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New developments in the application of Pontryagin's Principle for the hydrothermal optimization

L. BAYÓN†, J. M. GRAU, M. M. RUIZ AND P. M. SUÁREZ
Department of Mathematics, E.U.I.T.I., Oviedo University, C./ Manuel Llaneza 75, 33208 Gijón, Asturias, Spain

In this paper we have developed a much simpler theory than previous ones that resolves the problem of the optimization of hydrothermal systems. The problem involves non-holonomic inequality constraints. In particular, we have established a necessary condition for the stationary functions of the functional. We shall use Pontryagin's Minimum Principle as the basis for proving this theorem, setting out our problem in terms of optimal control in continuous time, with the Lagrange-type functional. This theorem allows us to elaborate the optimization algorithm that leads to the determination of the optimal solution of the hydrothermal system. We generalize the problem, taking into account a cost associated with the water, to then set out and solve the corresponding Bolza's problem. Finally, we present an example employing the algorithm developed for this purpose with the 'Mathematica' package.

Keywords: optimal control; Bolza's problem; Pontryagin's Principle; scheduling; hydrothermal system.

1. Introduction

This paper studies the optimization of hydrothermal systems. A hydrothermal system is made up of hydraulic and thermal power plants that must jointly satisfy a certain demand in electric power during a definite time interval. Thermal plants generate power at the expense of fuel consumption, which is the object of minimization, while hydro-plants obtain power from the energy liberated by water, there being a limited quantity of water available during the optimization period.

The study of optimal conditions for the functioning of a hydrothermal system constitutes a complicated problem that has attracted significant interest in recent decades. Several techniques have been applied to solve this problem, such as functional analysis techniques (El-Hawary & Christensen, 1979), network techniques (Branlund *et al.*, 1986), fuzzy dynamic programming (Xiao *et al.*, 1997), sequential Monte-Carlo simulation (Allan *et al.*, 1998), a probabilistic algorithm (Puntel *et al.*, 1998), a Lagrangian relaxation technique (Ernan *et al.*, 1999; Ngundam *et al.*, 2000), Ritz's method (Bayón & Suárez, 2000), neural networks (Lee & Kim, 2002) or a simulated annealing algorithm (Mantawy *et al.*, 2003).

In this paper we propose Pontryagin's Minimum Principle (PMP) to solve the optimum scheduling problem of hydrothermal systems. Several applications of optimal control theory (OCT) in hydrothermal optimization have been reported in the literature. These range from the initial studies corresponding to El-Hawary & Christensen (1979), Papageorgio (1985) or Christensen *et al.* (1987) to more recent works such as that of Wong *et al.* (1993), who consider a class of discrete-time constrained optimal control problems in which the cost function is non-smooth, or Pursimo *et al.* (1998), who have developed an optimal feedback control method, and in which the constraints are considered using Lagrangian multipliers. What is more, some applications of PMP to hydrothermal systems have focused on a different problem, such as that of designing the configuration of multi-reservoir systems (Mousavi & Ramamurthy, 2002).

†Email: bayon@uniovi.es

The studies cited above employ concrete models both for the function of thermal cost as well as for the function of effective hydraulic generation. Hence, if the model changes, the obtained algorithms are not valid. Such a variety of mathematical models forces us to undertake a general study of the problem. We have done so thanks to the new developments in the application of PMP for hydrothermal optimization that we present in this paper, which generalize and take even further those presented in the literature to date.

In prior studies (Bayón *et al.*, 2002a), it was proven that the problem of optimization of the fuel costs of a hydrothermal system with m thermal power plants may be reduced to the study of a hydrothermal system made up of one single thermal power plant, called the thermal equivalent. In the present paper, we first of all consider a simple hydrothermal system with one hydraulic power plant and m thermal power plants that have been substituted by their thermal equivalent. Under these conditions, we present the problem from the electrical engineering perspective to then go on to resolve the mathematical problem thus formulated. We will call this problem: the H_1–T_1 problem. This type of approach is quite common, as we can see in the recent study of Bortolossi *et al.* (2002).

