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# New developments on equivalent thermal in hydrothermal optimization: an algorithm of approximation

L. Bayón*, J.M. Grau, M.M. Ruiz, P.M. Suárez

*University of Oviedo, Department of Mathematics. E.U.I.T.I., C./Manuel Llaneza 75, 33208 Gijón, Asturias, Spain*

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**Abstract**

In this paper we revise the classical formulation of the problem of the optimization of hydrothermal systems. First we demonstrate that a number of thermal plants can be substituted by a single one that behaves equivalently to the entire set. We then calculate the equivalent plant in the case where the cost functions are general (nonquadratic). We prove that the equivalent thermal plant is a second-order polynomial with piece-wise constant coefficients. Moreover, it belongs to the class $C^1$. Next we calculate the equivalent plant in the case of imposing constraints of minimum or maximum thermal power. Finally, we present an example and execute the proposed algorithm using Mathematica package.


*MSC:* 49K30

*Keywords:* Optimization; Hydrothermal systems; Equivalent thermal

## 1. Introduction

This paper studies the optimization of hydrothermal systems. A hydrothermal system is made up of hydraulic and thermal power plants that must jointly satisfy a certain demand in electric power during a definite time interval.

The idea of introducing an equivalent thermal plant has already appeared in several earlier studies. In [3] the authors consider it in application to purely thermal problems, though they did not notice the

* Corresponding author. Tel.: +34-985-18-22-42; fax: +34-985-18-22-40.
*E-mail address:* bayon@uniovi.es (L. Bayón)

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need to define the equivalent plant piece-wisely, since the restriction of power positivity is ignored. The idea has also been used in problems with hydraulic components. For example, [7] reports the application of the discrete maximum principle and [5] considers the application of a modified algorithm based on Pontryagin's maximum principle.

The concept of the equivalent thermal plant has been used up until now. Thus, [8] and [9] develop a short-term hydrothermal scheduling algorithm based on the simulated annealing technique, and an efficient short-term hydrothermal scheduling algorithm is proposed in [4] based on the evolutionary programming technique.

In a previous paper [1] we considered the possibility of substituting a problem with $m$ thermal plants and $n$ hydroplants $(\mathrm{H}_{\mathbf{n}} - \mathrm{T}_{\mathbf{m}})$ by an equivalent problem $(\mathrm{H}_{\mathbf{n}} - \mathrm{T}_{\mathbf{1}})$ with a single thermal power station: the equivalent thermal plant. In said paper, we calculated the equivalent minimizer in the case where the cost functions are second-order polynomials. We proved that the equivalent minimizer is a second-order polynomial with piece-wise constant coefficients; moreover, it belongs to the class $C^1$.

In this paper, we shall add various fundamental contributions. First we continue the theoretical studies of the equivalent thermal plant. We prove the existence and uniqueness of the equivalent minimizer, under certain assumptions. We then calculate the equivalent minimizer for a general (nonquadratic) model and go on to prove that it belongs to the class $C^1$.

Next we prove that, under certain hypotheses, the existence and uniqueness of the equivalent minimizer is guaranteed in the case of imposing constraints of minimum or maximum thermal power, and we go on to calculate the equivalent plant in this case. Finally, we present an example and perform the proposed algorithm using Mathematica package.

## 2. Description of the problem

Let us assume that a hydrothermal system accounts for $m$ thermal plants. We assume the following definitions throughout the paper.

Let $F_i : D_i \subseteq \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, m)$ be the cost functions of the thermal power plants. We assume that

$$\forall \xi \in D = D_1 + \cdots + D_m \subseteq \mathbb{R}, \quad \exists (x_1, \ldots, x_m) \in \prod_{i=1}^{m} D_i$$

the unique minimum of $\sum_{i=1}^{m} F_i(x_i)$ with the condition $\sum_{i=1}^{m} x_i = \xi$.

**Definition 1.** Let us call the $i$th distribution function, the function

$$\Psi_i : D_1 + \cdots + D_m \to D_i$$

defined by $\Psi_i(\xi) = x_i, \forall i = 1, \ldots, m$.

**Definition 2.** We will denote as the equivalent minimizer of $\{F_i\}_1^m$, the function $\Psi : D_1+\cdots+D_m \to \mathbb{R}$ defined by

$$\Psi(\xi) = \min \sum_{i=1}^{m} F_i(x_i)$$

with the constraint $\sum_{i=1}^m x_i = \xi$.

