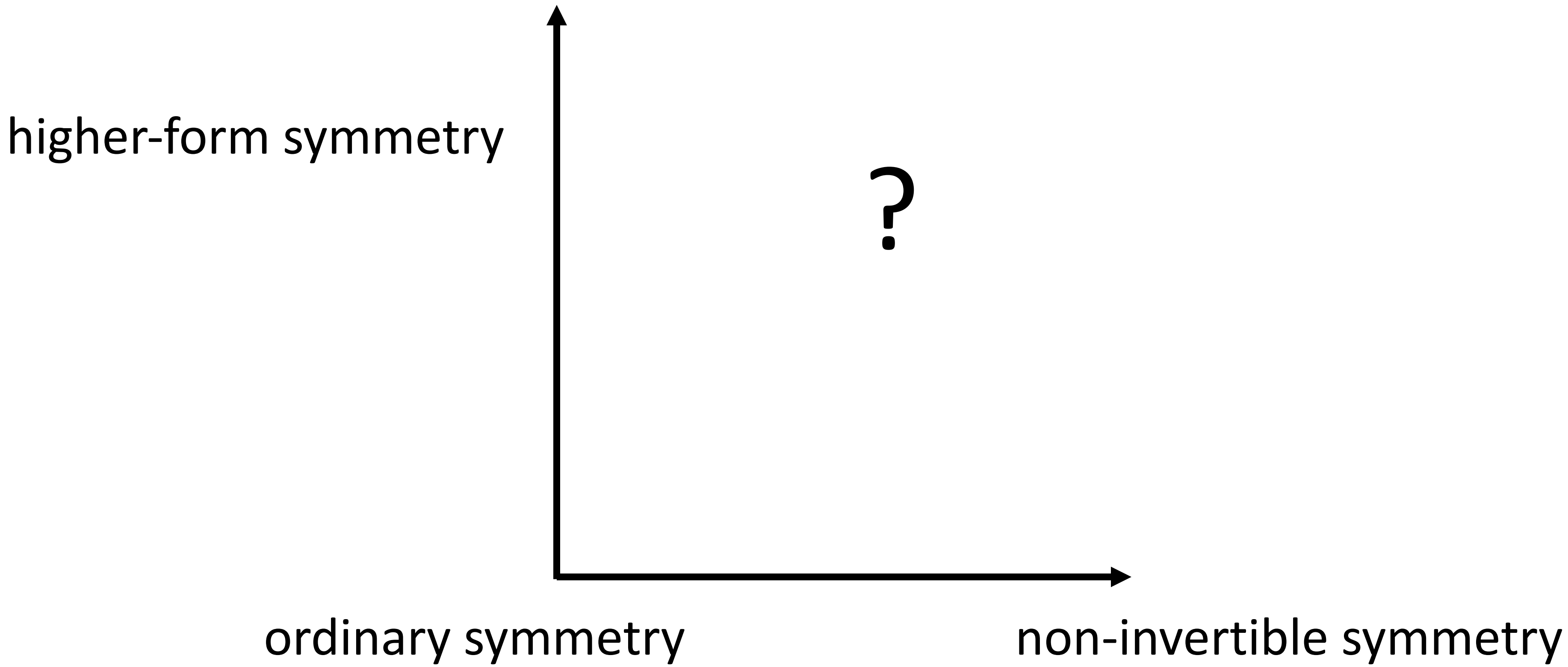
in 3+1d Maxwell theory is the charge of a $U(1)$ 1-form symmetry.

- Higher-form gauge symmetries are ubiquitous in string theory.





Two orthogonal generalizations?



Non-invertible higher-form symmetries

- It is however easier to construct non-invertible higher-form symmetries, i.e., non-invertible topological operators of codimension greater than 1.
- For instance, in a finite non-abelian G gauge theory, the topological Wilson lines W_R by an irreducible representation R of G. They are non-invertible $(d-2)$-form symmetry with fusion rule given by the representation ring of G.
- These non-invertible higher-form symmetries (or the absence thereof) play an important role in quantum gravity [Rudelius-SHS 2020, Heidenreich et al. 2021, AriasTamrgo-RodriguezGomez 2022]:

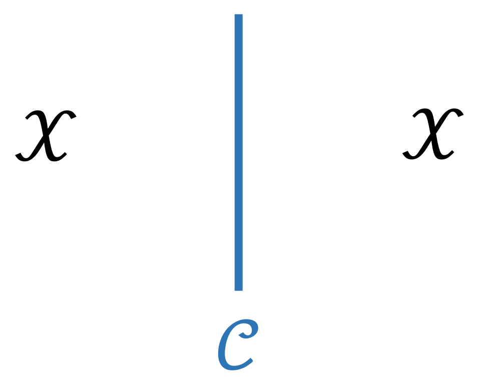
  no **generalized** global symmetry $\Longleftrightarrow$ completeness of gauge spectrum

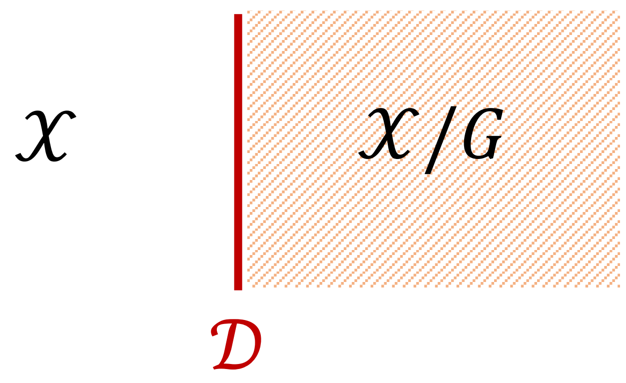
- Many other examples [Nguyen-Tanizaki-Unsal 2021, Wang-You 2021, Antinucci-Galati-Rizi 2022…].