In Section 2, we shall see that the H_1–T_1 problem consists in the minimization of a functional

$$F(z) = \int_0^T L(t, z(t), z'(t))\, \mathrm{d}t,$$

within the set of piecewise C^1 functions ($\hat{C}^1$) that satisfy $z(0) = 0$, $z(T) = b$ and the constraints

$$0 \leqslant H(t, z(t), z'(t)) \leqslant P_{\mathrm{d}}(t) \quad \forall\, t \in [0, T].$$

Hence, the problem involves non-holonomic inequality constraints. Variational problems in which the derivatives of the admissible functions must be subject to certain inequality constraints (differential inclusions) have traditionally been dealt with by recurring to a large number of diverse techniques (see, e.g. Clarke, 1983). Using classic mathematical methods, we shall focus in the present paper on the development of the applications of OCT to the specific problem of hydrothermal optimization.

In Section 3, we shall establish a necessary condition for the stationary functions of the functional and we shall use PMP as the basis for proving this theorem. We shall see that the treatment of the constraints of the problem using this new approach will be very simple.

The development is self-contained and extremely basic and also enables the construction, in Section 4, of the optimization algorithm that leads to the determination of the optimal solution of the hydrothermal system. In the said section, we shall also study the general case in which the system consists of n hydraulic power plants and we will call this problem the H_n–T_1 problem.

In Section 5, we shall assign a cost to the water, we shall generalize the problem and we shall set out the corresponding Bolza's problem. We shall thus show how it is possible using this technique to modify the cost functional and to obtain the solution of different problems in a simple way.

Finally, in Section 6, we present an example employing the algorithm developed for this purpose with the 'Mathematica' package.

2. Statement of the problem (H_1–T_1)

The H_1–T_1 problem consists in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0, T]$. The said cost may be represented by the functional

$$F(P) = \int_0^T \Psi(P(t))\, \mathrm{d}t,$$

where Ψ is the function of thermal cost of the thermal equivalent and $P(t)$ is the power generated by the said plant. Moreover, the following equilibrium equation of active power will have to be fulfilled

$$P(t) + H(t, z(t), z'(t)) = P_{\mathrm{d}}(t) \quad \forall t \in [0, T],$$

where $P_{\mathrm{d}}(t)$ is the power demand and $H(t, z(t), z'(t))$ is the power contributed to the system at the instant t by the hydro-plant, $z(t)$ being the volume that is discharged up to the instant t by the plant and $z'(t)$ the rate of water discharge of the plant at the instant t.

Taking into account the equilibrium equation, the problem reduces to calculating the minimum of the functional

$$F(z) = \int_0^T \Psi(P_{\mathrm{d}}(t) - H(t, z(t), z'(t)))\,\mathrm{d}t.$$

If we assume that b is the volume of water that must be discharged during the entire optimization interval, the following boundary conditions will have to be fulfilled:

$$z(0) = 0, \quad z(T) = b.$$

For the sake of convenience, we assume throughout the paper that these are sufficiently smooth and are subject to the following additional assumptions:
Function of thermal cost: Let us assume that the function of thermal cost $\Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\Psi'(x) > 0$, $\forall x \in \mathbb{R}^+$, and is thus strictly increasing. This constraint is absolutely natural: it reads more cost to more generated power. Let us also assume that $\Psi''(x) > 0$, $\forall x \in \mathbb{R}^+$, and is therefore strictly convex. The models traditionally employed meet this constraint.
Function of effective hydraulic generation: Let us assume that the hydraulic generation $H(t, z, z')$: $\Omega_H = [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing with respect to the rate of water discharge z', i.e. $H_{z'} > 0$. Let us also assume that $H(t, z, z')$ is concave with respect to z', i.e. $H_{z'z'} \leqslant 0$.

The real models meet these two restrictions; the former means more power to a higher rate of water discharge. We see that we only admit non-negative thermal power $P(t)$ and we will solely admit non-negative volumes $z(t)$ and rates of water discharge $z'(t)$.

Therefore, we may expound the mathematical problem in the following terms. We will call H_1–T_1 the problem of minimization of the functional

$$F(z) = \int_0^T L(t, z(t), z'(t))\,\mathrm{d}t,$$

with L having the form

$$L(t, z(t), z'(t)) = \Psi(P_{\mathrm{d}}(t) - H(t, z(t), z'(t))),$$

over the set Θ_b.

$$\Theta_b = \{z \in \hat{C}^1[0, T] | z(0) = 0,\ z(T) = b,\ 0 \leqslant H(t, z(t), z'(t)) \leqslant P_{\mathrm{d}}(t), \forall t \in [0, T]\}.$$

The assumptions we have made guarantee the fulfilment of the following inequalities: $L_{z'z'}(t, z, z') > 0$; $L_{z'}(t, z, z') < 0$. It is evident that in the set Θ_b technical constraints of the following type may also be considered

$$H(t, z(t), z'(t)) \leqslant H_{\max}.$$

To do so, it is sufficient to take the following function as the upper limit for $H(t, z(t), z'(t))$ at any instant

$$\min\{H_{\max}, P_{\rm d}(t)\},$$

and the theoretical development would be the same.

