# Scaling in holographic turbulence

Eurostrings in Gijón, 25.04.2023

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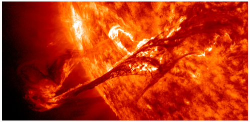
# Fluid Dynamics and Turbulence

# Turbulence as an ubiquitous phenomenon

Understanding turbulent flow is crucial in various fields and research areas



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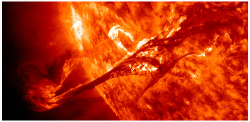

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**Aim: Geometrize the statistical properties of fluid flow and turbulence through the fluid/gravity duality.**

# Fluid dynamics and turbulent flow

(incompressible) Navier-Stokes equations:

$$\nabla_i u^i = 0$$
$$\partial_t u^i + \left(\mathbf{u} \cdot \nabla\right) u^i = -\nabla^i p + \nu \nabla^2 u^i$$

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- The driving force $f^i$ is fluctuating randomly.
- The fluid velocity $u^i$ and the pressure $p$ are random variables.
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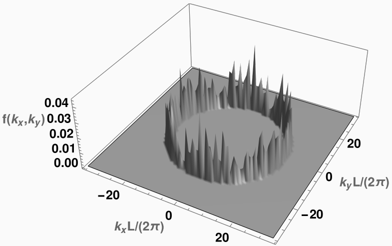


The Fourier transform of $f = |\mathbf{f}|$ peaks at $|\mathbf{k}| = k_f$, associated with a driving scale $2\pi/k_f$.

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For sufficiently large Reynolds number

$$R = \frac{\langle |\mathbf{u}| \rangle L}{\nu}$$

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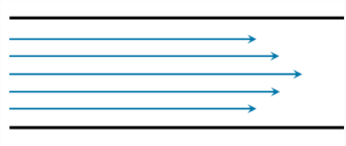
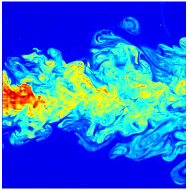


$R \sim 10^0 - 10^3$



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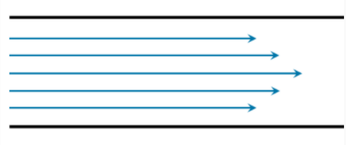

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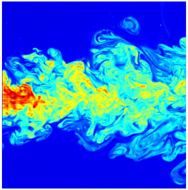


$10^3 \lesssim R$

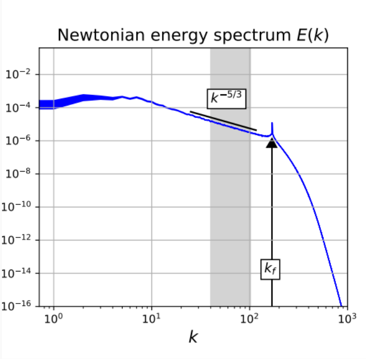
Kolmogrov (1941): For $1 \ll R$ and in the inertial range $2\pi/k_f \ll r \ll L$ statistical properties of the fluid velocity show an universal behavior.

$$\delta S = (\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})) \cdot \hat{\mathbf{r}}$$

$$\left\langle (\delta S)^n \right\rangle \sim r^{\zeta_n}$$

$$\zeta_n = n/3$$

# The energy power spectrum




Westernacher-Schneider (2017)

$$\left\langle (\delta S)^2 \right\rangle \sim r^{2/3}$$

$$E(k) \sim \partial_k \int_{k'<k} d^d k' \left\langle u^i(k') \bar{u}_i(k') \right\rangle \sim k^{-5/3}$$

$$\int E(k) dk = \int d^d x \frac{\rho}{2} u^i(x) u_i(x)$$

For $d = 2$:
inverse cascading energy
& direct cascading enstrophy

$\rightarrow$ Good agreement between theory and experiment for lower moments.

