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A New Formulation of the Equivalent Thermal in Optimization of Hydrothermal Systems

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In this paper, we revise the classical formulation of the problem depriving it of the concepts that are superfluous from the mathematical point of view. We observe that a number of power stations can be substituted by a single one that behaves equivalently to the entire set. Proceeding in this way, we obtain a variational formulation in its purest sense (without restrictions). This formulation allows us to employ the theory of calculus of variations to the highest degree. We then calculate the equivalent minimizer in the case where the cost functions are second-order polynomials. We prove that the equivalent minimizer is a second-order polynomial with piece-wise constant coefficients. Moreover, it belongs to the class C^1. Finally, we present various examples prompted by real systems and perform the proposed algorithms using Mathematica.

Key words: Optimization; Hydrothermal systems; Equivalent thermal; Restrictions

1 INTRODUCTION

This work is embedded in the line of research entitled "Optimization of hydrothermal systems". The study of optimal conditions for the functioning of a hydrothermal system constitutes a complicated problem which has attracted significant interest in recent decades. Various techniques have been applied to solve the problem, such as functional analysis techniques [1] and [2], calculus of variations [3] and [4], dynamic programming, both linear and nonlinear, approximation methods such as Ritz's method [5], neuronal nets, etc. These studies contain numerous mathematical models which approximate reality to various degrees of precision. For instance, one may consider hydraulic power stations of fixed or variable load, connected or disconnected, connected in a line or forming a net, with or without transport delays, with transmission losses taken into account or neglected, and so on.

Such a variety of mathematical models requires a comprehensive study of the problem that may be extended to all hydrothermal systems.

In this paper we prove that power stations can be substituted by a single one that behaves equivalently to the entire set. This supposes a significant simplification of hydrothermal models and will also be useful for any method used to study the problem. It allows us to develop algorithms that are simpler, more reliable, and which require less time for their execution; this last feature is of crucial importance, for instance, in the problem of Economic Dispatching. Apart from the computational advantages which consist in the possibility of reducing the number of unknowns, there are other benefits, because an incorrect formulation can lead to important errors.

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The idea of introducing an equivalent thermal station is relatively old. Reference [1] considers it in application to purely thermal problems, though the authors did not notice the need to define the equivalent thermal station piece-wisely. This is to be expected since the restriction of power positivity is forgotten. It is most important to stress the fact that proceeding in this way can lead to incorrect conclusions. We will see in the present study how to circumvent this problem in a definitive way.

The idea was also used in problems with hydraulic components, which is where it attains its true value. For example, [6] describe the application of the discrete maximum principle to the problem of optimum scheduling of multi-reservoir systems with a model two-reservoir system and one equivalent thermal plant.

On the other hand, [7] considers the application of a modified algorithm based on Pontryagin's maximum principle to the solution of the problem of optimum economical load distribution in a power system comprising an equivalent thermal power station and five river-driven hydro stations in series.

The concept of the equivalent thermal station has been used up until now. However, many authors do not take into account certain aspects of this approach that we discuss in this study. Thus, [8] and [9] develop a short-term hydrothermal scheduling algorithm based on the simulated annealing technique. In the algorithm, the load balance constraint, total water discharge constraint, reservoir volume limits and constraint on the operation limits of the hydrothermal generator and the equivalent thermal generator are fully accounted for. A relaxation method for checking the limits is proposed and included in the algorithm. In Ref. [10] an efficient short-term hydrothermal scheduling algorithm based on the evolutionary programming (EP) technique is proposed. In the algorithm, the thermal generating units in the system are represented by an equivalent unit. The power balance constraints, total water discharge constraint, reservoir volume constraints and the constraints on the operation limits of the equivalent thermal and hydro units are fully taken into account. The effectiveness of the proposed algorithm is demonstrated through an example system and the results are compared with those obtained by the classical gradient search and simulated annealing (SA) approaches.

We are going to add an important mathematical component to this study by proving substantial results ignored until now. Let us briefly present the contents of the rest of this study.

First, we revise the classical formulation of the problem depriving it of the concepts that are superfluous from the mathematical point of view. We observe that a number of power stations can be substituted by a single one that behaves equivalently to the entire set. Proceeding in this way, we obtain a variational formulation in its purest sense (without restrictions). This formulation allows us to employ the theory of calculus of variations to the highest degree, and the problem is thus afforded a significant simplification.

Then, we shall calculate the equivalent minimizer in the case where the cost functions are second-order polynomials. We prove that the equivalent minimizer is a second-order polynomial with piece-wise constant coefficients. Moreover, it belongs to the class C^1.

Finally, we present various examples prompted by real systems and perform the proposed algorithms using Mathematica. We consider a wide variety of situations, which in all the cases are resolved in a very satisfactory way, comparing with the methods employed by other authors and stressing the advantages obtained.

2 DESCRIPTION OF THE PROBLEM

A hydrothermal system is made up of hydraulic and thermal power stations which must jointly satisfy a certain demand for electrical power during a definite time interval.

Thermal stations generate power at the expense of fuel consumption (which is the object of minimization), while hydraulic stations obtain it from the energy liberated by water that moves a turbine; a limited amount of water being available during the optimization period. Let us now see the definitions of the elements which are present in any problem of hydrothermal optimization.

Let us assume that a hydrothermal system accounts for m thermal stations and n hydraulic stations.

DEFINITION *The cost function of the ith thermal power station is the mapping*

$$F_i: D_i \subset \mathbb{R} \to \mathbb{R}$$

that relates the instant consumption of the ith thermal station with the power it contributes. So that if we denote by $P_i(t)$ the power generated at the instant t, the consumption during the optimization interval $[0, T]$ is

$$\int_0^T F_i(P_i(t))\,\mathrm{d}t$$

By D_i we denote the set of values of the power which can be generated by the ith thermal power station.