**Remark 3.** It follows that $\sum_{i=1}^m \Psi_i(\xi) = \xi$ and $\sum_{i=1}^m F_i(\Psi_i(\xi)) = \Psi(\xi)$.

## 3. New theoretical developments

In this paper, we continue the theoretical studies of the equivalent thermal plant. First we prove, under certain assumptions, the existence and uniqueness of the equivalent minimizer $\Psi$.

**Theorem 4.** *Let $\{F_i\}_{i=1}^m \subset C^1[0,\infty)$ be a set of functions such that $F_i'$ is strictly increasing ($i=1,\ldots,m$), with $F_i'(0) \leqslant F_{i+1}'(0)$, and let the function $F : [0,\infty)^m \to R$ be $F(x_1,\ldots,x_m) := \sum_{i=1}^m F_i(x_i)$.*

*Let $C_a := \{(x_1,\ldots,x_m) \in \mathbb{R}^m \mid x_i \geqslant 0 \wedge \sum_{i=1}^m x_i = a\}$.*

*Then, there exists a unique set $\{\Psi_i\}_{i=1}^m$ such that*:

(1) $(\Psi_1(a),\ldots,\Psi_m(a))$ *is the minimum of F on $C_a$, $\forall a \geqslant 0$.*

(2) *It holds that*

$$(\Psi_1(a),\ldots,\Psi_m(a)) \in \mathring{C}_a \Leftrightarrow a > \left(\sum_{i=1}^{m} F_i'^{-1} \circ F_m'\right)(0)$$

$$\Leftrightarrow \left(\sum_{i=1}^{m} F_i'^{-1} \circ F_m'\right)^{-1}(a) > 0$$

*being*

$$\Psi_k(a) = \left(\sum_{i=1}^{m} F_i'^{-1} \circ F_k'\right)^{-1}(a)$$

(3) $(\Psi_1(a),\ldots,\Psi_m(a)) \notin \mathring{C}_a \Rightarrow$ *for certain* $i \in \{1,\ldots,m-1\}$

$$\Psi_i(a) = \Psi_{i+1}(a) = \cdots = \Psi_m(a) = 0.$$

**Proof.** (1) The existence of a minimum on $C_a$ is guaranteed by its compactness; the strict convexity of $F$ guarantees unicity.

(2 $\Rightarrow$) If $(\Psi_1(a),\ldots,\Psi_m(a)) \in \mathring{C}_a$ is the minimum of $F$ on $C_a$, then it is also a local minimum of $F$ on $\{(x_1,\ldots,x_m) \in (0,\infty)^m \mid \sum_{i=1}^m x_i = a\}$.

Consequently, for some $\lambda_a \in \mathbb{R}$, $(\Psi_1(a),\ldots,\Psi_m(a))$ is critical point of

$$F^*(x_1,\ldots,x_m) = F(x_1,\ldots,x_m) - \lambda_a \cdot (x_1+\cdots+x_m-a)$$

so, we will have

$$0 = \frac{\partial F^*(\Psi_1(a), \ldots, \Psi_m(a))}{x_i} = F_i'(\Psi_i(a)) - \lambda_a, \quad \forall i = 1, \ldots, m$$

therefore it follows that $\Psi_i(a) = F_i'^{-1}(\lambda_a)$ and, since $\sum_{i=1}^m \Psi_i(a) = a$, we have:

$$a = \sum_{i=1}^m F_i'^{-1}(\lambda_a) \Rightarrow \lambda_a = \left(\sum_{i=1}^m F_i'^{-1}\right)^{-1}(a)$$

and consequently

$$\Psi_k(a) = F_k'^{-1}\left(\sum_{i=1}^m F_i'^{-1}\right)^{-1}(a) = \left(\sum_{i=1}^m F_i'^{-1} \circ F_k'\right)^{-1}(a).$$

Now, since $0 < \Psi_i(a)$ and $F_i'$ and $F_i'^{-1}$are strictly increasing, we have

$$\left(\sum_{k=1}^m F_k'^{-1} \circ F_i'\right)(0) < \left(\sum_{k=1}^m F_k'^{-1} \circ F_i'\right)(\Psi_i(a)) = a.$$

(2) $\Leftarrow$) Let us consider

$$\Psi_k(a) = \left(\sum_{i=1}^m F_i'^{-1} \circ F_k'\right)^{-1}(a).$$

Let us see, firstly, that $\Psi_k(a) > 0$ for every $k \in \{1, \ldots, m\}$.