# Anomalous exponents

In the inertial range:

$$\delta S = (\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})) \cdot \hat{\mathbf{r}}$$

$$\left\langle (\delta S)^n \right\rangle = \langle e_r^{n/3} \rangle r^{n/3}$$

$$\left\langle (\delta S)^n \right\rangle \sim r^{\zeta_n} = r^{n/3+\Delta(n)}$$

where $\langle e_r^m \rangle$ is the $m$-th moment of the averaged energy dissipation:

$$\langle e_r^m \rangle = \left\langle \left( \frac{\int_{B_d(r)} d^d x' e(x')}{Vol(B_d(r))} \right)^m \right\rangle$$

$$e(x) = \frac{\nu}{2} \left( \partial_i u^j + \partial_j u^i \right)^2$$

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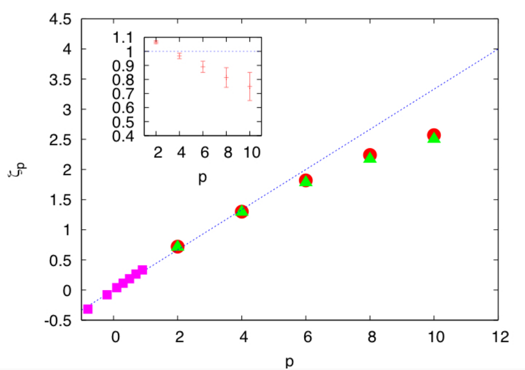


Chen et. al. (2005)

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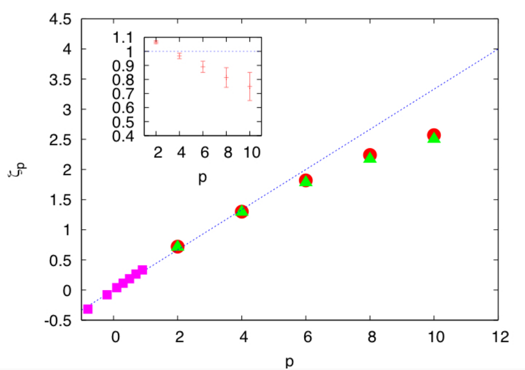


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**(One of) the unsolved problem(s) of turbulence is to get an analytic handle on $\Delta(n)$.**

# Turbulence in Holography

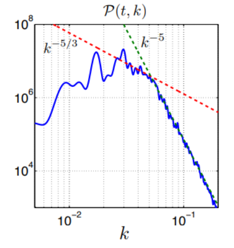
# Holographic turbulence

**Gravity in $d+1$ dimensions $\leftrightarrow$ Fluid dynamics in $d$ dimensions**

[Bhattacharyya et. al. (2007)]

- fluid in equilibrium $\leftrightarrow$ black hole with smooth horizon
- turbulent fluid $\leftrightarrow$ self-similar horizon structure [Adams et. al. (2013)]

Decaying holographic turbulence




- Irregular, chaotic fluid flow develops from **unstable initial conditions**
- Scaling in inertial range is short and transient

- $-5/3$-power law $\leftrightarrow$ fractal dimension of the horizon $\approx d + 2/3$ [Westernacher-Schneider (2017)]
- $\langle(\delta S)^{n>2}\rangle \leftrightarrow$ some geometric/horizon related objects (?)

# Geometrizing the local energy dissipation

In the inertial range:

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The horizon extrinsic curvature

$$\Theta_{\mu\nu} = \Pi_\mu^\alpha \Pi_\nu^\beta \nabla_\alpha n_\beta$$

can be related to the energy dissipation

$$e(x) = \frac{\nu}{2} \left(\partial_i u^j + \partial_j u^i\right)^2$$
$$\Theta_j^i \Theta_i^j \sim e(x) + \mathcal{O}(\partial^2) + \mathcal{O}(1/c^3)$$

[Eling, Oz (2009)]

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[Eling, Oz (2009)]

In infalling coordinates with $n_\mu dx^\mu = d\rho$ one can show that $\Theta^\mu_\nu \Theta^\nu_\mu = \Theta^i_j \Theta^j_i$, so that

$$\langle e_r^n \rangle = \left\langle \left( \frac{1}{Vol(B(r))} \int_{B(r)} d^{d-1}x' e(x') \right)^n \right\rangle$$

matches

$$\langle (e_r^h)^n \rangle = \left\langle \left( \frac{1}{Vol(\tilde{B}(r))} \int_{\tilde{B}(r)} d^{d-1}x' \Theta^\mu_\nu \Theta^\nu_\mu \right)^n \right\rangle$$

# From stochastic gravity to driven, incompressible Navier-Stokes equations

**stochastic gravity in AdS$_{d+1}$**

$R^{\mu\nu} - 1/2 R g^{\mu\nu} + \Lambda g^{\mu\nu} = 0$ with $g^{b}_{\alpha\beta} = \eta_{\alpha\beta} + Q_{\alpha\beta}$ as boundary condition & $Q_{\alpha\beta}$ random part of boundary metric

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**CFT in fluctuating background**