In a previous paper (Bayón *et al.*, 2003a), the problem to be solved was approximated substituting the constraint

$$0 \leqslant H(t, z(t), z'(t)) \leqslant P_{\rm d}(t), \tag{2.1}$$

by others of the type

$$0 \leqslant H(t, b, z'(t)), \quad H(t, 0, z'(t)) \leqslant P_{\rm d}(t). \tag{2.2}$$

To do so, bearing in mind the weak influence of volume $z(t)$, it was assumed that

$$H(t, b, z') \lesssim H(t, z, z') \lesssim H(t, 0, z') \quad \forall\, z \in \Theta_b,$$

thus obtaining an approximate solution. One of the main contributions of the present paper is that we shall now consider the original problem with constraints of type (2.1), without any additional simplifications.

If z satisfies Euler's equation for the functional $F(z)$, we have that, $\forall\, t \in [0, T]$, Euler's equation is fulfilled

$$L_z(t, z(t), z'(t)) - \frac{\rm d}{{\rm d}t}(L_{z'}(t, z(t), z'(t))) = 0. \tag{2.3}$$

Integrating (2.3), we have the integral form of Euler's equation, known as the Du Bois-Reymond equation

$$\int_0^t L_z(s, z(s), z'(s))\,{\rm d}s - L_{z'}(t, z(t), z'(t)) = -L_{z'}(0, z(0), z'(0)) = K \in \mathbb{R}^+ \quad \forall\, t \in [0, T]. \tag{2.4}$$

If we divide Euler's equation (2.3) by $L_{z'}(t, z(t), z'(t)) < 0,\ \forall\, t$, and integrating with

$$L(t, z(t), z'(t)) = \Psi(P_{\rm d}(t) - H(t, z(t), z'(t))),$$

we have that

$$-L_{z'}(t, z(t), z'(t)) \exp\left[-\int_0^t \frac{H_z(s, z(s), z'(s))}{H_{z'}(s, z(s), z'(s))}\,{\rm d}s\right] = -L_{z'}(0, z(0), z'(0)) = K \in \mathbb{R}^+. \tag{2.5}$$

We shall call relation (2.5) the coordination equation for $z(t)$, and the positive constant K will be termed the coordination constant of the extremal.

Let us now see the fundamental result (the main coordination theorem), which enables us to characterize the extremals of the problem and which is also the basis for elaborating the optimization algorithm that leads to determination of the optimal solution of the hydrothermal system. We shall use the above coordination equation (2.5) in the development of the proof of the theorem.

3. The main coordination theorem

We shall use PMP as the basis for proving this theorem, setting out our problem in terms of optimal control in continuous time, with the Lagrange-type functional.

In a previous paper (Bayón *et al.*, 2003a), the problem was set out considering the state variable to be $z(t)$, the control variable $u(t)$ and the state equation $z' = u$. The optimal control problem was therefore:

$$\min_{u(t)} \int_0^T L(t, z(t), u(t))\, \mathrm{d}t, \quad \text{with} \begin{cases} z' = u, \\ z(0) = 0, \quad z(T) = b, \\ u(t) \in \Omega^*(t) = \{x | 0 \leqslant H(t, b, x) \wedge H(t, 0, x) \leqslant P_\mathrm{d}(t)\}. \end{cases}$$

With this statement, the Du Bois-Reymond equation given by expression (2.4) was obtained by applying PMP, from which the necessary conditions for the extremals were consequently obtained. However, the real problem was not solved, but rather an approximation, since the weak influence of the volume was imposed so as to be able to consider the constraints defined by means of the set $\Omega^*(t)$.