Bearing in mind that $F_m'(0) \geqslant F_k'(0)$ for every $k \in \{1, \ldots, m\}$ and that $F_i'$ and $F_i'^{-1}$ are increasing

$$a > \left(\sum_{i=1}^m F_i'^{-1} \circ F_m'\right)(0) \geqslant \left(\sum_{i=1}^m F_i'^{-1} \circ F_k'\right)(0) \Rightarrow \Psi_k(a) > 0$$

so $(\Psi_1(a), \ldots, \Psi_m(a)) \in \mathring{C}_a$.

Taking into account the above considerations, $(\Psi_1(a), \ldots, \Psi_m(a))$ is a critical point of the convex functional

$$F^*(x_1, \ldots, x_m) = F(x_1, \ldots, x_m) - \lambda_a(x_1 + \cdots + x_m - a)$$

considered in $(0, \infty)^m$, where

$$\lambda_a = \left(\sum_{i=1}^m F_i'^{-1}\right)^{-1}(a).$$

So $(\Psi_1(a), \ldots, \Psi_m(a))$ is a minimum of $F^*$ and is consequently also a minimum of $F$ on $\mathring{C}_a$.

(3) Let us suppose that for certain $i \in \{1, \ldots, m-1\}$, $\Psi_i(a) = 0$ and that $\Psi_{i+1}(a) > 0$.

Let us consider the function $f : [0, \Psi_{i+1}(a)] \to \mathbb{R}$

$$f(\varepsilon) = F(\Psi_1(a), \dots, \Psi_i(a) + \varepsilon, \Psi_{i+1}(a) - \varepsilon, \dots, \Psi_m(a)).$$

Bearing in mind that

$$(\Psi_1(a), \dots, \Psi_i(a) + \varepsilon, \Psi_{i+1}(a) - \varepsilon, \dots, \Psi_m(a)) \in C_a$$

for every $\varepsilon \in [0, \Psi_{i+1}(a))$, it is enough to observe that $f'_+(0) < 0$, which is contradictory with the minimum character of $(\Psi_1(a), \dots, \Psi_m(a))$. Indeed

$$f'(\varepsilon) = F_i'(\Psi_i(a) + \varepsilon) - F_{i+1}'(\Psi_{i+1}(a) - \varepsilon) = F_i'(\varepsilon) - F_{i+1}'(\Psi_{i+1}(a) - \varepsilon),$$
$$f'(0) = F_i'(0) - F_{i+1}'(\Psi_{i+1}(a)) < F_i'(0) - F_{i+1}'(0) < 0. \qquad \square$$

In the above theorem we also obtain the distribution functions $\Psi_k$. Now we define the equivalent thermal plant piece-wisely, taking into account the restriction of power positivity.

**Theorem 5.** *Let $\{F_i\}_{i=1}^m$, $F$, and $C_a$ be defined as in Theorem 4. Then there exists $\{\delta_k\}_{k=1}^{m+1} \subset \overline{\mathbb{R}}$ (with $\delta_{m+1} = \infty$) and $\{\Psi_k\}_{k=1}^m \subset C[0, \infty)$ such that for every $a > 0$, the minimum of $F$ on $C_a$ attains at $(\Psi_1(a), \dots, \Psi_m(a))$, being*

$$\delta_k = \sum_{i=1}^{k} (F_i'^{-1} \circ F_k')(0) \leqslant \sum_{i=1}^{k+1} (F_i'^{-1} \circ F_{k+1}')(0) = \delta_{k+1},$$

$$\Psi_k(a) = \begin{cases} \left(\sum_{i=1}^{j} F_i'^{-1} \circ F_k'\right)^{-1}(a) & \text{if } \delta_k \leqslant \delta_j \leqslant a < \delta_{j+1}, \\ 0 & \text{if } a \leqslant \delta_k. \end{cases}$$

**Proof.** We will argue by induction. If $m = 1$, it is obvious that $\Psi_1(a) = a$. Let us assume that the theorem is true for $m - 1$ and let us see that this implies that it is true for $m$.