Evolution of $\overline{\langle T^{\mu\nu}\rangle} = \int DQ_{\alpha\beta} P(Q_{\alpha\beta}) \langle T^{\mu\nu}\rangle_{\eta+Q}$ in random background $g_{\alpha\beta} = \eta_{\alpha\beta} + Q_{\alpha\beta}$

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**relativistic hydrodynamics**

$\nabla_\mu T^{\mu\nu} = 0$ in a curved, stochasticly fluctuating background $\eta_{\alpha\beta} + Q_{\alpha\beta}$, such that $\partial_\mu T^{\mu\nu} = f^\nu(Q^{\alpha\beta}, T^{\mu\nu})$

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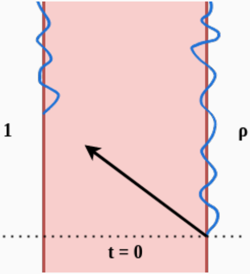
**non-relativistic fluid**

Incompressible ($\nabla_i u^i = 0$), driven Navier-Stokes equations
$\partial_t u^i = -(\mathbf{u} \cdot \nabla) u^i - \nabla^i p + \nu \nabla^2 u^i - f^i$

# Stochastic gravity and driven turbulence

The metric of $AdS_4$ in Eddington-Finkelstein coordinates

$$ds^2 = \Sigma(t,\vec{x},\rho)^2 \hat{g}_{ij}(t,\vec{x},\rho) dx^i dx^j \\ - 2dt\left(F_i(t,\vec{x},\rho)dx^i + A(t,\vec{x},\rho)dt + \omega_0(t,\vec{x})\frac{d\rho}{\rho^2}\right)$$



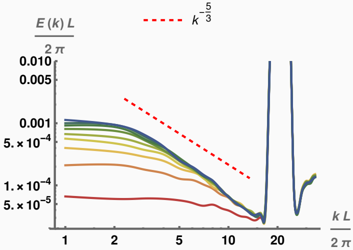

On the boundary:

$$g^b_{\alpha\beta} = \eta_{\alpha\beta} - \delta_{\alpha t}\delta_{\beta t}\, Q(t,\vec{x}) \\ Q(t,\vec{x}) = q(t,\vec{x}) + 3\left(\overline{q(t,\vec{x})q(t,\vec{x})}\right)^{1/2}$$

with Ornstein-Uhlenbeck process

$$\dot{q}(t,\vec{x}) = -\frac{q(t,\vec{x})}{\tau} + \frac{\xi(t,\vec{x})}{\tau} \\ \overline{\xi(t,\vec{x})\xi(t',\vec{x}')} = D\delta(t-t') \sum_{i,|\vec{k}_i|=k_f} \cos\left(\vec{k}_i\,(\vec{x}-\vec{x}')\right)$$

# Driven holographic turbulence



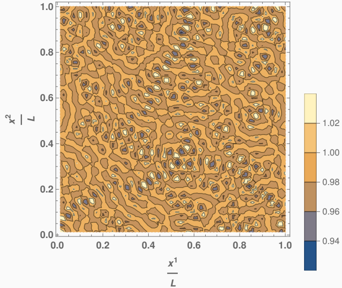

Radial positions of the horizon for one point in time and one sample:

- boundary perturbations imprinted on horizon
- Self-similar horizon with fluctuations dominated by driving scale

From

$$T^{\mu\nu}\,u_\mu = \epsilon\,u^\nu$$
$$T^{\mu\nu} \approx \epsilon\,(g^b)^{\mu\nu} + (\epsilon + p)u^\mu\,u^\nu + \dots$$

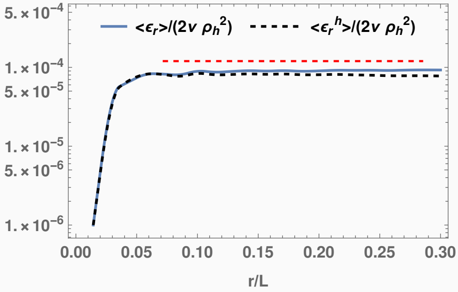
we determine the energy power spectrum $E(k) \sim k\,\langle u^i(k)\bar{u}^i(k)\rangle$.



