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# A new algorithm for the optimization of a simple hydrothermal problem

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**Abstract**

This paper falls within the scope of studies concerning the optimization of the functioning of hydrothermal systems. We have developed a much simpler theory than previous ones that resolves the problem of minimization of a functional

$$F(z) = \int_0^T L(t, z(t), z'(t))\,\mathrm{d}t$$

within the set of piecewise $C^1$ functions $(\widehat{C}^1)$ that satisfy $z(0) = 0$, $z(T) = b$ and the constraints

$$0 \leqslant H(t, z(t), z'(t)) \leqslant P_{\mathrm{d}}(t) \quad \forall t \in [0, T]$$

In particular, we have established a necessary condition for the stationary functions of the functional. This theorem allows us to elaborate the optimization algorithm that leads to determination of the optimal functioning of the hydroplant and of the whole hydrothermal system. Finally, we present a example employing the algorithm realized to this end with the "Mathematica" package. The program developed is very simple and easy to use.


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## 1. Introduction

A hydrothermal system is made up of hydraulic and thermal power plants that must jointly satisfy a certain demand in electric power during a definite time interval. Thermal plants generate

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power at the expense of fuel consumption (which is the object of minimization), while hydraulic plants obtain power from the energy liberated by water that moves a turbine; there being a limited quantity of water available during the optimization period. In prior studies [1,2], it was proven that the problem of optimization of the fuel costs of a hydrothermal system with $m$ thermal power plants without transmission losses may be reduced to the study of a hydrothermal system made up of one single thermal power plant, called the *thermal equivalent*. In the present paper, we consider a simple hydrothermal system with one hydraulic power plant and $m$ thermal power plants without transmission losses that have been substituted by their thermal equivalent. Under these conditions, we present the problem from the Electrical Engineering perspective to then go on to resolve the mathematical problem thus formulated. We will call this problem: the $H_1$–$T_1$ problem.

## 2. Hydrothermal statement of the $H_1$–$T_1$ problem

The problem consists in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0,T]$. Said cost may be represented by the functional

$$F(P(t)) = \int_0^T \Psi(P(t))\,\mathrm{d}t$$

where $\Psi$ is the function of thermal cost of the thermal equivalent and $P(t)$ is the power generated by said plant. Moreover, the following equilibrium equation of active power will have to be fulfilled

$$P(t) + H(t, z(t), z'(t)) = P_{\mathrm{d}}(t) \quad \forall t \in [0,T]$$

where $P_{\mathrm{d}}(t)$ is the power demand and $H(t, z(t), z'(t))$ is the power contributed to the system at the instant $t$ by the hydraulic plant, being: $z(t)$ the volume that is discharged up to the instant $t$ (in what follows, simply volume) by the plant, and $z'(t)$ the rate of water discharge of the plant at the instant $t$.

Taking into account the equilibrium equation, the problem reduces to calculating the minimum of the functional

$$F(z) = \int_0^T \Psi(P_{\mathrm{d}}(t) - H(t, z(t), z'(t)))\,\mathrm{d}t$$

If we assume that $b$ is the volume of water that must be discharged during the entire optimization interval, the following boundary conditions will have to be fulfilled

$$z(0) = 0, \quad z(T) = b$$

Traditional studies dealing with hydrothermal optimization employ concrete models both for the function of thermal cost $\Psi$, as well as for the function of effective hydraulic generation $H$. Hence, if the model changes, the algorithms obtained are not valid. The study of optimal conditions for the functioning of a hydrothermal system constitutes a complicated problem that has attracted significant interest in recent decades. Several techniques have been applied to solve this problem, such as functional analysis techniques [3] or Ritz's method [4]. Such a variety of mathematical models forces us to undertake a general study of the problem. The algorithms obtained with this study should be extensible to a large set of hydrothermal problems.