DEFINITION *The function $P: [0, T] \to \mathbb{R}$ is said to be admissible for F_i if*

$$\forall t \in [0, T] \quad P(t) \in D_i$$

The set of elements admissible for F_i is denoted by $\mathcal{F}_i$ and $\mathcal{F} = \prod_{i=1}^{m} \mathcal{F}_i$.

DEFINITION *We call function of effective hydraulic contribution the mapping $H: \Omega_H \to \mathbb{R}$ where*

$$H(t, z_1(t), z_2(t), \ldots, z_n(t), z_1'(t), z_2'(t), \ldots, z_n'(t))$$

is the power contributed to the system at the instant t by the set of hydraulic power stations, being: $z_i(t)$ the volume which is turbined by the instant t (in what follows simply volume) by the ith power station, $z_i'(t)$ the rate of water discharge at the instant t by the ith power station and $\Omega_H \subset [0, T] \times \mathbb{R}^{2n}$ the domain of definition of H.

DEFINITION *We say that $\vec{Z} = (z_1, z_2, \ldots, z_n)$ is admissible for H if:*

(i) *z_i belong to the class KC^1, $\forall i = 1, \ldots, n$, (Continuous with the piece-wise continuous derivatives).*
(ii) $(t, z_1(t), \ldots, z_n(t), z_1'(t), \ldots, z_n'(t)) \in \Omega_H, \ \forall t \in [0, T]$.

The assumption that the derivatives of the admissible functions need not be continuous is natural because it is equivalent to the assumption that the rate of water discharge has abrupt variations, which is, in turn, possible in practice.

The set of the elements admissible for H is denoted by $\mathcal{H}$.

OBSERVATION *All the models rely on the hypothesis that at every instant of time the generated power $H(t)$ is a function of the rate of water discharge $q(t)$ and of the height of the water jump*:

$$H(t) = \Phi(q(t), h(t))$$

We will try to unify from our general point of view the diverse studies which differ in the way of modeling the water height depending on the deposit geometry, in the technical characteristics of the power station, in the choice of the simplifying hypothesis of the model, etc.

DEFINITION *The function*

$$P_d\colon [0, T] \to \mathbb{R}$$

expressing the power that has to be delivered to the system at each instant of the optimization interval $[0, T]$ *is called the demanded power.*

DEFINITION *The volume b_i that can be turbined by the instant T is called the admissible volume of the ith hydraulic power station.*

DEFINITION *Under the above notation, let*

$$\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$$

be the vector of admissible volumes. We term the generalized hydrothermal problem the problem

$$\Gamma \equiv H_n\text{–}T_m\{P_d, \{F_i\}_1^m, H, \vec{b}\}$$

of minimizing the functional

$$F(y_1, \ldots, y_m, z_1, \ldots, z_n) = \int_0^T \sum_{i=1}^m F_i(y_i(t))\,\mathrm{d}t$$

over the set Ξ of admissible elements for Γ, defined in the following way:

$$\Xi = \{(y_1, \ldots, y_m, z_1, \ldots, z_n) \in \mathcal{T} \times \mathcal{H} \mid \text{satisfy (1) and (2)}\}$$

(1) *the equilibrium equation*

$$\sum_{i=1}^m y_i(t) + H(t, z_1(t), \ldots, z_n(t), z_1'(t), \ldots, z_n'(t)) = P_d(t), \quad \forall t \in [0, T]$$

(2) *restrictions on the admissible volume*

$$z_i(0) = 0, \quad z_i(T) = b_i \quad (i = 1, \ldots, n)$$

DEFINITION *We say that* $(P_1, \ldots, P_m, \vec{Q}) \in \Xi$ *is a solution of the problem if* $\forall (y_1, \ldots, y_m, \vec{R}) \in \Xi$

$$\int_0^T \sum_{i=1}^m F_i(P_i(t))\,\mathrm{d}t \le \int_0^T \sum_{i=1}^m F_i(y_i(t))\,\mathrm{d}t$$

3 REDUCTION OF PROBLEM H_n–T_m TO H_n–T_1

In this section we consider the possibility of substituting a problem with m thermal power stations (H_n–T_m) by an equivalent problem (H_n–T_1) with a single thermal power station: the equivalent thermal plant.

It should be stressed that the theoretical result we are going to establish cannot be found in the literature and that the theorem is of a general character because it does not depend on the models used.

Let $F_i: D_i \subseteq \mathbb{R} \to \mathbb{R} (i = 1, \ldots, m)$ be the cost functions of the thermal power stations.

Throughout the paragraph we assume that

$$\forall \xi \in D = D_1 + D_2 + \cdots + D_m \subseteq \mathbb{R} \quad \exists\, |(\xi_1, \ldots, \xi_m) \in \prod_{i=1}^m D_i \text{ such that}$$

$$\sum_{i=1}^m \xi_i = \xi \quad \text{and} \quad \sum_{i=1}^m F_i(\xi_i) = \min_{\sum_{i=1}^m x_i = \xi} \left[\sum_{i=1}^m F_i(x_i) \right]$$

In general, we shall assume that $\forall \xi \in D$, the problem of minimization of the function $\sum_{i=1}^m F_i(x_i)$ under the condition $\sum_{i=1}^m x_i = \xi$, has a unique[1] solution $(\xi_1, \ldots, \xi_m)$.

DEFINITION *Let us call the ith distribution function the function* $\Psi_i: D_1 + D_2 + \cdots + D_m \to D_i$ *defined by* $\Psi_i(\xi) = \xi_i$, $\forall i = 1, \ldots, m$ *where* $(\xi_1, \ldots, \xi_m)$ *is the unique minimum of* $\sum_{i=1}^m F_i(x_i)$ *subject to the condition* $\sum_{i=1}^m x_i = \xi$.