If $a > \delta_m$, by virtue of Section (2) of Theorem 4

$$\Psi_k(a) = \left(\sum_{i=1}^{m} F_i'^{-1} \circ F_k'\right)^{-1}(a) > \left(\sum_{i=1}^{m} F_i'^{-1} \circ F_k'\right)^{-1}(\delta_m) = 0, \quad \forall k.$$

If $a \leqslant \delta_m$, by virtue of the Section (3) of Theorem 4, $\Psi_m(a) = 0$ and we are under conditions of using the induction hypothesis according to which

$$(\Psi_1(a), \dots, \Psi_{m-1}(a))$$

minimizes $\sum_{i=1}^{m-1} F_i(x_i)$ constrained to $\sum_{i=1}^{m-1} x_i = a$. Therefore,

$$(\Psi_1(a), \dots, \Psi_m(a))$$

minimizes $\sum_{i=1}^{m} F_i(x_i)$ constrained to $\sum_{i=1}^{m} x_i = a$. $\square$

We shall also prove that for a general model the equivalent thermal plant belongs to the class $C^1$. Let us see the following lemma first.

**Lemma 6.** *Let* $\{F_i\}_{i=1}^{2} \subset C^1[0,\infty)$ *be a set of functions such that* $F_i'$ *is strictly increasing* $(i=1,2)$ *with* $F_1'(0) \leqslant F_2'(0)$, *let* $\delta$ *be such that* $F_1'(\delta) = F_2'(0)$ *and the function*

$$g(\xi) = \begin{cases} \xi & \text{if } \xi < \delta, \\ [(F_2')^{-1} \circ F_1' + Id]^{-1}(\xi) & \text{if } \xi \geqslant \delta. \end{cases}$$

*The following is verified*:

(i) *For every* $a > 0$, $(g(a), a - g(a))$ *it provides the minimum value of* $F(x,y) = F_1(x) + F_2(y)$ *on*

$$\{(x,y) \mid x \geqslant 0 \wedge y \geqslant 0 \wedge x + y = a\}.$$

(ii) *The function* $\Psi(a) = F_1(g(a)) + F_2(a - g(a))$ *belongs to the class* $C^1$ *and* $\Psi'(0) = F_1'(0)$.

**Proof.** (i) It is Theorem 5 in the case of $m = 2$.

(ii) The only conflicting point is $\delta$. Now, bearing in mind that $g$ is continuous and that $g(\delta) = \delta$

$$\Psi(\delta-) = F_1(\delta) + F_2(0),$$
$$\Psi(\delta+) = F_1(g(\delta)) + F_2(\delta - g(\delta)) = F_1(\delta) + F_2(0)$$

so $\Psi$ is continuous. Let us see the lateral derivatives at $\delta$

$$\Psi'(\xi-) = g'(\xi-)F_1'(g(\xi-)) + (1 - g'(\xi-))F_2'(\xi - g(\xi-)),$$
$$\Psi'(\delta-) = F_1'(\delta),$$
$$\Psi'(\delta+) = g'(\delta+)F_1'(\delta+) + (1 - g'(\delta+))F_2'(0),$$
$$\Psi'(\delta+) = g'(\delta+)[F_1'(\delta+) - F_2'(0)] + F_2'(0).$$
$$\Psi'(\delta+) = F_2'(0) = F_1'(\delta).$$

Therefore $\Psi'(\delta+) = \Psi'(\delta-)$. Finally

$$\Psi'(x) = F_1'(g(x))g'(x) + F_2'(x - g(x))(1 - g'(x)),$$
$$\Psi'(0) = F_1'(g(0))g'(0) + F_2'(0)(1 - g'(0)) = F_1'(0). \qquad \square$$

We shall also prove that for a general model the equivalent thermal plant belongs to the class $C^1$.

**Theorem 7.** *Let* $\{F_i\}_{i=1}^{m} \subset C^1[0,\infty)$ *be a set of functions defined as in Theorem* 4. *Then the function*

$$\Psi(a) = \sum_{k=1}^{m} F_k(\Psi_k(a)) = \min_{v \in C_a} F(v)$$

*belongs to the class* $C^1$ *and* $\Psi'(0) = F_1'(0)$.