One of the main contributions of this paper is that the method is valid for any model of power plants, since we will try to consider the functions $P_{\mathrm{d}}$, $\Psi$ and $H$ as general as possible without any restrictions, except those that are natural for problems of this type. For the sake of convenience, we assume throughout the paper that they are sufficiently smooth and are subject to the following additional assumptions:

*Function of thermal cost*. Let us assume that the function of thermal cost $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies: $\Psi'(x) > 0\ \forall x \in \mathbb{R}^+$ and thus is strictly increasing. This restriction is absolutely natural: it reads more cost to more generated power. Let us assume as well that $\Psi''(x) > 0\ \forall x \in \mathbb{R}^+$ and is therefore strictly convex. The models traditionally employed meet this restriction.

*Function of effective hydraulic generation*. Let us assume that the hydraulic generation $H(t, z, z') : \Omega_{\mathrm{H}} = [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing with respect to the rate of water discharge $z'$, that is $H_{z'} > 0$. Let us also assume that $H(t, z, z')$ is concave with respect to $z'$, that is $H_{z'z'} \leqslant 0$. The suppositions we have made guarantee the fulfilment of the following inequalities: $L_{z'z'}(t, z, z') > 0$; $L_{z'}(t, z, z') < 0$.

The real models meet these two restrictions; the former means more power to a higher rate of water discharge. We see that we only admit non-negative thermal power $P(t)$ and we will solely admit non-negative volumes $z(t)$ and rates of water discharge $z'(t)$, therefore we may expound the mathematical problem in the following terms.

## 3. Variational statement of the $H_1$–$T_1$ problem

We will call $H_1$–$T_1$ the problem of minimization of the functional

$$F(z(t)) = \int_0^T L(t, z(t), z'(t))\, \mathrm{d}t$$

with $L$ having the form

$$L(t, z(t), z'(t)) = \Psi(P_{\mathrm{d}}(t) - H(t, z(t), z'(t)))$$

over the set $\Theta_b$

$$\{z \in \widehat{C}^1[0, T]/z(0) = 0, \quad z(T) = b, \quad 0 \leqslant H(t, z(t), z'(t)) \leqslant P_{\mathrm{d}}(t) \quad \forall t \in [0, T]\}$$

We also suppose that

$$H(t, b, z'(t)) \leqslant H(t, z(t), z'(t)) \leqslant H(t, 0, z'(t)) \quad \forall z \in \Theta_b$$

These suppositions are fulfilled in all real hydrothermal problems, and bearing in mind the weak influence of $z(t)$, ($H(t, b, z') \simeq H(t, z, z') \simeq H(t, 0, z')$), it is reasonable to substitute the restriction

$$0 \leqslant H(t, z(t), z'(t)) \leqslant P_{\mathrm{d}}(t) \tag{1}$$

by others of the type

$$0 \leqslant H(t, b, z'); \quad H(t, 0, z') \leqslant P_{\mathrm{d}}(t) \tag{2}$$

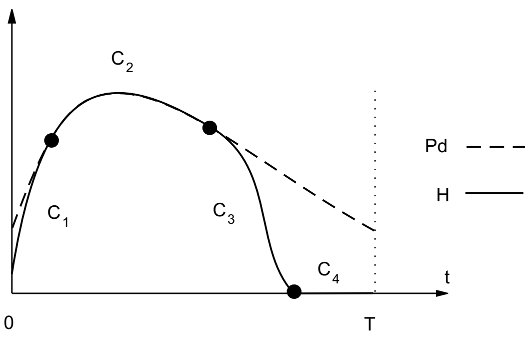


Fig. 1. Boundary and interior arcs.

The solution to the problem with restrictions (2) will be very close to that obtained with restrictions (1), the advantage being that the mathematical treatment of those of type (2) is much simpler that those of type (1) (see [5]). So the problem involves non-holonomic inequality constraints. Recently, a general optimization problem with inequality constraints has been studied [6–8] using diverse techniques. In the present paper, we have developed a simple theory that solves the hydrothermal $H_1$–$T_1$ problem. The development is hence self-contained and extremely basic and also enables the construction of the optimal solution.