DEFINITION *We will call equivalent minimizer of* $\{F_i\}_1^m$, *the function* $\Psi: D_1 + D_2 + \cdots + D_m \to \mathbb{R}$ *defined by*

$$\Psi(\xi) = \min_{\sum_{i=1}^m x_i = \xi} \left[\sum_{i=1}^m F_i(x_i) \right]$$

OBSERVATION *It follows that* $\sum_{i=1}^m \Psi_i(\xi) = \xi$ *and* $\sum_{i=1}^m F_i(\Psi_i(\xi)) = \Psi(\xi)$

THEOREM 1 *Let* Ψ *be the equivalent minimizer of* $\{F_i\}_1^m$ *and* $\{\Psi_i\}_1^m$ *be the distribution functions. If* $(P(t), \vec{Q}(t))$ *is a solution of the problem*

$$\Lambda^* \equiv H_n\text{–}T_1\{P_d, \{\Psi\}, H, \vec{b}\}$$

[1]The uniqueness is not essential. Its lack can be understood as the existence of various distribution functions.

then $(\Psi_1(P(t)), \Psi_2(P(t)), \ldots, \Psi_m(P(t)), \vec{Q}(t))$ *is a solution of the problem*

$$\Lambda \equiv \mathrm{H}_n\text{–}\mathrm{T}_m\{P_d, \{F_i\}_1^m, H, \vec{b}\}$$

Proof If $P(t), \vec{Q}(t))$ is a solution of Λ^*, then

$$\begin{cases} P(t) + H(t, \vec{Q}(t), \vec{Q}'(t)) = P_d(t), & \forall t \in [0, T] \\ \vec{Q}(0) = \vec{0}, \quad \vec{Q}(T) = \vec{b}. \end{cases}$$

Let us first check that $(\Psi_1(P(t)), \Psi_2(P(t)), \ldots, \Psi_m(P(t)), \vec{Q}(t))$ is admissible and then prove that it is a solution. Indeed, $(\Psi_1(P(t)), \Psi_2(P(t)), \ldots, \Psi_m(P(t)), \vec{Q}(t))$ is admissible for Λ because

$$\sum_{i=1}^{m} \Psi_i(P(t)) = P(t), \quad \forall t \in [0, T]$$

which yields the validity of the equilibrium equation for Λ, and $\vec{Q}$ readily implies the fulfillment of the boundary conditions.

It remains to see that $(\Psi_1(P(t)), \Psi_2(P(t)), \ldots, \Psi_m(P(t)), \vec{Q}(t))$ minimizes the functional

$$\int_0^T \sum_{i=1}^{m} F_i(y_i(t))\,\mathrm{d}t$$

We shall argue to this end by contradiction.

Let us assume that there exists $(\tilde{P}_1, \ldots, \tilde{P}_m, \vec{R}(t))$, admissible for Λ,

$$\sum_{i=1}^{m} \tilde{P}_i(t) + H(t, \vec{R}(t), \vec{R}'(t)) = P_d(t), \quad \forall t \in [0, T]$$

with $\vec{R}(0) = \vec{0}$, $\vec{R}(T) = \vec{b}$, satisfying

$$\int_0^T \sum_{i=1}^{m} F_i(\tilde{P}_i(t))\,\mathrm{d}t < \int_0^T \sum_{i=1}^{m} F_i(\Psi_i(P(t))\,\mathrm{d}t$$

This affords the desired contradiction.

Indeed, if we consider $\tilde{P}(t) = \sum_{i=1}^{m} \tilde{P}_i(t)$, then $\forall t \in [0, T]$, $\tilde{P}(t)$ is admissible for Λ^* because

$$\begin{cases} \tilde{P}(t) + H(t, \vec{R}(t), \vec{R}'(t)) = P_d(t), & \forall t \in [0, T] \\ \vec{R}(0) = \vec{0}, \quad \vec{R}(T) = \vec{b} \end{cases}$$

Moreover, we have that

$$\Psi(\tilde{P}(t)) = \sum_{i=1}^{m} F_i(\Psi_i(\tilde{P}(t))) = \min_{\sum_{i=1}^{m} x_i = \tilde{P}(t)} \left[\sum_{i=1}^{m} F_i(x_i) \right] \le \sum_{i=1}^{m} F_i(\tilde{P}_i(t)), \quad \forall t \in [0, T]$$

hence

$$\int_0^T \Psi(\tilde{P}(t))\,\mathrm{d}t \le \int_0^T \sum_{i=1}^{m} F_i(\tilde{P}_i(t))\,\mathrm{d}t < \int_0^T \sum_{i=1}^{m} F_i(\Psi_i(P(t)))\,\mathrm{d}t = \int_0^T \Psi(P(t))\,\mathrm{d}t$$

which contradicts the assumption that $P(t)$ delivers the minimum. ■

OBSERVATION *In the case of the "purely thermal" problem* $\mathrm{H}_0\text{–}\mathrm{T}_m\{P_d, \{F_i\}_1^m\}$, *one can consider the equivalent problem* $\hat{\Gamma} \equiv \mathrm{H}_0\text{–}\mathrm{T}_1\{P_d, \{\Psi\}\}$, *where* Ψ *represents the equivalent minimizer for* $\{F_i\}_1^m$.

So that if $P(t)$ is a solution of problem $\hat{\Gamma}$ and Ψ_i are the distribution functions, then

$$(\Psi_1(P(t)), \Psi_2(P(t)), \ldots, \Psi_m(P(t))) \text{ is a solution of } \Gamma.$$

Now taking into account the fact that $\hat{\Gamma}$ can only have a sole admissible function, say, P_d, this function will be the solution. This fact implies that the solution of the original problem will be

$$(\Psi_1(P_d(t)), \Psi_2(P_d(t)), \ldots, \Psi_m(P_d(t)))$$

OBSERVATION *The possibility of constructing an equivalent minimizer is due to the fact that knowing the instant contribution of each of the thermal power stations of the set, one can assign to every thermal power station the generation of power that delivers minimum to the global instant cost of the fuel. Such a possibility does not exist in the case of hydraulic stations. This is because there is no reasonable way to organize "the distribution" on the basis of the knowledge of the power generated by all the hydraulic power stations of the set. Nonetheless, this is true if the inverse problem is considered: to minimize the total water consumption under the assumption that the fuel cost is prescribed.*

