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A NECESSARY CONDITION FOR BROKEN EXTREMALS IN PROBLEMS INVOLVING INEQUALITY CONSTRAINTS

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ABSTRACT. The Weiertrass-Erdmann conditions are essential when calculating extremals with corner points for functionals of the type $F(z) = \int_a^b L(t, z(t), z'(t))dt$. What is more, when it is assumed that $L_{z'z'} \neq 0$, the first condition allows the existence of extremals with corner points to be rejected. This is well known even when the restriction for the functions that may admissibly remain below (or above) a particular curve of class C^1 is considered (obstacle problem).

However, when the restrictions for the admissible functions are of the inequality non-holonomic type, similar results are unknown. In this paper, we present a necessary condition for extremals with corner points that is valid for diverse problems involving inequality constraints. This condition has been obtained by adapting a novel, unpublished proof of the first Weiertrass-Erdmann condition, which we also present.

Keywords. Calculus of Variations, Optimization, Weierstrass-Erdmann, Broken Extremals, Inequality Constraints.

AMS (MOS) Subject Classification. 49K24, 49K30

1. INTRODUCTION

The extremal values of the functional $F(z) = \int_a^b L(t, z(t), z'(t))dt$ on

$$D = \{z \in \widehat{C}^1[a, b] \mid z(a) = \alpha,\ z(b) = \beta\}$$

may be achieved in functions with corner points (piecewise C^1).

The Weierstrass-Erdmann conditions (W-E conditions) show that the discontinuities of q' that are permitted at corner points of a local extremal q are limited to those which preserve the continuity of both

$$\begin{cases} \text{(i) } L_{z'}(t, q(t), q'(t)) \\ \text{(ii) } L(t, q(t), q'(t)) - q'(t)L_{z'}(t, q(t), q'(t)). \end{cases}$$

Although these two conditions of continuity have been known since the end of the 19th century [1], they have been expounded on diverse occasions with insufficient care. Both are correct when dealing with strong extremals, but only the first is true for weak extremals. An incorrect formulation of the second of these conditions was

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given by the authors of [2] and [3], who assumed that the condition was true for the weak minima. The counterexamples presented in [4, 5] show that this assumption was incorrect.

When it is assumed that $L_{z'z'} \neq 0$, the first condition allows the existence of extremals with corner points to be rejected. This is well known [6], even when the restriction for the functions that may admissibly remain below (or above) a particular curve of class C^1 is considered. When the restrictions for the admissible functions are inequality and of a non-holonomic type, similar results are unknown.

In this paper, we present a necessary condition for extremals with corner points, for problems involving inequality constraints. Said condition affirms that at each corner point t_0 of an extremal, it holds that:

$$(q'(t_{0-}) - q'(t_{0+})) \cdot (L_{z'}(t_0, q(t_0), q'(t_{0-})) - L_{z'}(t_0, q(t_0), q'(t_{0+}))) \leq 0.$$

The proof of which has been obtained by adapting a novel, unpublished proof of the first Weiertrass-Erdmann condition, which we also present.

The classic proofs of the first W-E condition are based on the use of the variation of the functional in the general case in which the end-points are variable [6], or employ the Du Bois-Reymond equation [7] satisfied by the extremals

$$L_{z'}(t, q(t), q'(t)) = const. + \int_a^t L_z(t, q(s), q'(s))ds. \tag{1}$$

These techniques do not work if constraints are taken into account, because the functional need not admit bilateral variations at the extremum, or equation (1) is simply not satisfied.

As a partial case, we give a novel proof of the fact that if $L_{z'}$ is strictly increasing in z', then the solution of the obstacle problem is a C^1 curve. This is proven without assuming that $L_{z'z'} \neq 0$ and moreover, we do not even assume the existence of $L_{z'z'}$ We have also attained an identical result for variational problems with velocity constraints and other, more general problems.

Moreover, the advantage of our technique is that the problem is easier to solve than by means of the methods of optimal control or via the equivalent Caratheodory formulation [8].

2. A NOVEL PROOF OF THE FIRST WEIERSTRASS-ERDMANN CONDITION

Let us present the proof of the Weierstrass-Erdmann condition based on the analysis of the Gâteaux variation in certain directions which we will call $h_\varepsilon^{t_0}$.