If we did not have the restrictions $0 \leqslant H(t, z(t), z'(t)) \leqslant P_d(t)$, we could use the shooting method to resolve the problem. In this case, we would use the integral form of Euler's equation (Du Bois–Reymond equation) $\forall t \in [0, T]$

$$\int_0^t L_z(s, z(s), z'(s))\,\mathrm{d}s - L_{z'}(t, z(t), z'(t)) = -L_{z'}(0, z(0), z'(0)) = K > 0$$

Varying the initial condition of the derivative $z'(0)$ (initial flow rate), we would search for the extremal that fulfills the second boundary condition $z(T) = b$ (final volume). However, we cannot use this method in our case, as due to the restrictions, the extremals may not admit bilateral variations, i.e. they may present boundary arcs.

The following questions arise: Do all the interior arcs ($C_1$ and $C_3$ in Fig. 1) have the same constant $K$ in the Du Bois–Reymond equation? At what moments does the boundary have to be penetrated and be abandoned? In the following section, we shall develop the theory needed to respond to these questions.

## 4. The main coordination theorem

We shall use Pontryagin's maximum principle as the basis for demonstrating this theorem, setting out our problem in terms of optimal control in continuous time, with the Lagrange-type functional. Prior to this, we define the following function:

**Definition 1.** Let us term the coordination function of $q \in \Theta_b$ the function in $[0, T]$, defined as follows

$$\mathbb{Y}_q(t) = \int_0^t L_z(s, q(s), q'(s))\,\mathrm{d}s - L_{z'}(t, q(t), q'(t))$$

**Theorem 1** (The main coordination theorem). *If $q \in \widehat{C}^1$ is a solution of problem $H_1$–$T_1$, then $\exists K$ such that*

(i) *If $0 < H(t, q(t), q'(t)) < P_\mathrm{d}(t)$ ($t$ is not a boundary point) $\Rightarrow \mathbb{Y}_q(t) = K$.*
(ii) *If $H(t, b, q'(t)) = 0 \Rightarrow \mathbb{Y}_q(t) \leqslant K$.*
(iii) *If $H(t, 0, q'(t)) = P_\mathrm{d}(t) \Rightarrow \mathbb{Y}_q(t) \geqslant K$.*

**Proof.** We present the problem considering the state variable to be $z(t)$, the control variable $u(t)$, and the state equation $z' = u$. The optimal control problem is thus:

$$\min_{u(t)} \int_0^T L(t, z(t), u(t))\,\mathrm{d}t \quad \text{with} \quad \begin{cases} z' = u \\ z(0) = 0, \quad z(T) = b \\ 0 \leqslant H(t, b, u) \wedge H(t, 0, u) \leqslant P_\mathrm{d}(t) \end{cases}$$

We shall term the optimal control $u_{\mathrm{opt}}$, which we see in our case is the optimal flow rate $q'(t)$, therefore the optimal state will be $q$. Let $\hbar$ be the Hamiltonian associated with the problem

$$\hbar(z, u, \lambda, t) = L(t, z, u) + \lambda \cdot u$$

In virtue of Pontryagin's principle, there exists a piecewise $C^1$ function $\lambda_{\mathrm{opt}}$ (co-state variable) that satisfies the two following conditions:

$$\lambda'_{\mathrm{opt}}(t) = -\frac{\partial \hbar}{\partial z} = -L'_z(t, q(t), u_{\mathrm{opt}}(t)) \tag{3}$$

$$\hbar(q(t), u_{\mathrm{opt}}(t), \lambda_{\mathrm{opt}}(t), t) \leqslant \hbar(q(t), u, \lambda_{\mathrm{opt}}(t), t) \quad \forall u; \begin{cases} 0 \leqslant H(t, b, u) \\ H(t, 0, u) \leqslant P_\mathrm{d}(t) \end{cases} \tag{4}$$

From (3) it follows that

$$\lambda_{\mathrm{opt}}(t) = -\int_0^t L'_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s + K$$