Now that it has been demonstrated that any problem of the type ($\mathrm{H}_n\text{–}\mathrm{T}_m$) can be substituted by another of the form ($\mathrm{H}_n\text{–}\mathrm{T}_1$), we can concentrate on the problems of this type which constitute the true variational problem. Moreover, this variational problem has a binding condition (the equilibrium equation)

$$y(t) + H(t, \vec{Z}(t), \vec{Z}'(t)) = P_d(t), \quad \forall t \in [0, T]$$

which can be omitted, together with the unknown function $y(t)$, thus yielding the problem of minimizing the functional

$$F(\vec{Z}) = F(z_1, \ldots, z_n) = \int_0^T \Psi(P_d(t) - H(t, \vec{Z}(t), \vec{Z}'(t)))\,\mathrm{d}t$$

with the boundary conditions $\vec{Z}(0) = \vec{0}, \vec{Z}(T) = \vec{b}$.

The unknown $y(t)$, which disappear in the new formulation and represents the contribution of the thermal power stations of the system, can be recovered once the values of the other unknowns are established. To define the partial contribution of each of these, one has to make use of the contribution functions.

4 EQUIVALENT MINIMIZER FOR THE COST FUNCTIONS: $F_i(x) = \alpha_i + \beta_i x + \gamma_i x^2$

Let us calculate the equivalent minimizer in the case where the cost functions are second-order polynomials. This assumption is systematically accepted in the studies on this topic. Moreover, let us impose the natural restriction of positivity of the thermal power.

Let $F_i(x) = \alpha_i + \beta_i x + \gamma_i x^2 \ \forall i = 1, \dots, m$ where $\beta_i \geq 0, \gamma_i > 0$ and $D_i = [0, \infty)$ are their respective domains of definition. Without loss of generality, we assume that $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$.

LEMMA 1 *If $\beta_i \leq \beta_j$ and the function $F: D_1 \times D_2 \times \cdots \times D_m \to \mathbb{R}$,*

$$F(x_1, \dots, x_m) = \sum_{i=1}^{m} F_i(x_i)$$

attains at $(a_1, \dots, a_m)$ the minimum over the set

$$C_a = \left\{ (x_1, \dots, x_m) \mid x_i \geq 0 \wedge \sum_{i=1}^{m} x_i = a \right\}$$

then

$$a_i = 0 \Longrightarrow a_j = 0$$

Proof We shall argue by contradiction. Let us assume that $\beta_i \leq \beta_j$, $a_i = 0$ and that $a_j > 0$. Consider the function

$$f(\varepsilon) = F(a_1, \dots, a_i + \varepsilon, \dots, a_j - \varepsilon, \dots, a_m) - F(a_1, \dots, a_m)$$
$$f(\varepsilon) = F_i(a_i + \varepsilon) + F_j(a_j - \varepsilon) - F_i(a_i) - F_j(a_j)$$

It is clear that if $(a_1, \dots, a_m) \in C_a$, then $(a_1, \dots, a_i + \varepsilon, \dots, a_j - \varepsilon, \dots, a_m) \in C_a$ for $0 \leq \varepsilon < a_j$.

Let us show the existence of an ε such that $f(\varepsilon) < 0$, which contradicts the fact that F has a minimum in $(a_1, \dots, a_m)$ within C_a.

We have that f is continuous and derivable at zero with $f(0) = 0$; therefore it suffices to observe that $f'(0) < 0$. In fact,

$$f'(\varepsilon) = F_i'(a_i + \varepsilon) - F_j'(a_j - \varepsilon)$$
$$f'(\varepsilon) = \beta_i - \beta_j + 2\gamma_i(a_i + \varepsilon) - 2\gamma_j(a_j - \varepsilon)$$

and, taking into account the fact that $a_i = 0$, we have

$$f'(0) = \beta_i - \beta_j - 2\gamma_j \cdot a_j < 0 \qquad \blacksquare$$

The significance of this Lemma is the following: to provide the optimal functioning of the thermal power stations once one of them is disconnected (*i.e.* generates zero), we have to disconnect as well all the stations whose coefficients β_i are greater than or equal to the coefficient corresponding to the inactive station.

LEMMA 2 *If $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$, then the parameters*

$$\delta_k = \frac{1}{2}\left[\beta_k \sum_{j=1}^{k} \frac{1}{\gamma_j} - \sum_{j=1}^{k} \frac{\beta_j}{\gamma_j}\right]$$

satisfy

$$0 = \delta_1 \leq \delta_2 \leq \cdots \leq \delta_m$$

Proof

$$\delta_{k+1} = \frac{1}{2}\left[\beta_{k+1} \sum_{j=1}^{k+1} \frac{1}{\gamma_j} - \sum_{j=1}^{k+1} \frac{\beta_j}{\gamma_j}\right] = \frac{1}{2}\left[\beta_{k+1} \sum_{j=1}^{k} \frac{1}{\gamma_j} - \sum_{j=1}^{k} \frac{\beta_j}{\gamma_j}\right] \geq \delta_k$$ ■

LEMMA 3 *The function*

$$F(x_1, \ldots, x_m) = \sum_{i=1}^{m} F_i(x_i)$$

attains the minimum over the set

$$C_\xi = \left\{(x_1, \ldots, x_m) \mid x_i \geq 0 \wedge \sum_{i=1}^{m} x_i = \xi\right\}$$

at the point $(a_1, \ldots, a_m) \in \mathring{C}_\xi (a_i > 0\ \forall i)$ if

$$\xi > \frac{1}{2}\left[\beta_m \sum_{j=1}^{m} \frac{1}{\gamma_j} - \sum_{j=1}^{m} \frac{\beta_j}{\gamma_j}\right] = \delta_m$$