Definition 1. *Let us take* $t_0 \in (a,b)$ *and* $\varepsilon > 0$. *We consider the auxiliary function* $h_\varepsilon^{t_0}$ *defined on* $[a,b]$:

$$h_\varepsilon^{t_0}(t) = \begin{cases} 0 & \text{if} \quad t \in [a, t_0-\varepsilon] \cup [t_0+\varepsilon, b] \\ (t-t_0+\varepsilon) & \text{if} \quad t \in [t_0-\varepsilon, t_0] \\ -(t-t_0-\varepsilon) & \text{if} \quad t \in [t_0, t_0+\varepsilon] \end{cases}$$

Notice that $h_\delta^{t_0} \in \widehat{C}^1[a,b]$, $0 \le h_\delta^{t_0}(t) \le \delta$, $\forall t \in [a,b]$, and

$$(h_\varepsilon^{t_0})'(t) = \begin{cases} 0 & \text{if} \quad t \in [a, t_0-\varepsilon) \cup (t_0+\varepsilon, b] \\ 1 & \text{if} \quad t \in (t_0-\varepsilon, t_0) \\ -1 & \text{if} \quad t \in (t_0, t_0+\varepsilon) \end{cases}$$

Theorem 1. *If* $L(t,z,z') \in C^1([a,b] \times \mathbb{R}^2)$ *and* $q \in \widehat{C}^1[a,b]$ *provides a (weak) local extremal value for*

$$F(z) = \int_a^b L(t, z(t), z'(t))dt$$

on

$$D = \{z \in \widehat{C}^1[a,b] \mid z(a) = \alpha,\ z(b) = \beta\}$$

then, $\forall t \in [a,b]$, *the first W-E condition holds:*

$$L_{z'}(t, q(t), q'(t_-)) = L_{z'}(t, q(t), q'(t_+)).$$

Proof. We shall suppose that for some $t_0 \in [a,b]$ (corner point)

$$L_{z'}(t_0, q(t_0), q'(t_{0-})) \ne L_{z'}(t_0, q(t_0), q'(t_{0+}))$$

and we will arrive at the contradiction that $\delta F(q; h_\varepsilon^{t_0}) \ne 0$, for some $\varepsilon > 0$.

Suppose firstly that

$$L_{z'}(t_0, q(t_0), q'(t_{0-})) < L_{z'}(t_0, q(t_0), q'(t_{0+})).$$

Bearing in mind that $0 \le h_\varepsilon^{t_0}(t) \le \varepsilon$, $\forall t \in [a,b]$, the existence of $\varepsilon > 0$ (sufficiently small) is obvious for which the following inequality is verified:

$$\sup_{t \in (t_0-\varepsilon, t_0)} L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) <$$
$$< \inf_{t \in (t_0, t_0+\varepsilon)} L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))$$

and from this relation, we derive the following chain of inequalities

$$I_1 = \int_{t_0-\varepsilon}^{t_0} [L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))]dt \le$$
$$\le \varepsilon \cdot \sup_{t \in (t_0-\varepsilon, t_0)} L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) <$$
$$< \varepsilon \cdot \inf_{t \in (t_0, t_0+\varepsilon)} L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) \le$$
$$\le \int_{t_0}^{t_0+\varepsilon} [L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))]dt = I_2.$$

It is well-known that $\forall w \in \widehat{C}^1[a,b]$

$$\delta F(q;w) = \int_a^b [w(t)\cdot L_z(t,q(t),q'(t)) + w'(t)\cdot L_{z'}(t,q(t),q'(t))]dt.$$

Taking into account now that

$$0 = h_\varepsilon^{t_0}(t) = (h_\varepsilon^{t_0})'(t),\ \forall t \in [a, t_0-\varepsilon]\cup[t_0+\varepsilon, b]$$

is fulfilled

$$\delta F(q;h_\varepsilon^{t_0}) = \int_{t_0-\varepsilon}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + (h_\varepsilon^{t_0})'(t)\cdot L_{z'}(t,q(t),q'(t))]dt$$

and consequently

$$\begin{aligned}\delta F(q;h_\varepsilon^{t_0}) &= \int_{t_0-\varepsilon}^{t_0} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + 1\cdot L_{z'}(t,q(t),q'(t))]dt + \\ &\quad + \int_{t_0}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + (-1)\cdot L_{z'}(t,q(t),q'(t))]dt\end{aligned}$$

i.e.