Non-invertible symmetry in higher dimensions

- In 2021, it was realized that non-invertible 0-form global symmetry can be realized in a large class of familiar 3+1d quantum systems [Koide-Nagoya-Yamaguchi 2021, Choi-Cordova-Hsin-Lam-SHS 2021, Kaidi-Ohmori-Zheng 2021].
- Some useful constructions:

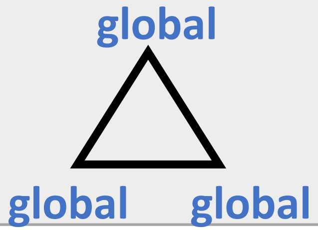
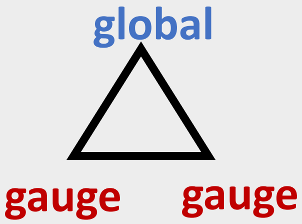
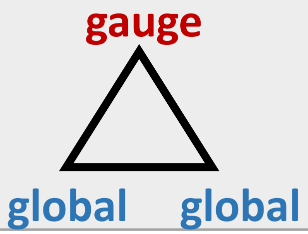
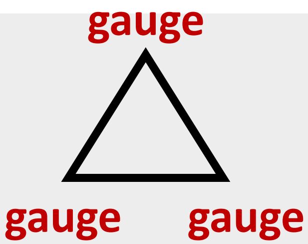
1. Higher gauging [Roumpedakis-Seifnashri-SHS 2022] : Gauging a higher-form symmetry on a higher codimensional manifold. See [Michele's talk].
2. Half gauging [Choi-Cordova-Hsin-Lam-SHS 2021]: Gauging a higher-form symmetry on half of spacetime.
3. Gauging a symmetry with mixed anomalies [Kaidi-Ohmori-Zheng 2021]
4. Other constructions from gauging [Tachikawa 2017, Bhardwaj-Bottini-SchaferNameki-Tiwari x2 2022, Bhardwaj-SchaferNameki-Wu 2022, Bartsch-Bullimore-Ferrari-Pearson x2 2022].





Non-invertible symmetry from anomalies

- A related construction of non-invertible symmetries is from gauging a global symmetry with mixed anomalies [Kaidi-Ohmori-Zheng 2021].
- Infinite, discrete non-invertible symmetries arise from the Adler-Bell-Jackiw anomaly [Choi-Lam-SHS 2022, Cordova-Ohmori 2022].

	't Hooft anomaly	**ABJ anomaly**	**2-group**	**gauge anomaly**
Is the QFT healthy?	Yes	Yes	Yes	No
What happens to the symmetry?	ordinary symmetry	non-invertible symmetry	2-group	no symmetry
				

Chiral symmetry in QED

- Consider QED with a massless, unit charge Dirac fermion and $U(1)$ gauge group.

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu\Psi$$

- The classical $U(1)_A$ chiral symmetry acts as

$$\Psi \to \exp\left(\frac{i\alpha}{2}\gamma_5\right)\Psi \quad , \qquad \alpha \sim \alpha + 2\pi$$

- Note that $\alpha = 2\pi$ corresponds to the fermion parity, which is part of the gauge symmetry.
- The Adler-Bell-Jackiw anomaly implies that the classical $U(1)_A$ chiral symmetry **fails** to be an ordinary global symmetry quantum mechanically.

QED

- The axial current $j_\mu^A = \frac{1}{2}\overline{\Psi}\gamma_5\gamma_\mu\Psi$ obeys the anomalous conservation equation

$$d \star j^A = \frac{1}{8\pi^2} F \wedge F$$

Here the field strength is normalized such that $\oint F \in 2\pi\mathbb{Z}$.

- Adler defined a symmetry operator that is formally conserved, but is **not** gauge invariant:

$$\text{“ } \widehat{U}_\alpha(M) = \exp[i\alpha \oint_M (\star j^A - \frac{1}{8\pi^2} AdA)] \text{ ”}$$

Fact: The Chern-Simons action $\exp[i \oint_M (\frac{N}{4\pi} AdA)]$ is gauge invariant iff N is an integer.

Rational angles

- Let us be less ambitious, and assume the chiral rotation angle is a fraction:

$$\alpha = \frac{2\pi}{N}$$

$$\text{“ } \hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{N} \star j^A - \frac{i}{4\pi N} AdA)] \text{ ”}$$

- The operator $\hat{U}_{\frac{2\pi}{N}}(M)$ is still not gauge invariant because of the fractional Chern-Simons term.