Proof (*Necessity*) If $(a_1, \ldots, a_m)$ is an interior point where F attains its minimum, it is a point of relative minimum of F on the set

$$\left\{(x_1, \ldots, x_m) \mid \sum_{i=1}^{m} x_i = \xi\right\}$$

It follows that for some $\lambda \in \mathbb{R}$, it is a critical point of

$$F^*(x_1, \ldots, x_m) = F(x_1, \ldots, x_m) - \lambda(x_1 + \cdots + x_n - \xi)$$

We denote by Π the function $\prod_{j=1}^{m} \gamma_j$ and by Γ_i the function Π/γ_i. Using the Lagrange multipliers method we have that

$$\left.\begin{array}{r} \beta_1 + 2\gamma_1 x_1 - \lambda = 0 \\ \beta_2 + 2\gamma_2 x_2 - \lambda = 0 \\ \vdots \\ \beta_m + 2\gamma_m x_m - \lambda = 0 \\ x_1 + x_2 + \cdots + x_m = \xi \end{array}\right\} \Longrightarrow \left\{\begin{array}{r} \beta_1\Gamma_1 + 2\Pi x_1 - \lambda\Gamma_1 = 0 \\ \beta_2\Gamma_2 + 2\Pi x_2 - \lambda\Gamma_2 = 0 \\ \vdots \\ \beta_m\Gamma_m + 2\Pi x_m - \lambda\Gamma_m = 0 \\ x_1 + x_2 + \cdots + x_m = \xi \end{array}\right.$$

hence

$$\lambda = \frac{\sum_{i=1}^{m} \beta_i \Gamma_i + 2\Pi\xi}{\sum_{i=1}^{m} \Gamma_i} = \frac{\sum_{i=1}^{m} \beta_i(\Pi/\gamma_i) + 2\Pi\xi}{\sum_{i=1}^{m}(\Pi/\gamma_i)} = \frac{\sum_{i=1}^{m} \beta_i(1/\gamma_i) + 2\xi}{\sum_{i=1}^{m}(1/\gamma_i)}$$

Let us consider $\Psi_k(\xi)$ as a function of the unknown x_k

$$\Psi_k(\xi) = x_k = \frac{\sum_{i=1}^{m} \beta_i(1/\gamma_i) + 2\xi}{2\gamma_k \sum_{i=1}^{m}(1/\gamma_i)} - \frac{\beta_k}{2\gamma_k}$$

$$\Psi_k(\xi) = \frac{\sum_{i=1}^{m} \beta_i(1/\gamma_i) + 2\xi}{2\gamma_m \sum_{i=1}^{m}(1/\gamma_i)} - \frac{\beta_k}{2\gamma_k} = 0 \Longleftrightarrow \frac{1}{2}\left[\sum_{i=1}^{m}\frac{\beta_k}{\gamma_i} - \sum_{i=1}^{m}\frac{\beta_i}{\gamma_i}\right] = \xi$$

Letting

$$\Delta_k = \frac{1}{2}\left[\sum_{i=1}^{m}\frac{\beta_k}{\gamma_i} - \sum_{i=1}^{m}\frac{\beta_i}{\gamma_i}\right]$$

we see that

$$\Delta_1 \leq \Delta_2 \leq \cdots \leq \Delta_m = \delta_m$$

It is evident that for every k, the solution $\Psi_k(\xi)$ is strictly increasing as a function of ξ. Thus

$$\xi \leq \delta_m \Longrightarrow \Psi_m(\xi) = a_m \leq \Psi_m(\delta_m) = 0$$

or, conversely,

$$\Psi_m(\xi) = a_m > 0 \Longrightarrow \xi > \delta_m$$

(*Sufficiency*) Since C_ξ is compact, the minimum of F clearly exists. If $\xi > \delta_m$, then

$$\xi > \Delta_k, \quad \forall k = 1, \ldots, m$$

Let us now consider

$$(a_1, \ldots, a_m) = (\Psi_1(\xi), \ldots, \Psi_m(\xi))$$

a critical point of the convex functional

$$F^*(x_1, \ldots, x_m) = F(x_1, \ldots, x_m) - \lambda(x_1 + \cdots + x_n - \xi)$$

where

$$\lambda = \frac{\sum_{i=1}^{m} \beta_i(1/\gamma_i) + 2\xi}{\sum_{i=1}^{m}(1/\gamma_i)}$$

We have that $(a_1, \ldots, a_m)$ delivers the minimum value to F^* and, consequently, it is also the minimum of F under the restriction

$$\left\{(x_1, \ldots, x_n) \mid \sum_{i=1}^{m} x_i = \xi\right\}$$

Moreover,

$$\xi > \Delta_k \Longrightarrow \Psi_k(\xi) = a_k > 0$$

so that $(a_1, \ldots, a_m) \in \overset{\circ}{C}_\xi$. ■

PROPOSITION 1 *For every $k = 1, \ldots, m$ the kth distribution function is*

$$\Psi_k(\xi) = \begin{cases} \dfrac{\sum_{i=1}^{j}(\beta_i/\gamma_i) + 2\xi}{2\gamma_k \sum_{i=1}^{j}(1/\gamma_i)} - \dfrac{\beta_k}{2\gamma_k} & \text{if } \delta_k \leq \delta_j \leq \xi < \delta_{j+1} \\ 0 & \text{if } \xi < \delta_k \end{cases}$$

with the coefficients δ_k

$$\delta_k = \frac{1}{2}\left[\beta_k \sum_{i=1}^{k} \frac{1}{\gamma_i} - \sum_{i=1}^{k} \frac{\beta_i}{\gamma_i}\right]$$

Proof In view of Lemma 3, if $\xi > \delta_m$, then the distribution functions $\Psi_k(\xi)$ are strictly positive for all k and it remains to derive the expression for x_k. If $\delta_{m-1} < \xi \leq \delta_m$, then the minimum of $\sum_{i=1}^{m} F_i(x_i)$ cannot be attained in the interior. According to Lemma 1, at least x_m must vanish. Thus, $\Psi_m(\xi) = 0$.