$$\begin{aligned}\delta F(q;h_\varepsilon^{t_0}) &= \int_{t_0-\varepsilon}^{t_0} [L_{z'}(t,q(t),q'(t)) + h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]dt \\ &\quad - \int_{t_0}^{t_0+\varepsilon} [L_{z'}(t,q(t),q'(t)) - h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]dt \\ &= I_1 - I_2 < 0.\end{aligned}$$

In the other case,

$$L_{z'}(t_0,q(t_0),q'(t_{0-})) > L_{z'}(t_0,q(t_0),q'(t_{0+}))$$

by analogy with the previous argument, we will have that $\delta F(q;-h_\varepsilon^{t_0}) < 0$. □

3. A NECESSARY CONDITION FOR BROKEN EXTREMALS IN PROBLEMS INVOLVING INEQUALITY CONSTRAINTS

The proof of Theorem 1 ceases to be valid when the admissible functions are subject to certain constraints. This is because in this case the extremum need not have bilateral variations. Nonetheless, the method proposed for the proof can be adapted to study the extremum of the functional restricted to the sets where: $\lim\limits_{x\to 0^+} \frac{F(q+xh_\varepsilon^{t_0})-F(q)}{x} = \delta F(q;h_\varepsilon^{t_0})$ exists.

Let us establish the concept of a shapeable set of functions. This will allow us to introduce a class of constraints on the admissible functions under which the necessary condition for broken extremals that we present is satisfied.

Definition 2. *Let $q \in W$. We will say that ω is a W-admissible direction at q if $\exists \theta > 0$ such that $q + x\omega \in W$, $\forall x \in [0,\theta]$.*

Definition 3. *We will say that a set of functions* $\Omega \subset \widehat{C}^1[a,b]$ *is shapeable at* $t_0 \in (a,b)$ *if* $\forall q \in \Omega$

i) $q'(t_{0-}) < q'(t_{0+}) \Longrightarrow \exists \varepsilon > 0$ *such that* $h_\varepsilon^{t_0}$ *is an* Ω*-admissible direction at* q.

ii) $q'(t_{0-}) > q'(t_{0+}) \Longrightarrow \exists \varepsilon > 0$ *such that* $-h_\varepsilon^{t_0}$ *is an* Ω*-admissible direction at* q.

Theorem 2. (A Necessary Condition). *If* $L(t,z,z') \in C^1([a,b]\times\mathbb{R}^2)$, Ω *is shapeable at* t_0 *(corner point) and* q *provides a (weak) local minimum value for*

$$F(z) = \int_a^b L(t, z(t), z'(t))dt$$

on

$$D = \Omega \cap \{z \in \widehat{C}^1[a,b] \mid z(a) = \alpha \wedge z(b) = \beta\}$$

then it holds that:

$$(q'(t_{0-}) - q'(t_{0+})) \cdot (L_{z'}(t_0, q(t_0), q'(t_{0-})) - L_{z'}(t_0, q(t_0), q'(t_{0+}))) \leq 0. \tag{2}$$

Proof. Suppose that

$$(q'(t_{0-}) - q'(t_{0+})) \cdot (L_{z'}(t_0, q(t_0), q'(t_{0-})) - L_{z'}(t_0, q(t_0), q'(t_{0+}))) > 0.$$

i) If $q'(t_{0-}) < q'(t_{0+})$, then $L_{z'}(t_0, q(t_0), q'(t_{0-})) < L_{z'}(t_0, q(t_0), q'(t_{0+}))$.

Let $\varepsilon > 0$ be such that $h_\varepsilon^{t_0}$ is a D-admissible direction at q. Proceeding as in Theorem 1, we shall have that

$$\delta F(q; h_\varepsilon^{t_0}) = \lim_{x\to 0^+} \frac{F(q + x h_\varepsilon^{t_0}) - F(q)}{x} < 0$$

which contradicts the assumption that q provides a weak local minimum of F on D.

ii) If $q'(t_{0-}) > q'(t_{0+})$, then $L_{z'}(t_0, q(t_0), q'(t_{0-})) > L_{z'}(t_0, q(t_0), q'(t_{0+}))$.

Let $\varepsilon > 0$ be such that $-h_\varepsilon^{t_0}$ is a D-admissible direction at q. We shall now have

$$\delta F(q; -h_\varepsilon^{t_0}) = \lim_{x\to 0^+} \frac{F(q + x h_\varepsilon^{t_0}) - F(q)}{x} < 0$$

which once again is contradictory. □

Let us next see how by imposing a certain property on $L_{z'}$, the necessary condition (2) becomes the classic first W-E condition.