In this paper we present the problem considering the state variable to be $z(t)$ and the control variable $u(t) = H(t, z(t), z'(t))$. Moreover, as $H_{z'} > 0$, the equation $u(t) - H(t, z(t), z'(t)) = 0$ allows the state equation $z' = f(t, z, u)$ to be explicitly defined. The optimal control problem is thus:

$$\min_{u(t)} \int_0^T L(t, z(t), u(t))\, \mathrm{d}t, \quad \text{with} \begin{cases} z' = f(t, z, u), \\ z(0) = 0, \quad z(T) = b, \\ u(t) \in \Omega(t) = \{x | 0 \leqslant x \leqslant P_\mathrm{d}(t)\}, \end{cases}$$

with L having the form

$$L(t, z(t), u(t)) = \Psi(P_\mathrm{d}(t) - u(t)).$$

We shall see that with this new approach we shall arrive at the coordination equation (2.5). The main advantage of this study is that it is thus possible to consider the real constraint (2.1) by means of the set $\Omega(t)$.

It can be seen that from the relations $u(t) - H(t, z(t), z'(t)) = 0$ and $z' = f(t, z, u)$, we easily obtain

$$f_z = -\frac{H_z}{H_{z'}}, \quad f_u = \frac{1}{H_{z'}}.$$

Prior to proving the theorem, we define the following function.

DEFINITION 1 Let us term the coordination function of $q \in \Theta_b$ the function in $[0, T]$, defined as follows

$$\mathbb{Y}_q(t) = -L_{z'}(t, q(t), q'(t)) \exp\left[-\int_0^t \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))}\, \mathrm{d}s\right].$$

THEOREM 1 (The main coordination theorem) If $q \in \hat{C}^1$ is a solution of problem H_1–T_1, then there exists a constant $K \in \mathbb{R}^+$ such that

(i) If $0 < H(t, q(t), q'(t)) < P_\mathrm{d}(t)$ (t is not a boundary point) $\Rightarrow \mathbb{Y}_q(t) = K$.
(ii) If $H(t, q(t), q'(t)) = P_\mathrm{d}(t) \Rightarrow \mathbb{Y}_q(t) \geqslant K$.
(iii) If $H(t, q(t), q'(t)) = 0 \Rightarrow \mathbb{Y}_q(t) \leqslant K$.

Proof. We shall term the optimal control u_{opt}, which we see in our case is the function of effective hydraulic generation $H(t, z(t), z'(t))$, and therefore the optimal state will be $q(t)$. Let $\mathbb{H}$ be the Hamiltonian associated with the problem

$$\mathbb{H}(t, z, u, \lambda) = \Psi(P_\mathrm{d}(t) - u) + \lambda f(t, z, u).$$

In virtue of Pontryagin's Principle, there exists a piecewise C^1 function λ_{opt} (co-state variable) that satisfies the two following conditions:

$$\lambda'_{\mathrm{opt}}(t) = -\frac{\partial \mathbb{H}(t, q(t), u_{\mathrm{opt}}(t), \lambda_{\mathrm{opt}}(t))}{\partial z} = -\lambda_{\mathrm{opt}}(t)\, f_z(t, q(t), u_{\mathrm{opt}}(t)), \tag{3.1}$$

$$\mathbb{H}(t, q(t), u_{\mathrm{opt}}(t), \lambda_{\mathrm{opt}}(t)) \leqslant \mathbb{H}(t, q(t), u, \lambda_{\mathrm{opt}}(t)) \quad \forall u, \;\; 0 \leqslant u \leqslant P_{\mathrm{d}}(t). \tag{3.2}$$

From (3.1), it follows that

$$\lambda_{\mathrm{opt}}(t) = \lambda_{\mathrm{opt}}(0) \exp\left[-\int_0^t f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right]. \tag{3.3}$$

From (3.2), it follows that for each t, $u_{\mathrm{opt}}(t)$ minimizes the function

$$F(u) = \Psi(P_{\mathrm{d}}(t) - u) + \lambda_{\mathrm{opt}}(t)\, f(t, q(t), u), \quad \text{on } \{u | 0 \leqslant u \leqslant P_{\mathrm{d}}(t)\}.$$

Hence, in accordance with the Kuhn–Tucker Theorem, for each t there exist two real non-negative numbers, α and β, such that $u_{\mathrm{opt}}(t)$ is a critical point of

$$F^*(u) = \Psi(P_{\mathrm{d}}(t) - u) + \lambda_{\mathrm{opt}}(t)\, f(t, q(t), u) + \alpha(-u) + \beta(u - P_{\mathrm{d}}(t)),$$

it being verified that:

- if $H(t, q(t), q'(t)) > 0$, then $\alpha = 0$.
- if $H(t, q(t), q'(t)) - P_{\mathrm{d}}(t) < 0$, then $\beta = 0$.