# Numerically testing the 'geometrization' of $\langle e_r^n \rangle$

$$\langle e_r \rangle = \left\langle \frac{1}{Vol(B(r))} \int_{B(r)} d^{d-1}x' e(x') \right\rangle$$

$$\langle e_r^h \rangle = \left\langle \frac{1}{Vol(\tilde{B}(r))} \int_{\tilde{B}(r)} d^{d-1}x' \Theta^\mu_\nu \Theta^\nu_\mu \right\rangle$$




- Good match between $\langle e_r \rangle$ and $\langle e_r^h \rangle$
- $\langle e_r \rangle$ is constant in inertial range (indicated by red line), as expected for turbulence in 2 spatial dimensions

# Summary & Outlook

## Summary

- Driven, non-relativistic, turbulent flow $\leftrightarrow$ Stochastic gravity with slowly varying, low amplitude boundary fluctuations
- Size, autocorrelation time and driving frequency of the random boundary metric are fixed by scaling relations.
- Anomalous scaling exponents are encoded in higher moments of the horizon extrinsic curvature squared averaged over geodesic balls on the horizon

## Outlook

- Numerical holographic turbulence in higher dimensions
- Study relation to entanglement entropy
- Trailing strings in 'turbulent geometries'
- Study super-sonic turbulence/relativistic turbulence

# Thank You

# Backup slides

# Relativistic Hydrodynamics and the non-relativistic limit

Invariance under
space-time-translations implies

$$\nabla_\mu T^{\mu\nu} = 0 \quad \text{with}$$
$$T^{\mu\nu} = p\, g^{\mu\nu} + (\epsilon + p) u^\mu u^\nu + \mathcal{O}(\partial)$$

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setting $p = p_0 T^3$ and $a^\sigma = u^\lambda \partial_\lambda u^\sigma$:

$$\nabla_\mu u^\mu + 2u^\mu \nabla_\mu \ln T = \mathcal{O}(\partial^2)$$
$$a^\sigma + P^{\sigma\mu} \nabla_\mu \ln T = \mathcal{O}(\partial^2)$$

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for $u^i \ll c$:

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for $u^i \ll c$:

If $u^i = \epsilon$
$\partial_i \phi / \phi = \epsilon\, T$
$\partial_t \phi / \partial_i \phi = \epsilon$, then

$$\nabla_i u^i = \mathcal{O}(\epsilon^4)$$
$$\partial_t u^i + \left(\mathbf{u} \cdot \nabla\right) u^i = -\nabla^i p + \nu \nabla^2 u^i + \mathcal{O}(\epsilon^4)$$

# A stochastic background metric as driving force

**relativistic hydro in curved space**

Solve $\nabla_\mu T^{\mu\nu} = 0$ in a curved, stochasticly fluctuating background

$$g_{\mu\nu} = diag[g(x^\mu), 1, 1],$$

with

$$g = -(1 + 2\Phi)$$

such that

$$\partial_\mu T^{\mu\nu} = D^\nu(\Phi, \partial\Phi, T^{\alpha\beta})$$

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**non-relativistic, driven fluid**

Setting $\Phi \sim \epsilon^2$, leads to

$$\Gamma^0_{00} \sim \epsilon^4 \quad \Gamma^i_{00} \sim \partial^i\Phi + \mathcal{O}(\epsilon^4)$$
$$\Gamma^0_{0i} \sim -\partial_i\Phi + \mathcal{O}(\epsilon^4)$$

with $u^i \sim \epsilon$, $\partial_i \sim \epsilon$, $\partial_t \sim \epsilon^2$ and yields to leading order

$$\begin{aligned} \nabla_i u^i &= 0 \\ \partial_t u^i &= -\left(\mathbf{u} \cdot \nabla\right) u^i - \nabla^i p \\ &\quad + \nu \nabla^2 u^i - \partial^i \Phi \end{aligned}$$

# Stochastic integration

Consider the ordinary stochastic differential equation,

$$\frac{\partial}{\partial t}X(t) = g(X(t)) + h(X(t))\xi(t)\,,$$

where $\xi(t)$ is a random variable. Let us write

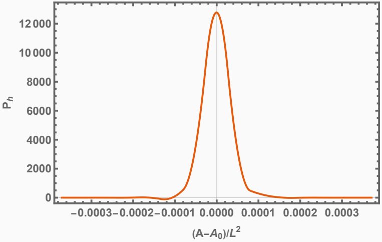
$$X(t) = X(t_0) + \int_{t_0}^{t} g(X(t'))dt' + \int_{t_0}^{t} h(X(t'))\xi(t')dt'\,.$$