**Proof.** We will argue by induction. It is obvious for $m = 1$.

Let us consider the operation

$$(F \odot G)(x) := \min_{a \in [0,x]} F(a) + G(x-a) = \min_{(a,b) \in C_x} F(a) + G(b).$$

It is easy to realize that $\odot$ is associative and commutative. In these terms

$$\Psi = F_1 \odot F_2 \odot \cdots \odot F_m = (F_1 \odot F_2 \odot \cdots \odot F_{m-1}) \odot F_m$$

now then, by induction hypothesis, $\Theta = F_1 \odot F_2 \odot \cdots \odot F_{m-1}$ belongs to class $C^1$, so we are under conditions to use the previous lemma and to arrive at the fact that $\Theta \odot F_m = \Psi$ belongs to class $C^1$.

Since $\odot$ is associative, $\Psi = F_1 \odot (F_2 \odot \cdots \odot F_m)$ and using the previous lemma: $\Psi'(0) = F_1'(0)$. □

## 4. Equivalent thermal plant with constraints

In this section, we analyze the situation that arises when the thermal plants are constrained to restrictions of the type

$$\left\{ (y_1, \ldots, y_m) \in \mathbb{R}^m | P^i_{\min} \leqslant y_i \wedge \sum_{i=1}^m y_i = a \right\},$$

$$\left\{ (y_1, \ldots, y_m) \in \mathbb{R}^m | y_i \leqslant P^i_{\max} \wedge \sum_{i=1}^m y_i = a \right\}.$$

From the economic point of view, it may be interesting for one plant to generate a minimum power $P_{\min}$ instead of stopping. On the other hand, technical restrictions of the type $P_{\max}$ also appears. The construction of the equivalent plant is similar to that already developed in Section 3. To abbreviate, we present only the results for the case $y_i \geqslant P^i_{\min}$. Using the new variables

$$y_i = x_i + P^i_{\min},$$
$$F_i(x_i) = G_i(y_i) = G_i(x_i + P^i_{\min})$$

the proofs become those already developed in the previous section. We will denote as the equivalent minimizer of $\{G_i\}_1^m$, the function

$$\Upsilon : D_1 + \cdots + D_m \to \mathbb{R}$$

defined by

$$\Upsilon(\xi) = \min \sum_{i=1}^m G_i(y_i)$$

with the constraints $\sum_{i=1}^m y_i = \xi$ and $y_i \geqslant P^i_{\min}$.

**Theorem 8.** *Let* $\{G_i\}_{i=1}^m \subset C^1[P^i_{\min}, \infty)$ *be a set of functions such that* $G_i'$ *is strictly increasing* $(i = 1, \ldots, m)$, *with* $G_i'(P^i_{\min}) \leqslant G_{i+1}'(P^{i+1}_{\min})$, *and let the function* $G : [P^1_{\min}, \infty) \times \cdots \times [P^m_{\min}, \infty) \to \mathbb{R}$ *be* $G(y_1, \ldots, y_m) := \sum_{i=1}^m G_i(y_i)$. *Let* $C_a := \{(y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_i \geqslant P^i_{\min} \wedge \sum_{i=1}^m y_i = a\}$.

*Then, there exists a unique set* $\{\Upsilon_i\}_{i=1}^m$ *such that*:

(1) $(\Upsilon_1(a), \ldots, \Upsilon_m(a))$ *is the minimum of* $G$ *on* $C_a$, $\forall a \geqslant \sum_{i=1}^m P_{\min}^i$.
(2) *It holds that*

$$(\Upsilon_1(a), \ldots, \Upsilon_m(a)) \in \mathring{C}_a \Leftrightarrow a > \left(\sum_{i=1}^m G_i'^{-1} \circ G_m'\right)(P_{\min}^m)$$
$$\Leftrightarrow \left(\sum_{i=1}^m G_i'^{-1} \circ G_m'\right)^{-1}(a) > P_{\min}^m$$

*being*

$$\Upsilon_k(a) = \left(\sum_{i=1}^m G_i'^{-1} \circ G_k'\right)^{-1}(a).$$

(3) $(\Upsilon_1(a), \ldots, \Upsilon_m(a)) \notin \mathring{C}_a \Rightarrow$ *for certain* $i \in \{1, \ldots, m\}$

$$\Upsilon_i(a) = P_{\min}^i, \Upsilon_{i+1}(a) = P_{\min}^{i+1}, \ldots, \Upsilon_m(a) = P_{\min}^m.$$