Fractional quantum Hall state

$$\text{“} -\frac{i}{4\pi N}\oint_M AdA \text{”}$$

- In condensed matter physics, this action is commonly used to describe the $\nu = 1/N$ fractional quantum Hall effect (FQHE) in 2+1d.
- It is however not gauge invariant. Fortunately, there is a well-known fix.
- The more precise, gauge invariant Lagrangian for the FQHE is

$$\oint_M \left(\frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)$$

  where a is a dynamical $U(1)$ gauge field living on the 2+1d manifold M.
- The two actions are related by illegally integrating out a to obtain

$$\text{“}a = -\frac{A}{N}\text{”}$$

Back to QED

[Choi-Lam-SHS 2022, Cordova-Ohmori 2022]

- Motivated by Fractional Quantum Hall Effects in 2+1d, we define a new operator in 3+1d QED:

$$\text{“ } \hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{N} \star j^A - \frac{i}{4\pi N} AdA)] \text{ ”}$$

a: auxiliary field on M
A: bulk gauge field

$$\downarrow$$

$$\mathcal{D}_{1/N}(M) \equiv \int [Da]_M \exp\left[\oint_M (\frac{2\pi i}{N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA) + \cdots\right]$$

- The gauge field a only lives on the defect – similar to the impurities discussed in [Mark’s talk].
- The new operator is gauge-invariant *and* conserved (topological).

*The FQH state **“cures”** the ABJ anomaly.*

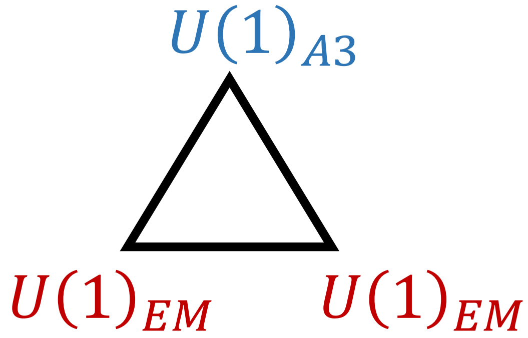
Non-invertible chiral symmetry in QED

[Choi-Lam-SHS 2022]

- The price we pay is that it **NOT** unitary:

$$\mathcal{D}_{1/N} \times \mathcal{D}_{1/N}^{\dagger} = \mathcal{C}$$
$$\equiv \int [Da]_M \int [D\bar{a}]_M \exp[\oint_M (\frac{iN}{4\pi} a da - \frac{iN}{4\pi} \bar{a} d\bar{a} + \frac{i}{2\pi}(a - \bar{a}) dA)]$$
$$\neq 1$$

- $\mathcal{C}$ is the condensation defect from higher gauging of the $\mathbb{Z}_N$ subgroup of the $U(1)$ magnetic one-form symmetry.
- It is easy to generalize this construction to an arbitrary rational chiral rotation $\alpha = 2\pi p/N$.



QCD and pion decay

[Choi-Lam-SHS 2022]

- Below the electroweak scale, the massless QCD Lagrangian for the up and down quarks has a chiral global symmetry (corresponding to π^0)

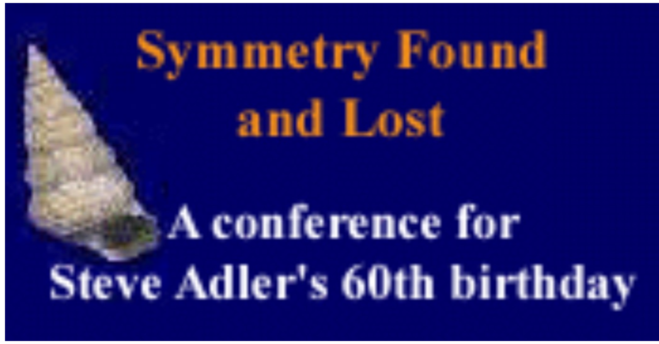
$$U(1)_{A3}\colon \begin{pmatrix} u \\ d \end{pmatrix} \to \exp(i\alpha\gamma_5\sigma_3) \begin{pmatrix} u \\ d \end{pmatrix}$$

- The ABJ anomaly with the electromagnetic $U(1)_{EM}$ gauge symmetry turns $U(1)_{A3}$ into the non-invertible global symmetry $\mathcal{D}_{1/N}$.
- The $\pi^0 F \wedge F$ coupling in the IR pion Lagrangian is necessary to match this non-invertible **global** symmetry in QCD.
- To put it in the maximally offensive way,

$\pi^0 \to \gamma\gamma$ *because of the non-invertible global symmetry.*

Symmetry Lost and Found

[Choi-Lam-SHS 2205.05086]



Year

$U(1)_A$ **IS** a (non-invertible) symmetry!