From (4) it follows that for each $t$, $u_{\mathrm{opt}}(t)$ minimizes the function

$$F(u) = L(t, q(t), u) + \lambda_{\mathrm{opt}}(t) \cdot u \quad \text{on} \quad \{u | 0 \leqslant H(t, b, u) \wedge H(t, 0, u) \leqslant P_\mathrm{d}(t)\}$$

Hence, in accordance with the Kuhn-Tucker theorem, for each $t$ there exists two real non-negative numbers, $\alpha$ and $\beta$, such that $u_{\mathrm{opt}}(t)$ is a critical point of

$$F^*(u) = L(t, q(t), u) + \lambda_{\mathrm{opt}}(t) \cdot u + \alpha \cdot (H(t, 0, u) - P_\mathrm{d}(t)) - \beta \cdot H(t, b, u)$$

it being verified that:

- if $H(t, 0, u_{\mathrm{opt}}(t)) - P_\mathrm{d}(t) < 0$, then $\alpha = 0$
- if $H(t, b, u_{\mathrm{opt}}(t)) > 0$, then $\beta = 0$.

We hence have

$$F^{*\prime}(u_{\text{opt}}(t)) = L'_u(t, q(t), u_{\text{opt}}(t)) + \lambda_{\text{opt}}(t) + \alpha \cdot H'_u(t, 0, u_{\text{opt}}(t)) - \beta \cdot H'_u(t, b, u_{\text{opt}}(t)) = 0$$

and the following cases:

*Case 1:* $0 < H(t, q(t), u_{\text{opt}}(t)) < P_{\text{d}}(t)$.
In this case, $\alpha = \beta = 0$ and hence

$$L'_u(t, q(t), u_{\text{opt}}(t)) + \lambda_{\text{opt}}(t) = 0$$

$$L'_u(t, q(t), u_{\text{opt}}(t)) - \int_0^t L'_z(s, q(s), u_{\text{opt}}(s))\, \mathrm{d}s + K = 0 \Rightarrow \mathbb{Y}_q(t) = K$$

*Case 2:* $H(t, b, u_{\text{opt}}(t)) = 0$.
In this case, $H(t, 0, u_{\text{opt}}(t)) - P_{\text{d}}(t) < 0$ and hence $\alpha = 0$.

$$L'_u(t, q(t), u_{\text{opt}}(t)) + \lambda_{\text{opt}}(t) - \beta \cdot H'_u(t, b, u_{\text{opt}}(t)) = 0$$

Bearing in mind now that $\beta \geqslant 0$ and $H'_u(t, b, u_{\text{opt}}(t)) \geqslant 0$, we have

$$L'_u(t, q(t), u_{\text{opt}}(t)) - \int_0^t L'_z(s, q(s), u_{\text{opt}}(s))\, \mathrm{d}s + K = \beta \cdot H'_u(t, b, u_{\text{opt}}(t)) \geqslant 0 \Rightarrow \mathbb{Y}_q(t) \leqslant K$$

*Case 3:* $H(t, 0, u_{\text{opt}}(t)) - P_{\text{d}}(t) = 0$.
In this case, $H(t, b, u_{\text{opt}}(t)) > 0$ and hence $\beta = 0$.

$$L'_u(t, q(t), u_{\text{opt}}(t)) + \lambda_{\text{opt}}(t) + \alpha \cdot H'_u(t, 0, u_{\text{opt}}(t)) = 0$$

Bearing in mind now that $\alpha \geqslant 0$ and $H'_u(t, 0, u_{\text{opt}}(t)) \geqslant 0$, we have

$$\begin{aligned} &L'_u(t, q(t), u_{\text{opt}}(t)) - \int_0^t L'_z(s, q(s), u_{\text{opt}}(s))\, \mathrm{d}s + K \\ &\quad = -\alpha \cdot H'_u(t, 0, u_{\text{opt}}(t)) \leqslant 0 \Rightarrow \mathbb{Y}_q(t) \geqslant K \qquad \square \end{aligned}$$

The constant $K$ will be termed the *coordination constant* of the solution $q$.