**Theorem 9.** *Let* $\{G_i\}_{i=1}^m$, $G$, *and* $C_a$ *be defined as in Theorem* 8. *Then there exists* $\{\delta_k\}_{k=1}^{m+1} \subset \overline{\mathbb{R}}$ (*with* $\delta_{m+1} = \infty$) *and* $\Upsilon_k \in C[P_{\min}^k, \infty, \forall k = 1, \ldots, m$, *such that for every* $a > \sum_{i=1}^m P_{\min}^i$, *the minimum of* $G$ *on* $C_a$ *attains at*

$$(\Upsilon_1(a), \ldots, \Upsilon_m(a))$$

*being*

$$\delta_k = \sum_{i=1}^k (G_i'^{-1} \circ G_k')(P_{\min}^k) + \sum_{i=k+1}^m P_{\min}^i$$
$$\leqslant \sum_{i=1}^{k+1} (G_i'^{-1} \circ G_{k+1}')(P_{\min}^{k+1}) + \sum_{i=k+2}^m P_{\min}^i = \delta_{k+1},$$

$$\Upsilon_k(a) = \begin{cases} \left(\sum\limits_{i=1}^j G_i'^{-1} \circ G_k'\right)^{-1} \left(a - \sum\limits_{i=j+1}^m P_{\min}^i\right) & \text{if } \delta_k \leqslant \delta_j \leqslant a < \delta_{j+1}, \\ P_{\min}^k & \text{if } a \leqslant \delta_k. \end{cases}$$

**Theorem 10.** *Let* $\{G_i\}_{i=1}^m$ *(with* $G_i \in C^1[P^i_{\min}, \infty)$ $\forall i = 1, \ldots, m$*) be a set of functions defined as in Theorem* 8. *Then the function*

$$\Upsilon(a) = \sum_{k=1}^{m} G_k(\Upsilon_k(a)) = \min_{v \in C_a} G(v)$$

*belongs to the class* $C^1$ *and* $\Upsilon'(\sum_{i=1}^m P^i_{\min}) = G_1'(P^1_{\min})$.

## 5. An algorithm of approximation

We have developed a new algorithm for the approximate calculus of the thermal equivalent of $m$ thermal power plants whose cost functional is general (nonquadratic). The outline is the following:

(i) We linearly approximate the derivative of the cost function of each thermal plant, $F_i'(x)$, $i=1, \ldots, m$ in the power generation interval of each plant. This approximation may be done as finely as one wishes by simply increasing the number of splines in said interval. The integration of these functions leads us to the piece-wise defined functions $\widetilde{\Psi}_i(x)$, $i = 1, \ldots, m$ that approximate the cost function of each thermal plant considered

$$\widetilde{\Psi}_i(x) = \begin{cases} \widetilde{\alpha}_{ik} + \widetilde{\beta}_{ik}x + \widetilde{\gamma}_{ik}x^2 & \text{if } \delta_{ik} \leqslant x < \delta_{ik+1}; \quad k = 1, \ldots, l-1, \\ \widetilde{\alpha}_{il} + \widetilde{\beta}_{il}x + \widetilde{\gamma}_{il}x^2 & \text{if } x \geqslant \delta_{il}. \end{cases}$$

(ii) We next demonstrate that each function $\widetilde{\Psi}_i(x)$ can be considered as the minimizing equivalent of $l$ fictitious thermal plants, whose cost functions, denoted by $\{F_{i1}(x), F_{i2}(x), \ldots, F_{il}(x)\}$, are second-order polynomials

$$F_{ik}(x) = \alpha_{ik} + \beta_{ik}x + \gamma_{ik}x^2; \quad k = 1, \ldots, l.$$

The aforementioned coefficients, deduced from those obtained in [7], are given by (with $k = 1, \ldots, l$)

$$\beta_{ik} = 2\widetilde{\gamma}_{ik}\delta_{ik} + \widetilde{\beta}_{ik},$$

$$\gamma_{ik} = \frac{\widetilde{\gamma}_{ik}}{1 - \widetilde{\gamma}_{ik}(\sum_{j=1}^{k-1} \frac{1}{\gamma_{ij}})},$$

$$\sum_{j=1}^{l} \alpha_{ij} = \widetilde{\alpha}_{ik} - \frac{\widetilde{\beta}^2_{ik}}{4\widetilde{\gamma}_{ik}} - \sum_{j=1}^{k} \frac{\beta^2_{ij}}{4\gamma_{ik}}.$$