2022

Fractional Quantum Hall Effect 1982

$U(1)_A$ is **NOT** a symmetry

Adler-Bell-Jackiw 1969

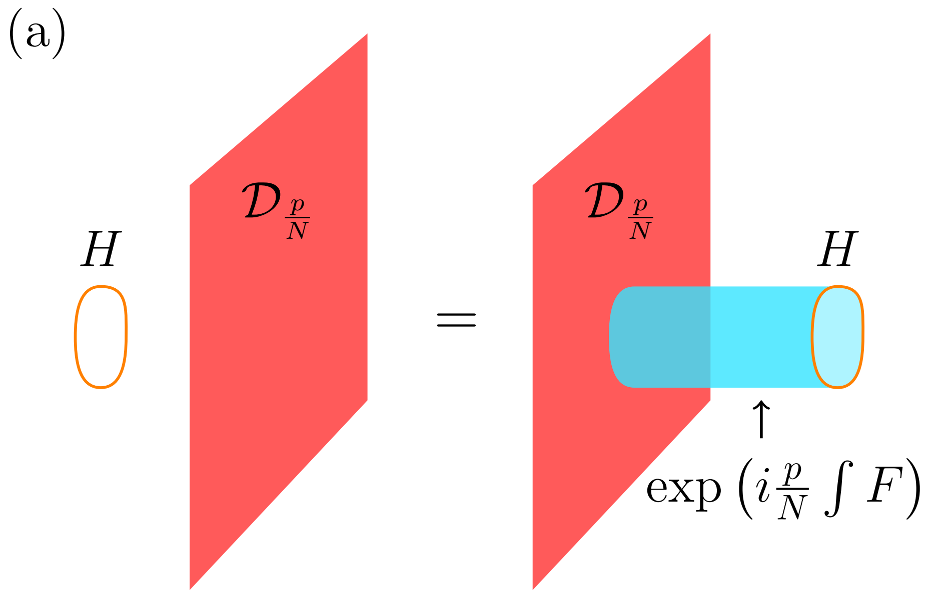
$U(1)_A$ is a symmetry

Selection rule in QED

- The operator $\mathcal{D}_{p/N}$ acts invertibly on the fermions as a chiral rotation with $\alpha = 2\pi p/N$. Helicity conservation.
- It acts non-invertibly on the 't Hooft lines $H(\gamma)$ by the Witten effect:

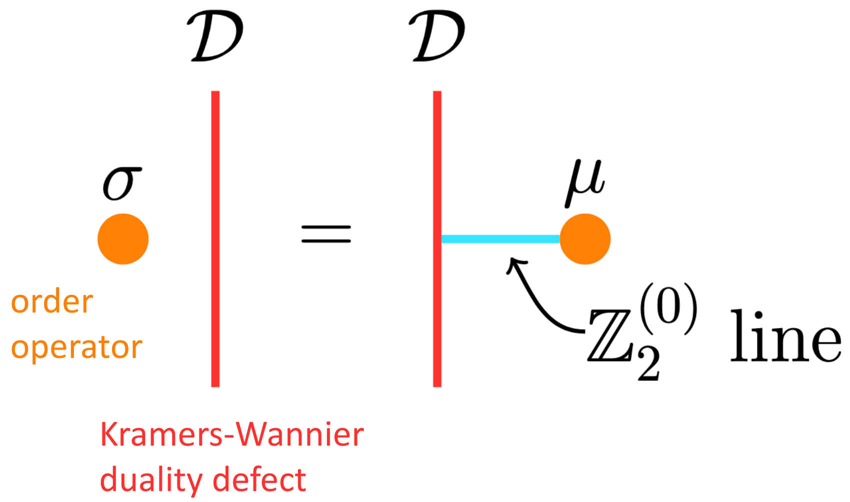
$$H(\gamma) \mapsto H(\gamma)\exp\left(\frac{ip}{N}\int F\right)$$

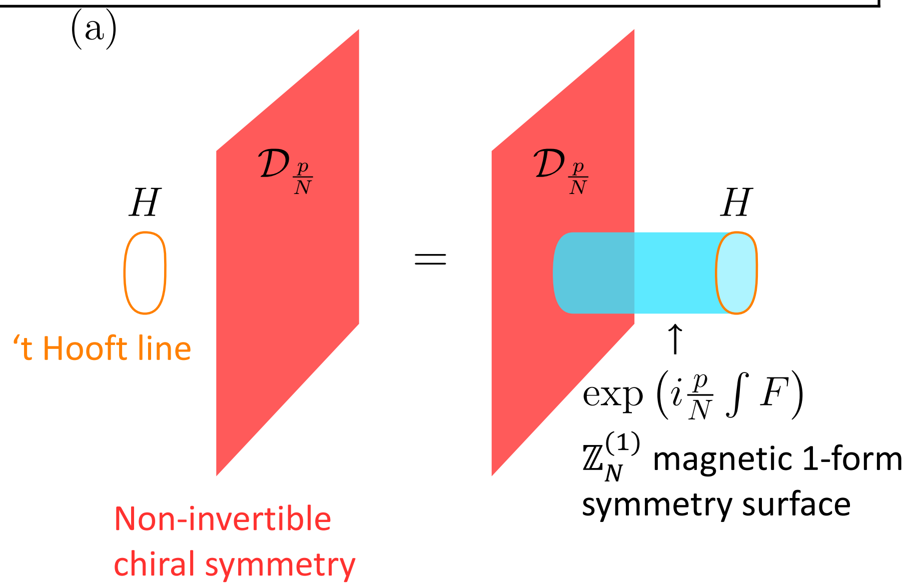
- The selection rule on the fermions on flat space amplitudes from $\mathcal{D}_{p/N}$ are the same as the naïve $U(1)_A$ symmetry.
- Note that there is no $U(1)$ instanton in flat space because $\pi_3\big(U(1)\big) = 0$.