Theorem 3. *If* $L(t,z,z') \in C^1([a,b]\times\mathbb{R}^2)$, $\psi(x) = L_{z'}(t_0, q(t_0), x)$ *is not decreasing,* Ω *is shapeable at* t_0*, and* q *provides a (weak) local minimum value for*

$$F(z) = \int_a^b L(t, z(t), z'(t))dt$$

on

$$D = \Omega \cap \{z \in \widehat{C}^1[a,b] \mid z(a) = \alpha \ \wedge \ z(b) = \beta\}$$

then the first W-E condition holds:

$$L_{z'}(t_0, q(t_0), q'(t_{0-})) = L_{z'}(t_0, q(t_0), q'(t_{0+})).$$

Proof. If $\psi(x) = L_{z'}(t_0, q(t_0), x)$ is not decreasing, then

$$\begin{aligned} q'(t_{0-}) &< q'(t_{0+}) \Rightarrow L_{z'}(t_0, q(t_0), q'(t_{0-}) \leq L_{z'}(t_0, q(t_0), q'(t_{0+}) \\ q'(t_{0-}) &> q'(t_{0+}) \Rightarrow L_{z'}(t_0, q(t_0), q'(t_{0-}) \geq L_{z'}(t_0, q(t_0), q'(t_{0+}) \end{aligned}$$

so

$$(q'(t_{0-}) - q'(t_{0+})) \cdot (L_{z'}(t_0, q(t_0), q'(t_{0-})) - L_{z'}(t_0, q(t_0), q'(t_{0+}))) \geq 0.$$

Now, bearing in mind (2), we have that

$$(q'(t_{0-}) - q'(t_{0+})) \cdot (L_{z'}(t_0, q(t_0), q'(t_{0-})) - L_{z'}(t_0, q(t_0), q'(t_{0+}))) = 0.$$

But at the corner point $q'(t_{0-}) \neq q'(t_{0+})$, so the first W-E condition holds. □

It is obvious now that the property that $L_{z'}$ is strictly increasing with respect z' allows the existence of extremals with corner points to be rejected.

Theorem 4. *If $L(t, z, z') \in C^1([a, b] \times \mathbb{R}^2)$, and $\psi(x) = L_{z'}(t, z, x)$ is strictly increasing $\forall (t, z) \in (a, b) \times \mathbb{R}$, Ω is shapeable at each $t \in [a, b]$, and q provides a (weak) local minimum value for*

$$F(z) = \int_a^b L(t, z(t), z'(t))dt$$

on

$$D = \Omega \cap \{z \in \widehat{C}^1[a, b] \mid z(a) = \alpha \wedge z(b) = \beta\}$$

then q is C^1.

Proof. Obvious from Theorem 3. □

4. CONSIDERATIONS FOR SHAPEABLE SETS

We shall next see, with examples, that the concept of the shapeable set embraces the constraints considered in the classic obstacle problem and in problems with velocity constraints, and is even more general.

4.1 Obstacle problem

Proposition 1. *If $G \in C^1[a, b]$, then the set*

$$\Omega = \{z \in \widehat{C}^1[a, b] \mid z(t) \geq G(t), \forall t \in [a, b]\}$$

is shapeable $\forall t_0 \in (a, b)$.

Proof. Let us take $q \in \Omega$ with $q'(t_{0+}) \neq q'(t_{0-})$. It is clear that it is sufficient to consider the points of union of q with the curve $G(t)$.

If $q(t_0) = G(t_0)$, then it is easy to see that $q'(t_{0+}) > q'(t_{0-})$ and it should be noted that $\forall x > 0$ and $\forall \varepsilon > 0$

$$q(t) + x h_\varepsilon^{t_0}(t) \geq q(t) \geq G(t), \ \forall t \in [a, b]$$

so that $h_\varepsilon^{t_0}$ is an Ω-admissible direction at q, $\forall \varepsilon > 0$. □

Proposition 2. *If* $G \in C^1[a,b]$, *then the set*

$$\Omega = \{z \in \widehat{C}^1[a,b] \mid z(t) \leq G(t), \forall t \in [a,b]\}$$

is shapeable $\forall t_0 \in (a,b)$.