**Note 1.** With the hypothesis $L_{z'z'}(t, z, z') > 0$ (see [9]), the solution may also be guaranteed to be of class $C^1$.

## 5. Construction of the optimal solution

We have already mentioned the fact that if we did not have inequality restrictions, the solution could be constructed by means of the shooting method. We use the same framework in the present case, but the variation of the initial condition for the derivative, which now need not make sense, is substituted by the variation of the coordination constant $K$.

The problem will consist in finding for each $K$ the function $q_K$ which satisfies $q_K = 0$ and the conditions of the main coordination theorem, and from among these functions, the one which generates an admissible function ($q_K(T) = b$).

We will denote by $M$ the rate of water discharge at the instant $t = 0$ that is necessary for the hydraulic power station to satisfy the power demand: $H(0,0,M) = P_{\mathrm{d}}(0)$ and we will denote by $m$ the rate of water discharge at the instant $t = 0$ that is necessary for $H(0,0,m) = 0$. We also set

$$K_m = -L_{z'}(0,0,m); \quad K_M = -L_{z'}(0,0,M)$$

We observe that $\forall x \in (m,M)$ (with the hypothesis $L_{z'z'}(t,z,z') > 0$) we have

$$K_M < -L_{z'}(0,0,x) < K_m$$

To construct $q_K$, we proceed by the following steps:

[*Step 1*] (*the first arc*)

(i) If $K \geqslant K_m$, we set $q_K(t) = \omega(t)$, the solution of the differential equation

$$H(t,b,\omega'(t)) = 0 \quad \text{with} \quad \omega(0) = 0$$

in the maximal interval $[0,t_1]$, where

$$K \geqslant \mathbb{Y}_{\omega}(t) = \int_0^t L_z(s,\omega(s),\omega'(s))\,\mathrm{d}s - L_{z'}(t,\omega(t),\omega'(t)) \quad \forall t \in [0,t_1]$$

(The thermal power station generates all the power demanded in $[0,t_1]$).

(ii) If $K \leqslant K_M$, we set $q_K(t) = \omega(t)$, the solution of the differential equation

$$H(t,0,\omega'(t)) = P_{\mathrm{d}}(t) \quad \text{with} \quad \omega(0) = 0$$

in the maximal interval $[0,t_1]$, where

$$K \leqslant \mathbb{Y}_{\omega}(t) = \int_0^t L_z(s,\omega(s),\omega'(s))\,\mathrm{d}s - L_{z'}(t,\omega(t),\omega'(t)) \quad \forall t \in [0,t_1]$$

(The hydraulic power station generates all the power demanded in $[0,t_1]$.)

(iii) $K_M < K < K_m$ ($\exists x$ such that $K = -L_{z'}(0,0,x)$).

$q_K$ will be the arc of the interior extremal (with $q_K(0) = 0$) which satisfies Euler's equation in its maximal domain $[0,t_1]$ and therefore the coordination equation

$$K = \mathbb{Y}_{q_K}(t) = \int_0^t L_z(s,q_K(s),q'_K(s))\,\mathrm{d}s - L_{z'}(t,q_K(t),q'_K(t)) \quad \forall t \in [0,t_1)$$

[*i-th Step*] (*i-th arc*)

(A) If $q_K$ has an interior arc in $[t_{i-1},t_i]$, there are two possibilities:

(I) If $H(t_i,b,q'_K(t_i)) = 0$, we consider the maximal interval $[t_i,t_{i+1}]$ such that, $\forall t \in [t_i,t_{i+1}]$

$$K \geqslant \int_0^{t_i} L_z(s,q_K(s),q'_K(s))\,\mathrm{d}s + \int_{t_i}^t L_z(s,\omega(s),\omega'(s))\,\mathrm{d}s - L_{z'}(t,\omega(t),\omega'(t))$$

$\omega(t)$ being a solution of the differential equation

$$H(t,b,\omega'(t)) = 0 \quad \text{with} \quad \omega(t_i) = q_K(t_i)$$

If this is the case, we set $q_K(t) = \omega(t)\ \forall t \in [t_i, t_{i+1}]$.