The same argument applies to the remaining problem of dimension $m - 1$

$$\Psi_k(\xi) = \frac{\sum_{i=1}^{m-1}(\beta_i/\gamma_i) + 2\xi}{2\gamma_k \sum_{i=1}^{m-1}(1/\gamma_i)} - \frac{\beta_k}{2\gamma_k}$$

If $\delta_{m-2} < \xi \leq \delta_{m-1}$, then $\Psi_m(\xi) = 0$ and, arguing as above, $\Psi_{m-1}(\xi) = 0$, and for $k < m - 1$ one has

$$\Psi_k(\xi) = \frac{\sum_{i=1}^{m-2}(\beta_i/\gamma_i) + 2\xi}{2\gamma_k \sum_{i=1}^{m-2}(1/\gamma_i)} - \frac{\beta_k}{2\gamma_k}$$

Finally, repeating the argument once again, we have that if $\delta_j < \xi \leq \delta_{j+1}$, then the kth distribution function is equal to zero if $\xi < \delta_k$, and if $\delta_k \leq \xi (k = 1, \dots, j)$,

$$\Psi_k(\xi) = \frac{\sum_{i=1}^{j}(\beta_i/\gamma_i) + 2\xi}{2\gamma_k \sum_{i=1}^{j}(1/\gamma_i)} - \frac{\beta_k}{2\gamma_k} \qquad \blacksquare$$

In short, if $\delta_j \leq \xi < \delta_{j+1}$, then "the distribution" is performed between the first j thermal power stations, while the rest of them remains inactive.

THEOREM 2 *The equivalent minimizer is a second-order polynomial with piece-wise constant coefficients*

$$\Psi(\xi) = \sum_{i=1}^{m} F_i(\Psi_i(\xi)) = \tilde{\alpha}_k + \tilde{\beta}_k\xi + \tilde{\gamma}_k\xi^2 \quad \text{if } \delta_k \leq \xi < \delta_{k+1}$$

$$\tilde{\gamma}_k = \frac{1}{\sum_{i=1}^{k}(1/\gamma_i)}; \quad \tilde{\beta}_k = \tilde{\gamma}_k \sum_{i=1}^{k} \frac{\beta_i}{\gamma_i}; \quad \tilde{\alpha}_k = \sum_{i=1}^{m} \alpha_i + \frac{\tilde{\beta}_k^2}{4\tilde{\gamma}_k} - \sum_{i=1}^{k} \frac{\beta_i^2}{4\gamma_i}$$

Moreover, it belongs to the class C^1 and $\Psi'(\delta_k) = \beta_k$ for $i = 1, \dots, m$.

Proof It is evident that Ψ is piece-wise equal to a second-order polynomial. The coefficients $\tilde{\alpha}_m$, $\tilde{\beta}_m$ and $\tilde{\gamma}_m$ are obtained without difficulty in Ref. [1], where the question was treated without taking into account any restrictions. It is clear that in our case, the coefficients result from changing the sub-indexes m by k, the exception is the expression for $\tilde{\alpha}_k$, preserving the sum with the upper limit m which represents the fuel consumptions of the disconnected power stations (with zero generation).

Let us now see that the lateral limits of Ψ and Ψ' coincide with the only conflictive points δ_k. First of all, let us observe that

$$\delta_k = \frac{1}{2}\left[\beta_k \sum_{i=1}^{k} \frac{1}{\gamma_i} - \sum_{i=1}^{k} \frac{\beta_i}{\gamma_i}\right] = \frac{1}{2}\left[\frac{\beta_k - \tilde{\beta}_k}{\tilde{\gamma}_k}\right]$$

$$\delta_k = \frac{1}{2}\left[\beta_k \sum_{i=1}^{k-1} \frac{1}{\gamma_i} - \sum_{i=1}^{k-1} \frac{\beta_i}{\gamma_i}\right] = \frac{1}{2}\left[\frac{\beta_k - \tilde{\beta}_{k-1}}{\tilde{\gamma}_{k-1}}\right]$$

$$\sum_{i=1}^{k-1} \frac{\beta_k^2}{4\gamma_i} - \sum_{i=1}^{k-1} \frac{\beta_i^2}{4\gamma_i} = \sum_{i=1}^{k} \frac{\beta_k^2}{4\gamma_i} - \sum_{i=1}^{k} \frac{\beta_i^2}{4\gamma_i}$$

Let us check that the lateral limits of Ψ coincide at δ_k:

$$\Psi(\delta_k^+) = \tilde{\alpha}_k + \tilde{\beta}_k \frac{1}{2}\left[\frac{\beta_k - \tilde{\beta}_k}{\tilde{\gamma}_k}\right] + \tilde{\gamma}_k \frac{1}{2}\left[\frac{\beta_k - \tilde{\beta}_k}{\tilde{\gamma}_k}\right]^2$$

$$\Psi(\delta_k^+) = \sum_{i=1}^{m} \alpha_i + \frac{\tilde{\beta}_k^2}{4\tilde{\gamma}_k} - \sum_{i=1}^{k} \frac{\beta_i^2}{4\gamma_i} + \frac{[\beta_k^2 - \tilde{\beta}_k^2]}{4\tilde{\gamma}_k} = \sum_{i=1}^{m} \alpha_i - \sum_{i=1}^{k} \frac{\beta_i^2}{4\gamma_i} + \frac{\beta_k^2}{4\tilde{\gamma}_k}$$

By analogy

$$\Psi(\delta_k^-) = \tilde{\alpha}_{k-1} + \tilde{\beta}_{k-1}\frac{1}{2}\left[\frac{\beta_k - \tilde{\beta}_{k-1}}{\tilde{\gamma}_{k-1}}\right] + \tilde{\gamma}_{k-1}\frac{1}{2}\left[\frac{\beta_k - \tilde{\beta}_k}{\tilde{\gamma}_k}\right]^2$$

$$\Psi(\delta_k^-) = \sum_{i=1}^{m}\alpha_i - \sum_{i=1}^{k-1}\frac{\beta_i^2}{4\gamma_i} + \frac{\beta_k^2}{4\tilde{\gamma}_{k-1}}$$