1+1d Ising CFT vs. 3+1d QED

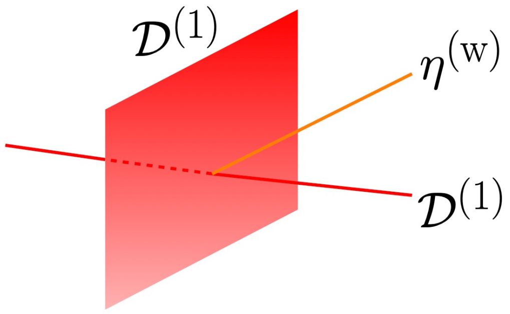
1+1d Ising CFT	3+1d QED
non-invertible Kramers-Wannier defect	non-invertible chiral symmetry
$\mathbb{Z}_2^{(0)}$ 0-form symmetry	$\mathbb{Z}_N^{(1)}$ magnetic 1-form symmetry
order operator σ	't Hooft line H
disorder operator μ	dyonic line





Non-invertible higher "groups"

[Choi-Lam-SHS 2212.04499]



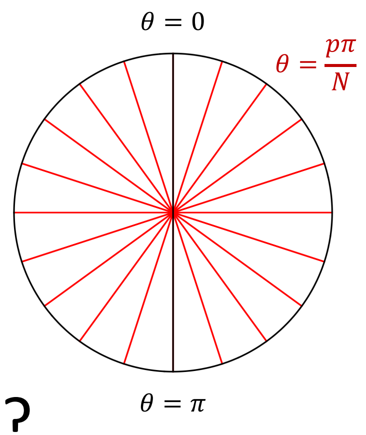
- The non-invertible symmetries in QED and in axions have a higher symmetry structure analogous to higher groups [many refs].
- In particular, it is usually the case that one non-invertible symmetry is subordinate to another higher-form symmetry, in the sense that the former cannot exist without the latter.
- It leads to universal inequalities on the symmetry breaking scales. For instance, in axions physics [Choi-Lam-SHS 2212.04499],

$$m_{electric} \lesssim \min(m_{monopole}, \sqrt{T})$$

T: axion string tension. This generalizes [Brennan-Cordova 2020] for invertible higher groups.

- See [Hsin 2022] and [Michele's talk] for more on the higher structure.

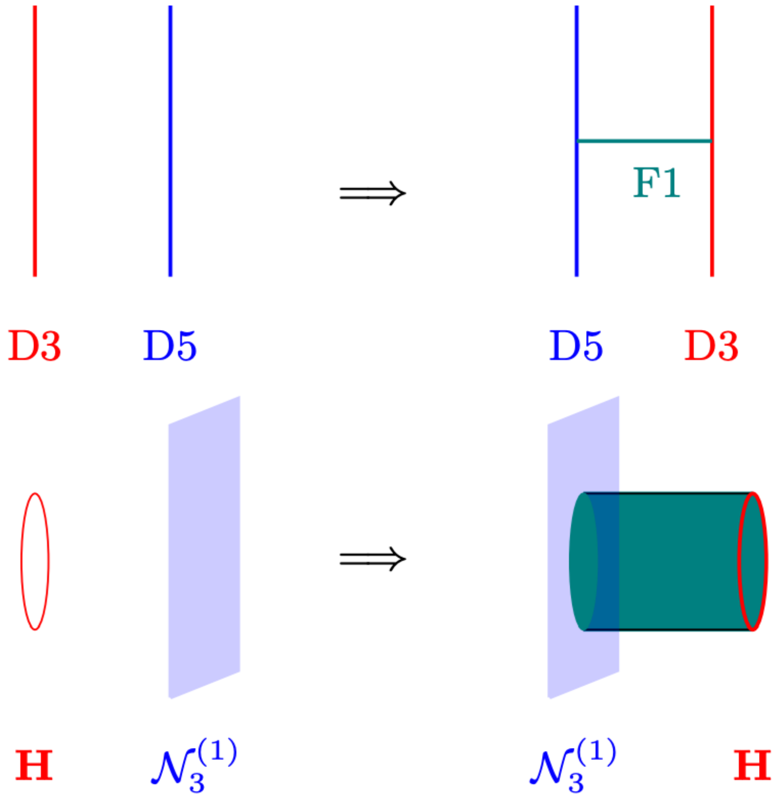
Generalizations of infinite non-invertible chiral symmetry



- Non-invertible CP symmetry in QED [Choi-Lam-SHS 2022]. Strong CP?
- 3d [AguileraDamia-Argurio-Tizzano 2022] and 5d [AguileraDamia-Argurio-GarciaValdecasas 2022]
- Sigma models [Chen-Tanizaki 2022, Hsin 2022]
- Related constructions in QED [Karasik 2022, GarciaEtxebarria-Iqbal 2022]
- Non-invertible Gauss laws in axion physics [Choi-Lam-SHS 2022, Yokokura 2022]
- Page charge as non-invertible symmetry [Choi-Lam-SHS 2022]. Supergravity [GarciaValdecasas 2023]
- Non-invertible symmetry from mixed gravitational anomaly [Putrov-Wang 2023]