(II) If $H(t_i, 0, q'_K(t_i)) = P_\mathrm{d}(t_i)$, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K \leqslant \int_0^{t_i} L_z(s, q_K(s), q'_K(s))\,\mathrm{d}s + \int_{t_i}^{t} L_z(s, \omega(s), \omega'(s))\,\mathrm{d}s - L_{z'}(t, \omega(t), \omega'(t))$$

$\omega(t)$ being a solution of the differential equation

$$H(t, 0, \omega'(t)) = P_\mathrm{d}(t) \quad \text{with} \quad \omega(t_i) = q_K(t_i)$$

If this is the case, we set $q_K(t) = \omega(t)\ \forall t \in [t_i, t_{i+1}]$.

(B) If $[t_{i-1}, t_i]$ is the boundary interval, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K = \int_0^{t_i} L_z(s, q_K(s), q'_K(s))\,\mathrm{d}s + \int_{t_i}^{t} L_z(s, \omega(s), \omega'(s))\,\mathrm{d}s - L_{z'}(t, \omega(t), \omega'(t)) \tag{5}$$

$\omega(t)$ being an interior arc of the extremal, with $\omega(t_i) = q_K(t_i)$, which satisfies Euler's equation in its maximal domain $[t_i, t_{i+1}]$ and therefore satisfies the coordination equation (5). Now, we set $q_K(t) = \omega(t)\ \forall t \in [t_i, t_{i+1}]$.

From the computational point of view, the construction of $q_K$ can be performed with the same procedure as in the shooting method, with the use of a discretized version of Eq. (5). The exception is that at the instant when the values obtained for $z$ and $z'$ do not obey the restrictions, we force the solution $q_K$ to belong to the boundary until the moment when the conditions of leaving the domain (established in the main coordination theorem) are fulfilled.

## 6. A numerical example

A program was elaborated using the Mathematica package that resolves the optimization problem and was then applied to a hydrothermal system made up of the thermal equivalent and a hydraulic plant.

For the fuel cost model of the equivalent thermal plant, we use the quadratic model

$$\Psi(P(t)) = \alpha + \beta P(t) + \gamma P(t)^2$$

The units for the coefficients are: $\alpha$ in (\$/h); $\beta$ in (\$/h Mw); $\gamma$ in (\$/h MW$^2$).

The hydro-plant's active power generation is given by

$$P_\mathrm{h}(t) = -A(t)z'(t) - Bz'(t)z(t) - Cz'(t)^2$$

where the coefficients $A$, $B$ and $C$ are

$$A(t) = \frac{-1}{G}B_y(S_0 + t \cdot i), \quad B = \frac{B_y}{G}, \quad C = \frac{B_T}{G}$$

We consider that the transmission losses for the hydro-plant are expressed by Kirchmayer's model, with the following loss equation: $b_\mathrm{l} \cdot (P_\mathrm{h}(t))^2$. So,

Table 1
Coefficients of the thermal and hydraulic plants

| $\alpha$ | $\beta$ | $\gamma$ | | | |
|---|---|---|---|---|---|
| 0 | 4 | 0.001 | | | |
| $G$ | $i$ | $S_0$ | $B_T$ | $B_y$ | $b_l$ |
| 526 315 | 10 190 000 | $200 \times 10^9$ | $581.740 \times 10^{-10}$ | $149.5 \times 10^{-12}$ | 0.0002 |

$$H(t) = P_h(t) - b_l \cdot (P_h(t))^2$$

The units for the coefficients of the hydro-plant are: the efficiency $G$ in ($m^4$/h Mw), the restriction on the volume $b$ in ($m^3$), the loss coefficient $b_l$ in ($Mw^{-1}$), the natural inflow $i$ in ($m^3$/h), the initial volume $S_0$ in ($m^3$), the coefficients $B_T$ in ($m^{-2}$ h) and the coefficients $B_y$ in ($m^{-2}$) (parameters that depend on the geometry of the tanks).