Finally:

$$\Psi(\delta_k^-) = \sum_{i=1}^{m}\alpha_i - \sum_{i=1}^{k-1}\frac{\beta_i^2}{4\gamma_i} + \frac{1}{4}\sum_{i=1}^{k-1}\frac{\beta_k^2}{\gamma_i} = \sum_{i=1}^{m}\alpha_i - \sum_{i=1}^{k}\frac{\beta_i^2}{4\gamma_i} + \frac{1}{4}\sum_{i=1}^{k}\frac{\beta_k^2}{\gamma_i} = \Psi(\delta_k^+)$$

Finally, let us see that the lateral derivatives of Ψ at the points δ_k coincide as well:

$$\Psi'(\delta_k^+) = \tilde{\beta}_k + 2\tilde{\gamma}_k\delta_k = \tilde{\beta}_k + \beta_k - \tilde{\beta}_k = \beta_k$$

$$\Psi'(\delta_k^-) = \tilde{\beta}_{k-1} + 2\tilde{\gamma}_{k-1}\delta_k = \tilde{\beta}_{k-1} + \beta_k - \tilde{\beta}_{k-1} = \beta_k$$

■

5 EXAMPLES

Let us now see a few examples which illustrate the practical importance of the results established. First, let us calculate the equivalent minimizer for the following four cost functions under the restriction of their positivity throughout the domain. The functions are second-order polynomials and the coefficients are:

i	α_i	β_i	γ_i
1	0	4	0.001
2	0	4.2	0.002
3	0	4.5	0.001
4	0	4.7	0.002

The units for the coefficients of the thermal plants are: α in ($\$/h$); β in ($\$/h.Mw$) and γ in ($\$/h.MW^2$). The coefficients δ_k, which appeared in the previous section, and the coefficients $\tilde{\alpha}_k$, $\tilde{\beta}_k$ and $\tilde{\gamma}_k$ of the second-order polynomial Ψ are in this case:

k	δ_k	$\tilde{\alpha}_k$	$\tilde{\beta}_k$	$\tilde{\gamma}_k$
1	0	0	4	0.001
2	100	−3.33333	4.06667	0.000666667
3	325	−31.5	4.24	0.0004
4	575	−53.5417	4.31667	0.000333333

and the equivalent minimizer $\Psi(\$/h)$ with ξ in Mw:

$$\Psi(\xi) = \begin{cases} 4\xi + 0.001\xi^2 & \text{if} \quad 0 \le \xi \le 100 \\ -3.33333 + 4.06667\xi + 0.000666667\xi^2 & \text{if } 100 \le \xi \le 325 \\ -31.5 + 4.24\xi + 0.0004\xi^2 & \text{if } 325 \le \xi \le 575 \\ -53.5417 + 4.31667\xi + 0.000333333\xi^2 & \text{if } 575 \le \xi \end{cases}$$

is a second-order polynomial with piece-wise constant coefficients as is shown in Figure 1.

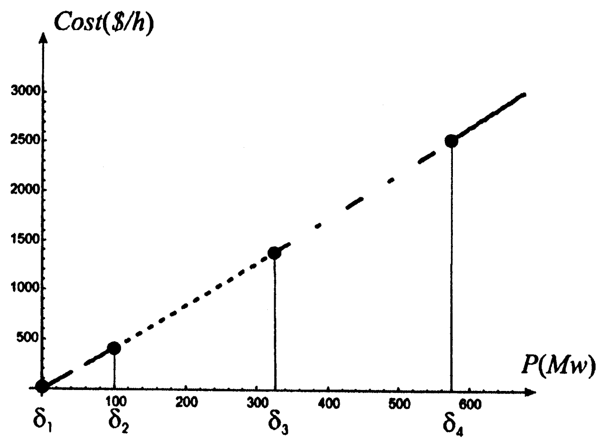

FIGURE 1 The equivalent minimizer.

On the other hand, since the restriction of power positivity has been omitted, the classical solution is:

$$\Psi_C(\xi) = -53.5417 + 4.31667\xi + 0.000333333\xi^2 \quad \text{for all } \xi$$

We now compare both solutions for different values of ξ, calculating the four distribution functions $\Psi_1(\xi)$, $\Psi_2(\xi)$, $\Psi_3(\xi)$, $\Psi_4(\xi)$:

ξ		$\Psi_1(\xi)$	$\Psi_2(\xi)$	$\Psi_3(\xi)$	$\Psi_4(\xi)$
75	$\Psi(\xi)\rightarrow$	75	0	0	0
	$\Psi_C(\xi)\rightarrow$	183.333	41.6667	−66.6667	−83.3333
150	$\Psi(\xi)\rightarrow$	133.333	16.6667	0	0
	$\Psi_C(\xi)\rightarrow$	208.333	54.1667	−41.6667	−70.8333
400	$\Psi(\xi)\rightarrow$	280	90	30	0
	$\Psi_C(\xi)\rightarrow$	291.667	95.8333	41.6667	−29.1667
600	$\Psi(\xi)\rightarrow$	358.333	129.167	108.333	4.16667
	$\Psi_C(\xi)\rightarrow$	358.333	129.167	108.333	4.16667

Let us see the important errors that can be committed by the classical solution Ψ_C, if the restriction of power positivity is omitted, and how negative thermal power can sometimes be produced, as is shown in Figure 2 for Ψ_4. It can be seen that the different solutions only coincide for $\xi \geq \delta_4$.