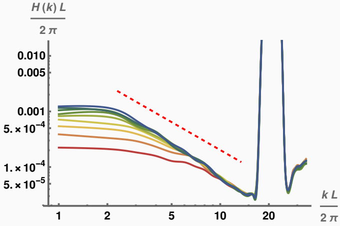
In Stratonovich prescription

$$\int_{t_0}^{t} h(X(t'))\xi(t')dt' = \lim_{\Delta t\to 0}\sum_{n=0}^{N-1} h\left(\frac{X(t_{n+1}) + X(t_n)}{2}\right)\int_{t_0+n\Delta t}^{t_0+(n+1)\Delta t} \xi(t')dt'\,.$$

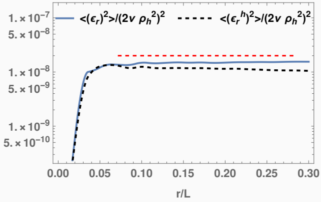
# More results

The horizon distribution and horizon powerspectrum:




and the second moments of $e_r$ & $e_r^h$:



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&- 2dt \left( F_i(t,\vec{x},\rho) dx^i + A(t,\vec{x},\rho) dt + \omega_0(t,\vec{x}) \frac{d\rho}{\rho^2} \right)
\end{aligned}
$$

initially fulfills

$$
\begin{aligned}
A|_{t=0} &= 1/2(1/\rho^2 - \rho) \\
\Sigma\Big|_{t=0} &= 1/\rho \\
F_i\Big|_{t=0} &= 0 \\
\hat{g}_{ij}|_{t=0} &= \delta_{ij} \\
\omega_0|_{t=0} &= 1.
\end{aligned}
$$

# Stochastic gravity and driven turbulence

In Eddington-Finkelstein coordinates the metric of $AdS_4$

$$
\begin{aligned}
ds^2 = & \Sigma(t,\vec{x},\rho)^2 \hat{g}_{ij}(t,\vec{x},\rho) dx^i dx^j \\
& - 2dt \left( F_i(t,\vec{x},\rho) dx^i + A(t,\vec{x},\rho) dt + \omega_0(t,\vec{x}) \frac{d\rho}{\rho^2} \right)
\end{aligned}
$$

initially fulfills

$$
\begin{aligned}
A|_{t=0} &= 1/2(1/\rho^2 - \rho) \\
\Sigma\Big|_{t=0} &= 1/\rho \\
F_i\Big|_{t=0} &= 0 \\
\hat{g}_{ij}|_{t=0} &= \delta_{ij} \\
\omega_0|_{t=0} &= 1.
\end{aligned}
$$

corresponding to the thermal expectation value

$$
\langle T^{\mu\nu} \rangle = \mathrm{Tr}\left( \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})} T^{\mu\nu} \right)
$$

with temperature $T = \pi\, \rho_h$.

# Stochastic gravity and driven turbulence

In Eddington-Finkelstein coordinates the metric of $AdS_4$

$$ds^2 = \Sigma(t,\vec{x},\rho)^2 \hat{g}_{ij}(t,\vec{x},\rho) dx^i dx^j - 2dt\left(F_i(t,\vec{x},\rho)dx^i + A(t,\vec{x},\rho)dt + \omega_0(t,\vec{x})\frac{d\rho}{\rho^2}\right)$$

has near boundary expansion

$$A = \frac{Q}{2\rho^2} - \frac{\tilde{R}Q^2}{4} + \mathcal{O}(\rho)$$

$$\Sigma = \frac{1}{\rho} + \mathcal{O}(\rho^5)$$

$$F_i = \frac{\partial_i \omega_0}{\rho} + \mathcal{O}(\rho)$$

$$\hat{g}_{ij} = \delta_{ij} + \mathcal{O}(\rho^3)$$

$$\omega_0 = \sqrt{Q}.$$

$$g^b_{\alpha\beta} = \eta_{\alpha\beta} - \delta_{\alpha t}\delta_{\beta t}\, Q(t,\vec{x})$$

$$Q(t,\vec{x}) = q(t,\vec{x}) + 3\left(\overline{q(t,\vec{x})q(t,\vec{x})}\right)^{1/2}$$

with Ornstein-Uhlenbeck process

$$\dot{q}(t,\vec{x}) = -\frac{q(t,\vec{x})}{\tau} + \frac{\xi(t,\vec{x})}{\tau}$$

$$\overline{\xi(t,\vec{x})\xi(t',\vec{x}')} = D\delta(t-t') \sum_{i,|\vec{k}_i|=k_f} \cos\left(\vec{k}_i\left(\vec{x}-\vec{x}'\right)\right)$$