The data for the thermal and hydraulic plants are summarized in Table 1.

The values of the power demand (in MW) were adjusted to the following curve:

$$P_d(t) = 1000 + 3(1 + 2\mathrm{Sin}[4\pi t/24] - t(24 - t)2\mathrm{Cos}[4\pi t/24])$$

Firstly, an optimization interval of 24 h. was considered, and a final volume $b = 30 \times 10^6$ $m^3$.

Fig. 2 presents the plots of power demand ($P_d$), thermal power ($P$) and effective hydraulic power ($H$). We can see that from 9 h until 15 h, corresponding to the hours of lowest power demand (i.e. with the most pronounced trough), the hydraulic plant stops functioning and the thermal plant assumes all the power demand. This is done to reserve water for when power demands are very high, which corresponds to the peaks that can be seen in the figure. In this case, the cost is \$120 848.

However, if we take a larger final volume, $b = 300 \times 10^6$ $m^3$, the solution is that depicted in Fig. 3. Here we see that as there is sufficient water, the hydraulic plant does not stop functioning at any time whatsoever, though the thermal plant shuts off in the most pronounced trough, i.e. from 11 h until 13 h. In this case, the fuel cost is \$51265.50, which logically is considerably lower.

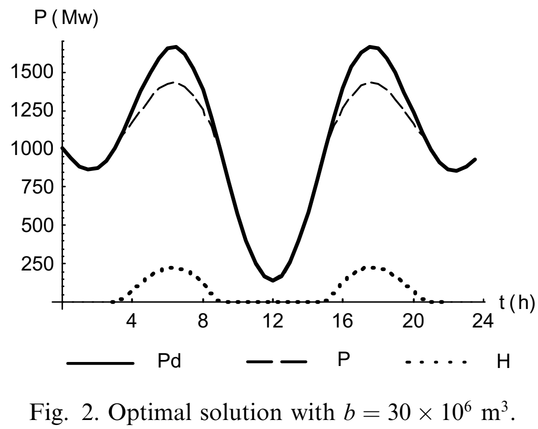


Fig. 2. Optimal solution with $b = 30 \times 10^6$ $m^3$.

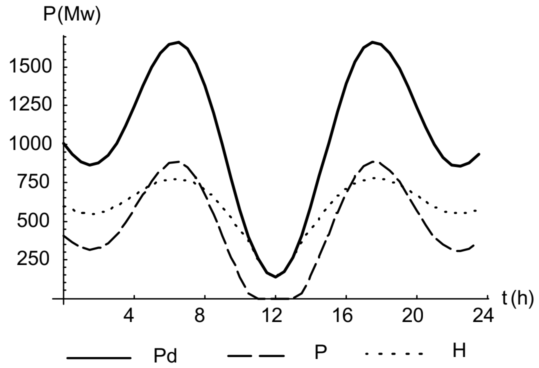


Fig. 3. Optimal solution with $b = 300 \times 10^6$ m$^3$.

## 7. Conclusions and future perspectives

From the Engineering perspective, one of the main contributions of this paper is that the implemented algorithm is independent of the models used both for thermal and for hydraulic power plants, in contrast to the majority of methods in this field, which use concrete models. What is more, we have obtained a very simple method that enables us to find an optimal solution in the presence of inequality constraints and which requires very little computational effort.

From the mathematical point of view, we have also obtained notable results. The main contribution of this paper is a property of the extremals in variational problems with non-holonomic constraints. Said property permits the solution to be constructed by means of a method inspired by the shooting method that is much simpler than those employed up until now for resolving this type of problem.

The algorithm presents a series of advantages. First of all, one does not have to start from specially selected initial values in order to run the method. Moreover, it shows a rapid convergence to the optimal solution, and its realization does not take much time due to the simplicity of the operations to be performed in this method.

As far as future perspectives are concerned, it would be most interesting to apply this method when the system is made up of $n$ hydraulic power plants.

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