Proof. The proof is similar to the previous proposition, taking into account the fact that if $q(t_0) = G(t_0)$, then

$$q'(t_{0+}) \neq q'(t_{0-}) \Longrightarrow q'(t_{0+}) < q'(t_{0-}).$$ □

Likewise, with the same technique, the following result is proven.

Proposition 3. *If* $G_1, G_2 \in C^1[a,b]$, *then the set*

$$\Omega = \{z \in \widehat{C}^1[a,b] \mid G_1(t) \leq z(t) \leq G_2(t), \forall t \in [a,b]\}$$

is shapeable $\forall t_0 \in (a,b)$.

4.2 Problems with velocity constraints

Proposition 4. *If* $G \in C[a,b]$, *then the set*

$$\Omega = \{z \in \widehat{C}^1[a,b] \mid z'(t) \leq G(t), \ \forall t \in [a,b]\}$$

is shapeable $\forall t_0 \in (a,b)$.

Proof. Let us take $q \in \Omega$. Firstly, suppose that $q'(t_{0-}) < q'(t_{0+}) \leq G(t_0)$.

Taking into account the fact that $q \in \widehat{C}^1[a,b]$ and the continuity of G at t_0

$$\exists \varepsilon > 0 \text{ such that } \sup_{[t_0-\varepsilon, t_0)} q'(t) < \inf_{[t_0-\varepsilon, t_0+\varepsilon]} G(t).$$

Therefore, taking

$$\theta = \inf_{[t_0-\varepsilon, t_0+\epsilon]} G(t) - \sup_{[t_0-\epsilon, t_0)} q'(t) > 0$$

we have that $\forall x \in [0,\theta)$

$$q'(t) + x < G(t), \ \forall t \in [t_0-\varepsilon, t_0)$$

$$q'(t) - x \leq G(t) - x \leq G(t), \ \ \forall t \in [t_0, t_0+\varepsilon]$$

and consequently,

$$q + x h_\varepsilon^{t_0} \in \Omega, \ \forall x \in [0,\theta) \Longrightarrow h_\varepsilon^{t_0} \text{ is an } \Omega\text{-admissible direction at } q.$$

If $q'(t_{0+}) < q'(t_{0-}) \leq G(t_0)$, by analogy with the previous argument, we will have that $\varepsilon > 0$ exists such that $-h_\varepsilon^{t_0}$ is an Ω-admissible direction at q. □

Proposition 5. *If* $G \in C[a,b]$, *then the set*

$$\Omega = \{z \in \widehat{C}^1[a,b] \mid z'(t) \geq G(t), \forall t \in [a,b]\}$$

is shapeable $\forall t_0 \in (a,b)$.

Proof. The proof is similar to the one given in the previous proposition. □

Likewise, with the same technique, the following result is proven.

Proposition 6. *If* $G_1, G_2 \in C[a,b]$, *then the set*

$$\Omega = \{z \in \widehat{C}^1[a,b] \mid G_1(t) \leq z'(t) \leq G_2(t), \forall t \in [a,b]\}$$

is shapeable $\forall t_0 \in (a,b)$.

4.3 Examples of non shapeables sets

Remark 1. (Internal point constraint).

The next set is not shapeable at $t_0 \in (a,b)$

$$\Omega_\alpha = \{z \in \widehat{C}^1[a,b] \mid z(t_0) = \alpha\}.$$

Remark 2. (Problem of reflection of extremals).

Let us take $G \in C[a,b]$. The next set is nowhere shapeable

$$\Omega = \{z \in \widehat{C}^1[a,b] \mid z(t) \leq G(t) \wedge \exists| \; t_0 \text{ such that } z(t_0) = G(t_0)\}.$$

5. EXAMPLE 1

Let us take $L \in C^1(\mathbb{R})$ with L' strictly increasing. Let us consider the problem of minimizing

$$F(z) = \int_0^2 L(z'(t))dt$$

for every L, on the set

$$D = \Omega \cap \{z \in \widehat{C}^1[0,2] \mid z(0) = 0, \; z(2) = \frac{7}{6}\}$$

where

$$\Omega = \{z \in \widehat{C}^1[0,2] \mid z'(t) \leq 2t - t^2, \; \forall t \in [0,2]\}.$$

It is necessary for the solution q to account for the arcs of the extremal $(C_1 + C_2 t)$ and the boundary arcs $(C + t^2 - \frac{t^3}{3})$, hence, since Ω is shapeable at every point, and by virtue of Theorem 4, its derivative must be continuous and can only be of the form

$$q'(t) = \begin{cases} 2t - t^2 & \text{if} \quad t \in [0,\alpha] \\ 2\alpha - \alpha^2 & \text{if} \quad t \in [\alpha, 2-\alpha] \\ 2t - t^2 & \text{if} \quad t \in [2-\alpha, \alpha] \end{cases}$$

for a certain α.