Stringy realizations

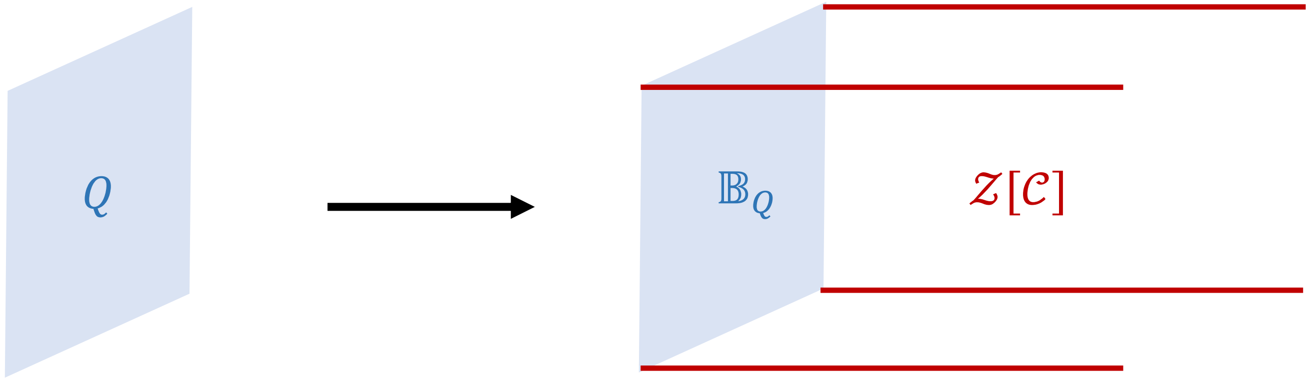
- The gauging constructions can be naturally embedded into string theory and supersymmetric theories.
- Realization from string/M-theory, holography [AguileraDamia-Argurio-GarciaValdecasa, GarciaEtxebarria, Apruzzi-Bah-Bonetti-SchaferNameki, Heckman-Hubner-Torres-Zhang, Antinucci-Benini-Copetti-Galati-Rizi, Heckman-Hubner-Torres-Yu, Etheredge-GarciaEtxebarria-Heidenreich-Rauch].
- 6d construction for 4d theories [Gukov-Hsin-Pei, Bashmakov-DelZotto-Hasan, Bashmakov-DelZotto-Hasan-Kaidi, Antinucci-Copetti-Galati-Rizi, Carta-Giacomelli-Mekareeya-Mininno].



Hanay-Witten = Non-invertible action
Figure taken from 2208.07373

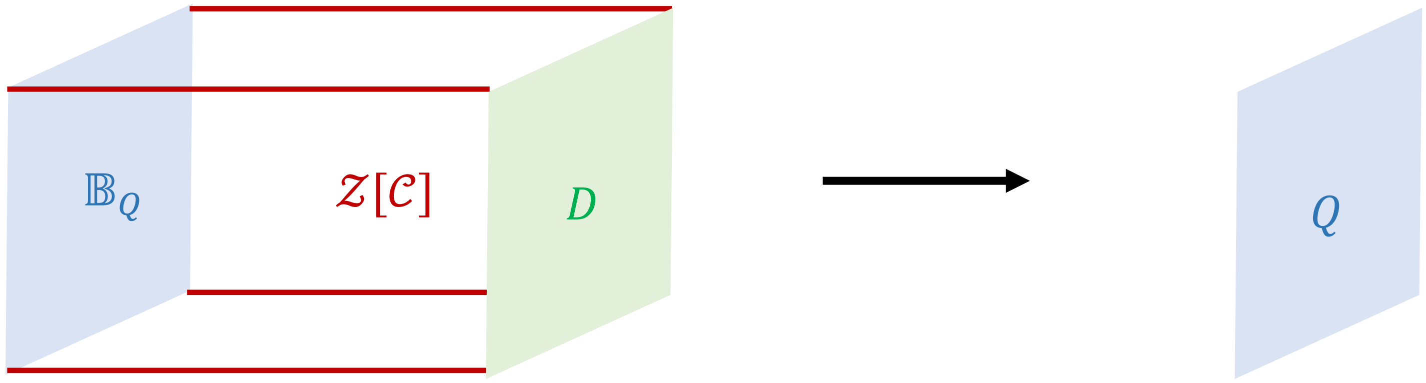
A bulk viewpoint

- Given a (finite) generalized global symmetry $\mathcal{C}$, possibly with an anomaly, in a d-dimensional QFT Q. We can couple it to a $d+1$-dimensional topological field theory (TFT) $\mathcal{Z}[\mathcal{C}]$ by gauging $\mathcal{C}$.
- When $\mathcal{C}$ is an invertible finite symmetry, $\mathcal{Z}[\mathcal{C}]$ is a finite group gauge theory, with the Dijkgraaf-Witten twist given by the anomaly.
- In $d=2$, $\mathcal{Z}[\mathcal{C}]$ is known as the Turaev-Viro TFT, or the Drinfeld center of $\mathcal{C}$.
- The original QFT Q is mapped a (non-topological) boundary condition $\mathbb{B}_Q$ for the Symmetry TFT $\mathcal{Z}[\mathcal{C}]$.