We hence have

$$F^{*\prime}(u_{\mathrm{opt}}(t)) = -\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t)) + \lambda_{\mathrm{opt}}(t)\, f_u(t, q(t), u_{\mathrm{opt}}(t)) - \alpha + \beta = 0,$$

and the following cases:

Case 1. $0 < u_{\mathrm{opt}}(t) = H(t, q(t), q'(t)) < P_{\mathrm{d}}(t)$. In this case, $\alpha = \beta = 0$ and hence

$$\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t)) = \lambda_{\mathrm{opt}}(t)\, f_u(t, q(t), u_{\mathrm{opt}}(t)).$$

From (3.3), we have

$$\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t)) = f_u(t, q(t), u_{\mathrm{opt}}(t))\lambda_{\mathrm{opt}}(0) \exp\left[-\int_0^t f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right],$$

$$\frac{\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t))}{f_u(t, q(t), u_{\mathrm{opt}}(t))} \exp\left[\int_0^t f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right] = \lambda_{\mathrm{opt}}(0).$$

Bearing in mind that

$$\frac{\Psi'}{f_u} = \Psi' H_{z'} = -L_{z'} \quad \text{and} \quad f_z = -\frac{H_z}{H_{z'}}, \tag{3.4}$$

the following relation is fulfilled

$$-L_{z'}(t, q(t), q'(t)) \exp\left[-\int_0^t \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))}\,\mathrm{d}s\right] = \lambda_{\mathrm{opt}}(0) \Rightarrow \mathbb{Y}_q(t) = K.$$

Case 2. $u_{\mathrm{opt}}(t) = H(t, q(t), q'(t)) = P_{\mathrm{d}}(t)$, then $\beta \geqslant 0$ and $\alpha = 0$. In this case,

$$-\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t)) + \lambda_{\mathrm{opt}}(t) f_u(t, q(t), u_{\mathrm{opt}}(t)) + \beta = 0.$$

Bearing in mind now that $\beta \geqslant 0$ and $f_u > 0$, we have

$$-\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t)) + \lambda_{\mathrm{opt}}(t) f_u(t, q(t), u_{\mathrm{opt}}(t)) \leqslant 0,$$

$$\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t)) \geqslant f_u(t, q(t), u_{\mathrm{opt}}(t))\lambda_{\mathrm{opt}}(0) \exp\left[-\int_0^t f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right],$$

$$\frac{\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t))}{f_u(t, q(t), u_{\mathrm{opt}}(t))} \exp\left[\int_0^t f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right] \geqslant \lambda_{\mathrm{opt}}(0).$$

Applying (3.4), we have

$$\mathbb{Y}_q(t) \geqslant K.$$

Case 3. $u_{\mathrm{opt}}(t) = H(t, q(t), q'(t)) = 0$, then $\alpha \geqslant 0$ and $\beta = 0$. In this case

$$-\Psi'(P_{\mathrm{d}}(t) - u_{\mathrm{opt}}(t)) + \lambda_{\mathrm{opt}}(t) f_u(t, q(t), u_{\mathrm{opt}}(t)) - \alpha = 0.$$

By analogous reasoning, we have

$$\mathbb{Y}_q(t) \leqslant K.$$

□

Note. With the hypothesis $L_{z'z'}(t, z, z') > 0$ (Bayón *et al.*, 2003b), the solution may also be guaranteed to be of class C^1.

4. Construction of the optimal solution. Generalization to the problem (H_n–T_1)

If we did not have the constraints $0 \leqslant H(t, z(t), z'(t)) \leqslant P_{\mathrm{d}}(t)$, we could use the shooting method to resolve the problem. In this case, we would use the coordination equation (2.5), $\forall\, t \in [0, T]$

$$-L_{z'}(t, z(t), z'(t)) \exp\left[-\int_0^t \frac{H_z(s, z(s), z'(s))}{H_{z'}(s, z(s), z'(s))}\,\mathrm{d}s\right] = -L_{z'}(0, z(0), z'(0)) = K \in \mathbb{R}^+.$$

Varying the initial condition of the derivative $z'(0)$ (initial flow rate), we would search for the extremal that fulfils the second boundary condition $z(T) = b$ (final volume). However, we cannot use this method in our case, as due to the restrictions, the extremals may not admit bilateral variations, i.e. they may present boundary arcs.

