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# Resúmenes del Congreso de la Real Sociedad Matemática Española

Oviedo, 4 a 7 de febrero de 2009

**Sesión especial 7: Geometría real y optimización**

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# Convexity and semi-algebraic geometry

**Jean B. Lasserre**

Let $\mathbb{R}[x_1, \ldots, x_n]\,(= \mathbb{R}[x])$ be the ring of real polynomials and let $\mathbf{K} \subset \mathbb{R}^n$ be the basic closed semi-algebraic set $\mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \ldots, m\}$ for some $(g_j) \subset \mathbb{R}[x]$. In this talk we gather some results from [1, 2, 3, 4] that show how convexity permits to derive specialized results in real algebraic geometry.

I. We first provide a specific representation of convex polynomials that are nonnegative on $\mathbf{K}$ when the polynomials that define $\mathbf{K}$ are concave (including the case when $\mathbf{K} \equiv \mathbb{R}^n$). This representation is the convex specialization of the important Putinar's Positivstellensatz.

II. We next consider the *semidefinite representation* of the convex hull $\mathrm{co}(\mathbf{K})$ of $\mathbf{K}$ (or of $\mathbf{K}$ itself if $\mathbf{K}$ is convex). We show that when what we call the PP-BDR property (from Putinar Prestel Bounded Degree Property) of the polynomials that define $\mathbf{K}$ holds, then $\mathrm{co}(\mathbf{K})$ has a semidefinite representation given explicitly in terms of those polynomials. We then provide several sufficient conditions on the $g_j$'s for the PP-BDR property to hold. In addition, if Slater's condition and a certain nondegeneracy assumption on the boundary of $\mathbf{K}$ both hold, then we also provide a necessary and sufficient condition for $\mathbf{K}$ to be convex. This condition can be checked numerically by solving finitely many semidefinite programs.

**Keywords:** Convex polynomials, sums of squares, convex sets, Jensen's inequality

**Mathematics Subject Classification 2000:** 14P10 90C22 11E25 12D15

## Referencias

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# Certificates of positivity in the multivariate Bernstein's basis

**Marie-Françoise Roy**

The Bernstein basis on a simplex of polynomials of degree less than $d$ can be easily defined.

It is well known that in the univariate case, the Bernstein coefficients of a polynomial define a control line, so called because the shape of the graph of the polynomial can be easily controled using the Bernstein coeffiicients.

In the multivariate case, even the definition of the control polytope cannot be given immediately from the Bernstein coefficients.

In order to define the control polytopel and study its distance to the graph of the polynomial, it is convenient to introduce a combinatorial construction : the standard triangulation of a simplex, and to characterize combinatorially the convex pieciwise linear functions based on this triangulation. Two ways of making the approximation between the control polytope and the graphe closer are considered: by elevation of the degree, or by subdivision.

Shorter and adaptative certificates making visible that a polynomial is positivie on a simplex follow from the subdivision method.

The talk is based on Richard Leroy's recent Ph D thesis in Rennes.

## Referencias

[1] Richard Leroy. *Certificats de positivité et minimisation polynomiale dans la base de Bernstein multivariée.* Thèse, Université de Rennes 1, 2008.

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# Using the iterated ring of bounded elements for global optimization of polynomials

**Markus Schweighofer**

We consider the problem of computing numerically the global infimum of a real polynomial in several variables. The most naïve approach is to follow the direction of the gradient. There are several obvious problems with this method. For example one might end up in a local minimum which is not a global one. A completely different approach is to maximize a global lower bound for the polynomial which is certified in the sense that the polynomial minus this bound is a sum of squares of polynomials. This problem can be formulated as a semidefinite program (a generalization of a linear program) and therefore be solved efficiently. This approach has also big drawbacks which result from the fact that many globally positive polynomials can not be written as sums of squares of polynomials. Nie, Demmel and Sturmfels [1] have combined the advantages of both methods by working with sums of squares modulo the gradient ideal. Their method however cannot deal with the particularly hard class of polynomials that do not attain a minimum. In [2], we gave a variant of their method whose proof of correctness needs amongst others the whole algebraic theory of iterated rings of bounded elements [2]. Here we will present an even considerably better variant recently given by Hà Huy Vui and Pham Tién Son [3] that uses similar ideas and the same theoretic underpinning.

**Keywords:** global optimization, polynomial, preorder, sum of squares, semidefinite programming

**Mathematics Subject Classification 2000:** 13J30, 14P10, 90C26

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# Computing real radical ideals and real solving polynomial equations with semidefinite programming

**M. Laurent**[1]

We propose a numerical method for finding the real solutions to a system of polynomial equations $h_1 = 0, \ldots, h_m = 0$, as well as a (Gröbner or border) basis of the real radical $\sqrt[\mathbb{R}]{I}$ of the ideal $I = (h_1, \ldots, h_m)$ in the ring $\mathbb{R}[x_1, \ldots, x_n]$. This method assumes only that the number of real roots is finite and does not enumerate the complex roots whose number could be infinite. We represent the real radical ideal $\sqrt[\mathbb{R}]{I}$ as the kernel of a suitable positive semidefinite moment matrix, and use semidefinite optimization for finding such a matrix, combined with linear algebra techniques. The termination of our algorithm relies on results about moment matrices, in particular, about flat extensions and finite rank moment matrices. This algorithm can be adapted to find the complex roots of $I$, by omitting the positive semidefinite condition, in which case it returns a border base of an ideal $J$ nested between $I$ and its radical $\sqrt{I}$, with $J = I$ precisely when $\mathbb{R}[x_1, \ldots, x_n]/I$ is a Gorenstein algebra. The first stopping criterion for our algorithm is based on flat extensions of moment matrices; the second one is inspired by the elimination algorithm of Zhi and Reid [4] for complex roots and has earlier termination.

This work is joint with J.B. Lasserre and P. Rostalksi.

**Keywords:** Polynomial equations, real radical ideal, semidefinite programming

**Mathematics Subject Classification 2000:** 14P05, 13P10, 90C22

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# Exposed Faces and Lasserre's Semidefinite Representation of Sets

**Tim Netzer**[1]

Sets defined by *linear matrix inequalities* (LMI) have been studied intensively in recent years, see for example [1]. The reason is that they allow to use efficient algorithms for polynomial optimization. *Projections* of such sets, called *semidefinite representable sets*, are still of much interest. So far, no necessary conditions except convexity is known for semialgebraic sets to be semidefinite representable.

Lasserre has given an explicit lifted LMI-construction for a certain class of convex semialgebraic sets (see [2]). The construction uses generalized sums of squares representations of nonnegative polynomials.

We show that for a basic closed convex semi-algebraic set $S$, Lasserre's construction can only work if all faces of $S$ are exposed. This result matches well with the known fact that LMI-sets have only exposed faces. We discuss some explicit examples.

The result is joint work with Daniel Plaumann and Markus Schweighofer ([3]).

**Keywords:** semidefinite programming, convex sets, sums of squares, real algebra

**Mathematics Subject Classification 2000:** 90C22, 52A20, 11E25

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# Sard type theorems in Tame Variational Analysis

**Jerome Bolte**[1], **Aris Daniilidis**[†2], **Adrian Lewis**[3], **Masahiro Shiota**[4]

After a short preliminary introduction in Variational Analysis we shall present a Sard-type result concerning the Clarke critical values of a (nonsmooth) Lipschitz continuous tame function [1]. An interesting example showing the limitations of the theory will be discussed [2].

**Keywords:** Nonsmooth analysis, Subanalytic geometry, Sard theorem.

**Mathematics Subject Classification 2000:** 49J52, 35B38, 32B20

## Referencias

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