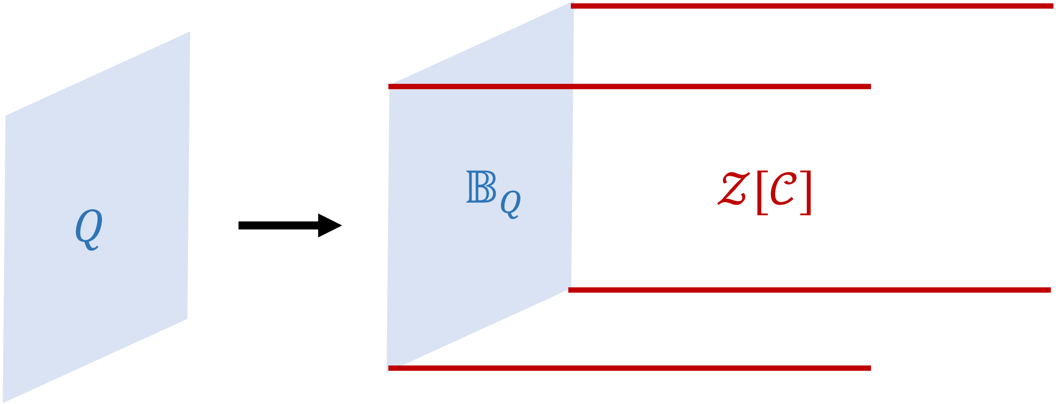
Symmetry TFT

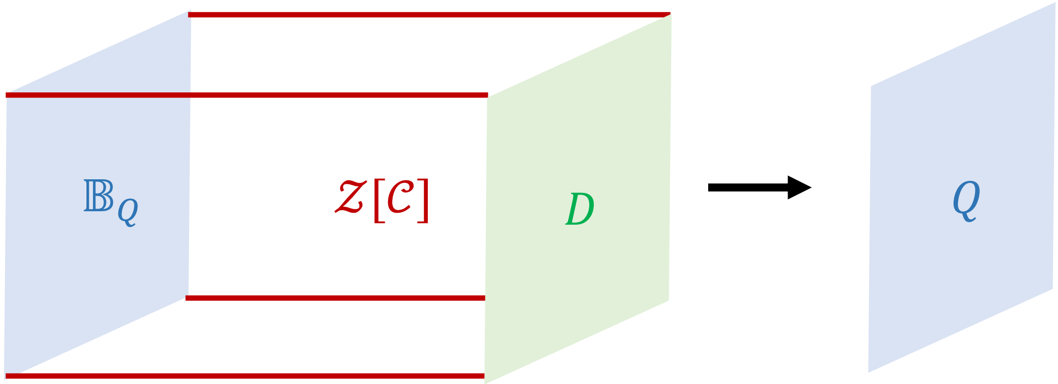
- What's the inverse map?
- The Symmetry TFT $\mathcal{Z}[\mathcal{C}]$ generally has several topological boundary conditions. One of them is a topological Dirichlet boundary condition D with respect to $\mathcal{C}$.
- Placing the SymTFT on an interval and collapse the topological boundary condition D to retrieve the original QFT Q.

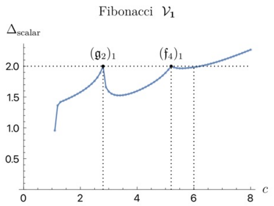


Quiche and sandwich

- Sandwich/quiche construction [Gaiotto-Kapustin-Seiberg-Willett 2014, Gaiotto-Kulp 2020, Freed-Moore-Teleman 2022, Lin-Okada-Seifnashri-Tachikawa 2022]. See also [Ji-Wen 2019, Kong-Lan-Wen-Zhang-Zheng 2020] for parallel developments in CMT.
- Powerful tool to characterize generalized symmetries.
- SymTFT for non-invertible symmetry and computations of anomalies [delZotto-GarciaEtxebarria 2022, Kaidi-Ohmori-Zheng 2022, Kaidi-Nardoni-Zafrir-Zheng 2023, Zhang-Cordova 2023].
- Embedding into string theory [Apruzzi-Bonetti-GarciaEtxebarria-Hosseini-SchaferNameki 2021,…]
- SymTFT bootstraps non-invertible symmetries [Lin-SHS 2023].