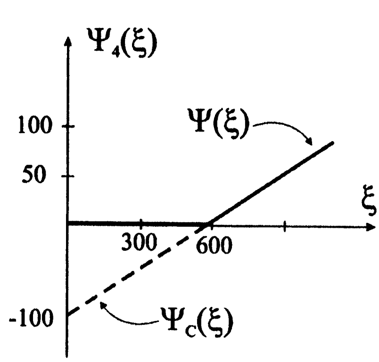

FIGURE 2 Error in classical solution.

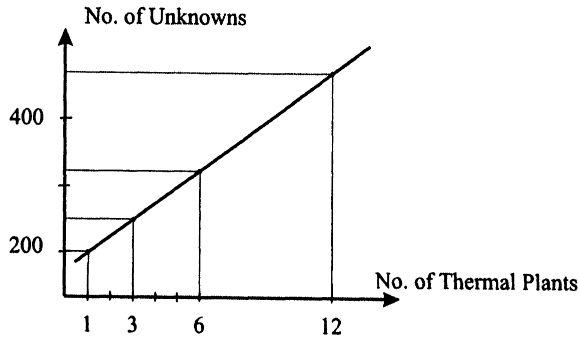

FIGURE 3 Influence of no. of thermal plants.

Secondly, we analyze the influence of the equivalent thermal station on the number of unknowns, and consequently, on the solvability of the problem. To this end, we take the Ritz method [5] as the reference in resolving a hydrothermal system. The system consists of six hydro-plants and we shall consider a variable number of thermal power stations. The set of equations defining optimality consists of

No. of thermal plants	1	3	6	12
No. of unknowns	198	246	318	462

We see that once the equivalent thermal station is taken into consideration, the number of unknowns drastically decreases. This fact yields a series of benefits. Firstly, we reduce the time that the CPU needs to solve the problem. Secondly, and even more importantly, the method is more reliable and its convergence is assured. The latter features lessen the importance of the choice of the initial values used in the calculation, which are always difficult to foresee.

6 CONCLUSIONS AND FUTURE DIRECTIONS

In this article, we establish the framework for a significant simplification of the study of optimization of hydrothermal systems, independently of the method used for their analysis, by means of the change from a system of m thermal stations to a system that involves a single thermal station called the equivalent thermal station. We obtain new theoretical results which cannot be found in the existing literature. Our theorem is of a general character as it does not depend on the choice of models.

Subsequently, we calculated the equivalent minimizer in the case where the cost functions are second-order polynomials. We proved that the equivalent minimizer is a second-order polynomial with piece-wise constant coefficients. Moreover, it belongs to the class C^1.

Finally, we apply the method to the study of various examples and exhibit its advantages compared to the methods employed by other authors: it reduces the time used by the CPU for the numerical solution, increases reliability and assures convergence, and lessens the importance of the choice of the initial values in the numerical realization. We have also seen that other methods are liable to commit serious errors, such as, for instance, those where negative thermal powers can appear.

As a perspective for further study, we may mention the problem of constructing the equivalent thermal station in the cases where models are distinct from second-order polynomials. To be exact, when any other type of model is used. This is of significant interest in the study of hydrothermal optimization, where the cost functional includes transmission losses, since in this case the functional ceases to be quadratic.

Moreover, now that it has been shown that any problem of the type (H_n–T_m) can be substituted by another of the form (H_n–T_1), which is followed by another "pure thermal" problem, we can concentrate on problems of the type (H_n–T_1), which constitute the true variational problem. This formulation allows us to employ the theory of calculus of variations.

References

[1] El-Hawary, M. E. and Christensen, G. S. (1979) *Optimal Economic Operation of Electric Power Systems.* Academic Press, pp. 126–129.

[2] Bayón, L. F., Suárez, P. M., Otero, J. A. and Dugnol, B. (1995) Fuel cost optimization of hydro-thermal systems, *Approximation and Optimization* (Vol. 8). Frankfurt, pp. 631–650.

[3] Bayón, L. F., Suárez, P. M., Otero, J. A., Grau, J. M. and Ruíz, M. M. (1999) Uso del cálculo variacional y el método de Han-Powel para la resolución de la coordinación hidrotérmica y el óptimo flujo de carga en un sistema hidrotérmico, *Proceedings IEEE-ANDESCON '99* (Vol. 1). Venezuela, pp. 268–291.

[4] Bayón, L. F., Suárez, P. M., Otero, J. A., Grau, J. M. and Ruíz, M. M. (1999) A new tool for control and diagnostic of electrical network, *Record of IEEE SDEMPED '99* (Vol. 1). Spain, pp. 323–328.

[5] Bayón, L. F. and Suárez, P. M. (2000) Multiple objective optimization of hydro-thermal systems using Ritz's method, *Mathematical Problems in Engineering*, **5**, 379–396.

[6] Nagrath, I. J. and Kothari, D. P. (1973) Application of the discrete maximum principle to the optimum scheduling of multi-reservoir systems, *Journal of the Institution of Engineers (India) Electrical Engineering Division*, **53**, 101–105.

[7] Langer, U. (1977) Economical load distribution in a hydro-thermal electrical power system, *Energiewirtschaftliche Tagesfragen*, Vol. 27, No. 11. Germany, pp. 731–734.

[8] Wong, K. P. and Wong, Y. W. (1993) Short-term hydrothermal scheduling with reservoir volume constraints. I. Simulated annealing approach, APSCOM-93, *Proceedings of the 2nd Int. Conf. on Advances in Power System Control, Operation and Management* (Vol. 2). London, UK: IEE, pp. 559–564.

[9] Wong, K. P., Wong, Y. W. and Yu, Y. (1993) A simulated annealing approach to short-term hydro scheduling, ANNPS-93, *Proceedings of the 2nd Int. Forum on Applications of Neural Networks to Power Systems.* New York: IEEE, pp. 323–328.

[10] Hota, P. K., Chakrabarti, R. and Chattopadhyay, P. K. (1999) Short-term hydrothermal scheduling through evolutionary programming technique, *Electric Power Systems Research*, **52**(2), 189–196.