(iii) Finally, we construct the equivalent minimizer of all the functions obtained

$$\{F_{ij}\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,l}}.$$

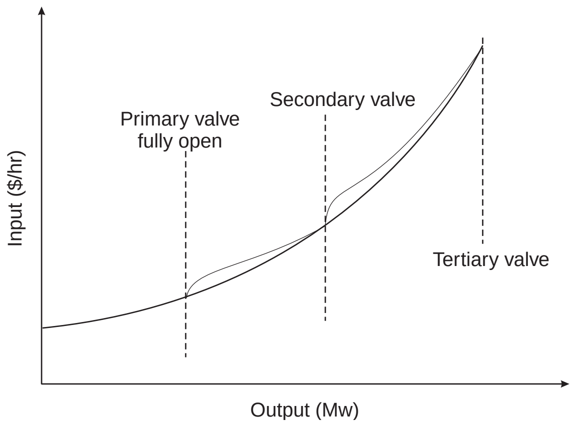


Fig. 1. Thermal plant input–output curve.

## 6. A example

Let us now see an example that illustrates the practical importance of the results established. Let us consider a thermal system that accounts for 3 thermal plants with piece-wise quadratic cost functions [6]. This model in the cost curves is due to sharp increases in throttle losses due to wire drawing effects occurring at valve points. These are loading (output) levels at which a new steam admission valve is being opened. The shape of the cost curve in the neighborhood of the valve points is difficult to determine by actual testing. Most utility systems find it satisfactory to represent the input–output characteristic by a smooth curve which can be defined by a polynomial or, even better, by means of piece-wise quadratic cost functions. We accept this more approximate model (Fig. 1).

The cost functions $F_i$ are piece–wise quadratic cost functions

$$F_i(x) = \alpha_i + \beta_i x + \gamma_i x^2$$

$$F_1(x) = \begin{cases} 1537.16 + 21.277x + 0.00286x^2 & \text{if } 0 \leqslant x < 51.049, \\ 1535.96 + 21.324x + 0.00239918x^2 & \text{if } x \geqslant 51.049, \end{cases}$$

$$F_2(x) = \begin{cases} 3240.78 + 6.347x + 0.09803x^2 & \text{if } 0 \leqslant x < 52.682, \\ 3008.08 + 15.181x + 0.0141888x^2 & \text{if } x \geqslant 52.682, \end{cases}$$

$$F_3(x) = \begin{cases} 2991.94 + 17.621x + 0.01325x^2 & \text{if } 0 \leqslant x < 80.151, \\ 2957.84 + 18.472x + 0.00794119x^2 & \text{if } 80.151 \leqslant x < 149.221, \\ 2802.69 + 20.5514x + 0.000973874x^2 & \text{if } x \geqslant 149.221 \end{cases}$$

and the units for the coefficients are: $\alpha$ in (\$/h); $\beta$ in (\$/h MW); $\gamma$ in (\$/h MW$^2$). The previous theoretical results of this paper establish the existence and uniqueness of the equivalent minimizer of any cost functions such that $F_i'$ are strictly increasing.

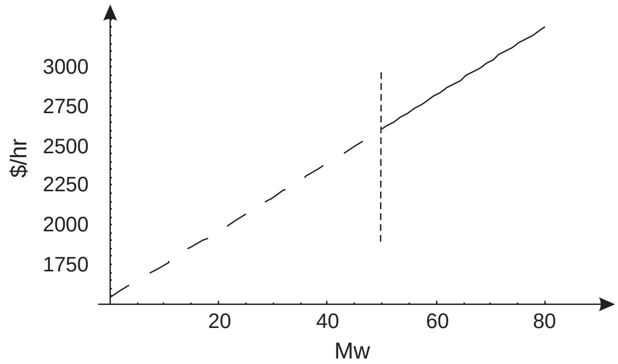


Fig. 2. The piece-wise quadratic cost function $F_1(x)$.