Taking into account the fact that $q(0) = 0$, $q(2) = \frac{7}{6}$ and that q is continuous, we have $\alpha = \frac{1}{2}$, so that the solution (*for every* L) is

$$q(t) = \begin{cases} t^2 - \dfrac{t^3}{3} & \text{if} \quad t \in [0, \frac{1}{2}] \\ \dfrac{3t}{4} - \dfrac{1}{6} & \text{if} \quad t \in [\frac{1}{2}, \frac{3}{2}] \\ -\dfrac{1}{6} + t^2 - \dfrac{t^3}{3} & \text{if} \quad t \in [\frac{3}{2}, 2] \end{cases}$$

This example shows how the assertion of Theorem 4 can exclude the presence of the corner points and therefore the unique solution is obtained in a much simpler way than by means of any of the traditional methods (for example, optimal control or an equivalent Caratheodory formulation).

6. EXAMPLE 2

The so-called first W-E condition, and even condition (2) presented in this paper, are not always satisfied in variational problems where the admissible functions are subject to certain constraints. For example, these conditions are not fulfilled in the problems of reflection of the extremals or in problems with point internal constraints. This is due to the fact that the constraint is not a result of shapeable sets.

We now consider the problem of minimization of the functional

$$F(z) = \int_{-1}^{1} L(t, z(t), z'(t))dt$$

with $L(t, z(t), z'(t)) = -z'(t)^2$, on the set

$$\Omega = \{z \in \widehat{C}^1[-1,1] \mid z(-1) = z(1) = 0 \wedge |z'(t)| \leq 1\}$$

it is obvious that

$$q(t) = \begin{cases} t+1 & \text{if} \quad t \in [-1,0] \\ 1-t & \text{if} \quad t \in [0,1] \end{cases}$$

is a solution of the problem with $1 = q'(0_-) \neq q'(0_+) = -1$. Nonetheless, the first W-E condition is not fulfilled:

$$L_{z'}(0, 1, q'(0_-)) = -2q'(0_-) \neq -2q'(0_+) = L_{z'}(0, 1, q'(0_+))$$

This is due to the fact that $L_{z'}$ is decreasing, and hence the hypotheses of Theorem 3 are not fulfilled. However, condition (2), which we have presented, is satisfied

$$(q'(t_{0-}) - q'(t_{0+})) \cdot (L_{z'}(t_0, q(t_0), q'(t_{0-})) - L_{z'}(t_0, q(t_0), q'(t_{0+}))) =$$
$$= (1 - (-1)) \cdot (-2 - (2)) \leq 0$$

since the set Ω is shapeable.

7. CONCLUSIONS AND FUTURE PERSPECTIVES

In this paper, we have presented a new method for proving the first W-E condition for extremals with corner points. This method is also applicable to certain variational problems with constraints on the admissible functions. The classic proofs are not applicable when constraints are imposed, because the functional need not admit bilateral variations at the extremum, or the Du Bois-Reymond equation is not fulfilled.

The novel proof proposed here is based on the analysis of the Gâteaux variation of the functional in certain directions and on the observation that it can only be zero if the first W-E condition is fulfilled. If constraints are imposed, the proof remains valid for the cases where these directions are admissible (shapeable sets) and under some unavoidable hypotheses of convexity on the functional.

We also present a novel, necessary condition for extremals with corner points, for problems involving very general inequality constraints.

We prove, for a general type of constraint, that the solution of the variational problem belongs to the class C^1. This result is obtained without assuming that $L_{z'z'} \neq 0$; moreover, we do not even assume that $L_{z'z'}$ exists. In certain problems, this sometimes implies the uniqueness of the solution and simplifies its calculation. We have thus solved the problem considered in example 1 in an extremely simple way, without the need to use the methods of optimal control or an equivalent Caratheodory formulation.

As a continuation of this work, we propose to employ this novel technique to problems with constraints of a more general type

$$G_1(t, z(t)) \leq z'(t) \leq G_2(t, z(t)).$$

We also propose to develop similar techniques for the second W-E condition.

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