Mathematical language

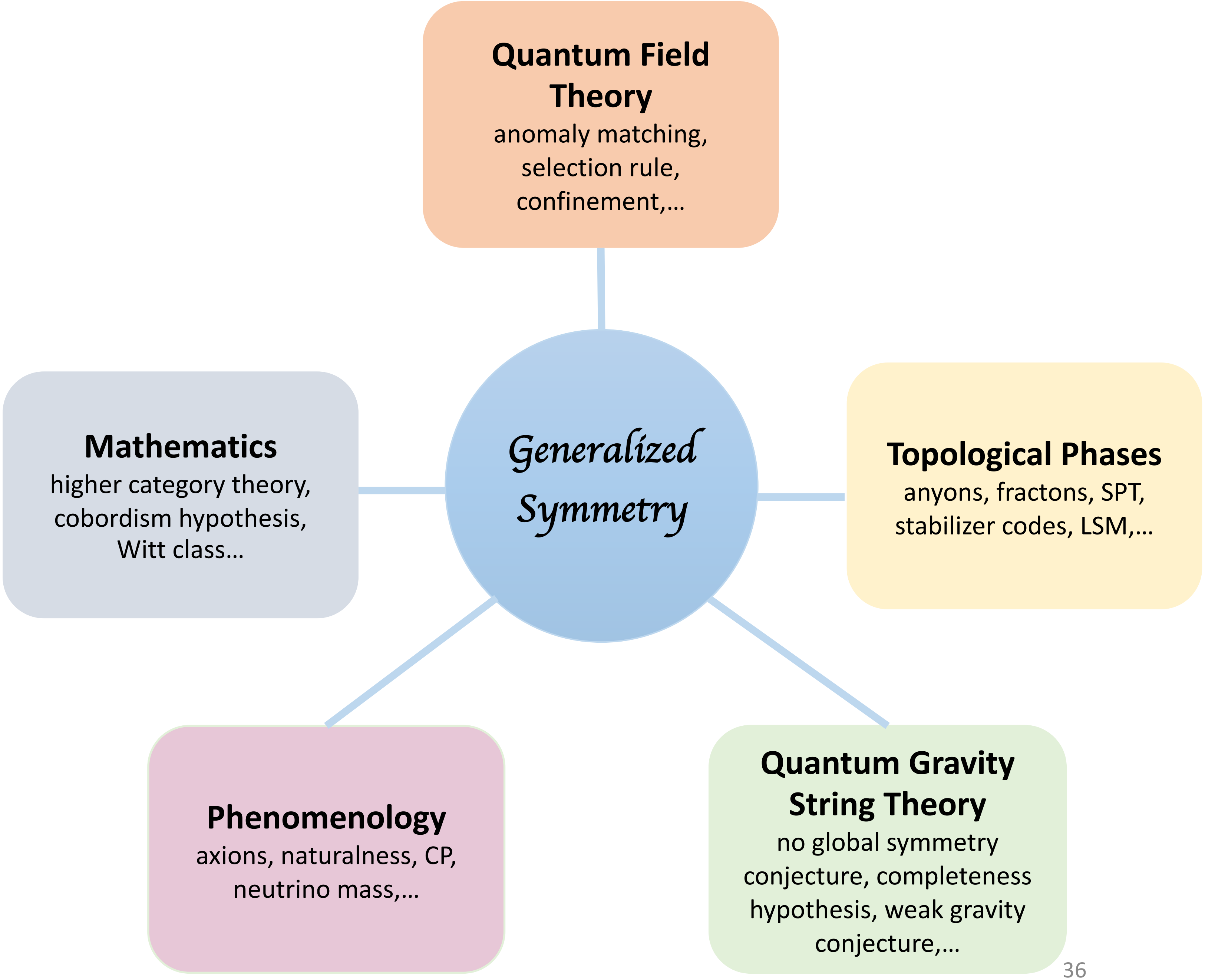
- Group theory is the mathematical language for symmetry in Quantum Mechanics.
- The recent developments in Quantum Field Theory is begging for a new mathematical language for generalized symmetries.
- In 1+1d, finite non-invertible symmetries are described by fusion category.
- In higher dimensions, it has been an active subject of research. Categorical symmetry [Bhardwaj-(Bottini)-SchaferNameki-Tiwari x3 2022, Bhardwaj-SchaferNameki-Wu 2022, Bartsch-Bullimore-Ferrari-Pearson 2022, Freed-Moore-Teleman 2022, Bhardwaj-SchaferNameki 2023, Bartsch-Bullimore-Grigoletto 2023,…]

Conclusion

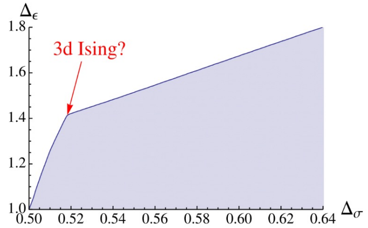
- Symmetry can be non-invertible. *What's done cannot be undone.*
- These non-invertible symmetries are everywhere, ranging from the Ising model to real-world QED. New examples discovered at a rapid pace.
- They are useful:
  - New constraints on RG flows.
  - Universal inequalities on symmetry breaking scales.
- They consolidate conjectures in quantum gravity.
- Interesting interplay between higher-form symmetries and non-invertible symmetries. The key connection is gauging.

Outlook

- New symmetries lead to new notions of naturalness. Implications for hierarchy and naturalness problems, such as strong CP, neutrino mass [Cordova-Hong-Koren-Ohmori 2022], and other phenomenological problems?
- New insights into monopole and axion string physics.
- New selection rules in scattering amplitudes.
- Classification and computation of their anomalies.
- Non-invertible symmetries in conventional lattice models?
- In higher dimensions, are there non-invertible that do **not** arise from generalized gauging?


Quantum Field Theory
anomaly matching, selection rule, confinement,…
Mathematics
higher category theory, cobordism hypothesis, Witt class…
Generalized Symmetry
Topological Phases
anyons, fractons, SPT, stabilizer codes, LSM,…
Phenomenology
axions, naturalness, CP, neutrino mass,…
Quantum Gravity
String Theory
no global symmetry conjecture, completeness hypothesis, weak gravity conjecture,…

Quantum Field Theory
anomaly matching, selection rule, confinement,…



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Topological Phases
anyons, fractons, SPT, stabilizer codes, LSM,…

Phenomenology
axions, naturalness, CP, neutrino mass,…

Quantum Gravity String Theory
no global symmetry conjecture, completeness hypothesis, weak gravity conjecture,…

Activities

- TASI 2023 TASI 2023 Aspects of Symmetry

  Aspects of Symmetry

- Aspen Workshop 2023

  Traversing the Particle Physics Peaks - Phenomenology to Formal

- KITP Workshop 2025

  Generalized Symmetries in QFT