First, using the proposed algorithm, we obtain the functions

$$\{F_{11}(x), F_{12}(x)\}, \{F_{21}(x), F_{22}(x)\} \quad \text{and} \quad \{F_{31}(x), F_{32}(x), F_{33}(x)\}$$

of which each $F_i(x)$, $i = 1, 2, 3$, is respectively equivalent minimizer. The equivalent plant of these new functions, $\Psi$($/h) (with $\xi$ in MW) is a second-order polynomial with piece-wise constant coefficients:

$$\Psi(\xi) = \begin{cases} 7769.88 + 6.347\xi + 0.09803\xi^2 & \text{if } 0 \leqslant \xi \leqslant 52.6829, \\ 7537.18 + 15.181\xi + 0.0141888\xi^2 & \text{if } 52.6829 \leqslant \xi \leqslant 85.9838, \\ 7482.94 + 16.4427\xi + 0.00685166\xi^2 & \text{if } 85.9838 \leqslant \xi \leqslant 240.983, \\ 7380.72 + 17.2911\xi + 0.00509155\xi^2 & \text{if } 240.983 \leqslant \xi \leqslant 348.71, \\ 6872.41 + 20.2064\xi + 0.000911324\xi^2 & \text{if } 348.71 \leqslant \xi \leqslant 587.374, \\ 6796.43 + 20.4651\xi + 0.000691106\xi^2 & \text{if } 587.374 \leqslant \xi \leqslant 798.63, \\ 6776.88 + 20.5141\xi + 0.000660452\xi^2 & \text{if } 798.63 \leqslant \xi \end{cases}$$

and is also the equivalent minimizer of the original cost functions $F_i(x)$ (Figs. 2, 3, 4).

The developed algorithm offers very good approximate results in comparison with prior methods, such as for instance [2].

## 7. Conclusions

In this paper, we present two fundamental contributions: firstly, new theoretical results relative to the equivalent thermal plant and, secondly, an algorithm for the approximate calculus for a general model.

With the new theoretical results, we establish the framework for a significant simplification of the study of optimization of hydrothermal systems, since our theorems are of a general character as they do not depend on the choice of models of cost functions of the thermal power plants. The algorithm presents

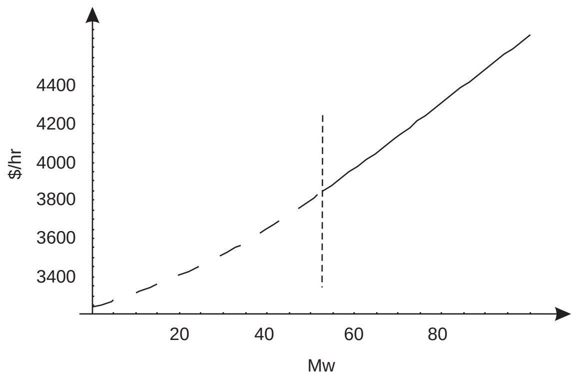


Fig. 3. The piece-wise quadratic cost function $F_2(x)$.

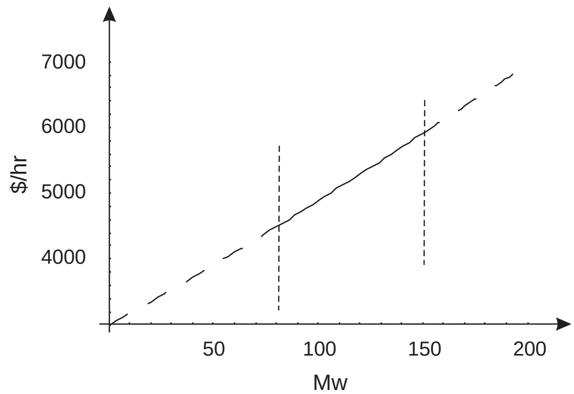


Fig. 4. The piece-wise quadratic cost function $F_3(x)$.

several advantages, such as: ease of implementation and minimum memory requirements (the program was developed on a PC with the Mathematica package). We also calculate the equivalent plant in the case of imposing constraints of minimum or maximum thermal power.

A major advantage of our method with respect to those previously employed is that it reduces the optimization of a system with $m$ thermal plants (general model) and $n$ hydraulic plants to a variational formulation without restrictions. This formulation allows us to employ the theory of calculus of variations to the highest degree, and the problem is thus afforded a significant simplification.

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