We use the same framework in the present case, but the variation of the initial condition for the derivative, which now need not make sense, is substituted by the variation of the coordination constant K.

The problem will consist in finding for each K the function q_K that satisfies $q_K(0) = 0$ and the conditions of the main coordination theorem and, from among these functions, the one that gives rise to an admissible function ($q_K(T) = b$).

We will denote by M the rate of water discharge at the instant $t = 0$ that is needed for the hydraulic power station to satisfy the power demand: $H(0, 0, M) = P_{\mathrm{d}}(0)$ and we will denote by m the rate of water discharge at the instant $t = 0$ that is needed for $H(0, 0, m) = 0$. We also set

$$K_m = -L_{z'}(0, 0, m), \quad K_M = -L_{z'}(0, 0, M).$$

We observe that $\forall\, x \in (m, M)$ (with the hypothesis $L_{z'z'}(t, z, z') > 0$), we have

$$K_M < -L_{z'}(0, 0, x) < K_m.$$

To construct q_K, we proceed by the steps shown in Appendix A.

From the computational point of view, the construction of q_K can be performed with the same procedure as in the shooting method, with the use of a discretized version of the coordination equation (2.5). The exception is that at the instant when the values obtained for z and z' do not obey the constraints, we force the solution q_K to belong to the boundary until the moment when the conditions of leaving the domain (established in the main coordination theorem) are fulfilled.

Having resolved the H_1–T_1 problem, we now generalize the study to the H_n–T_1 problem. Let us assume that a hydrothermal system accounts for n hydro-plants.

The mapping

$$H\colon \Omega_H \to \mathbb{R}; \quad H(t, z_1(t), z_2(t), \dots, z_n(t), z_1'(t), z_2'(t), \dots, z_n'(t))$$

is called the function of effective hydraulic contribution, and is the power contributed to the system at the instant t by the set of hydraulic plants, $z_i(t)$ being the volume that is discharged up to the instant t by the i-th power station, $z'{}_i(t)$ the rate of water discharge at the instant t by the i-th power station and $\Omega_H \subset [0, T] \times \mathbb{R}^{2n}$ the domain of definition of H. We say that $\vec{Z} = (z_1, z_2, \dots, z_n)$ is admissible for H if z_i belong to the class $\hat{C}^1[0, T]$, and $(t, z_1, \dots, z_n, z'{}_1, \dots, z'{}_n) \in \Omega_H, \forall\, t \in [0, T]$.

The volume b_i that should be discharged up to the instant T is called the admissible volume of the i-th hydraulic power station. Under the above notation, let $\vec{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ be the vector of admissible volumes.

Now we will call $\wp\{J, \vec{b}\}$, the problem of minimizing the functional

$$J(\vec{Z}) = \int_0^T L(t, \vec{Z}(t), \vec{z}'(t))\,\mathrm{d}t = \int_0^T \Psi(P_\mathrm{d}(t) - H(t, \vec{Z}(t), \vec{z}'(t)))\,\mathrm{d}t,$$

over the set $\Omega = \Omega_{b_1} \times \Omega_{b_2} \times \cdots \times \Omega_{b_n}$, being $\Omega_{b_i} \subset \{z_i \in \hat{C}^1[0, T]/z_i(0) = 0,\ z_i(T) = b_i\}$.

The problem of optimization of a hydrothermal system that involves various hydro-plants is highly complicated. One should not forget that the associated variational problem is related to solving a boundary-value problem for a system of differential equations. We have developed an algorithm of its numerical resolution prompted by the so-called method of cyclic coordinate descent. With the definitions presented in Appendix B, the following result is demonstrated (Bayón *et al.*, 2002b).

THEOREM 2 If the functional J is convex in Ω, and, in certain topology: (i) the minimizing mappings Φ_i are continuous, $\forall\, i = 1, \dots, n$, and (ii) the descending subsequences S_k^i are convergent, $\forall\, i = 1, \dots,$ n, then every descending subsequence converges to a solution of the problem $\wp\{J, \vec{b}\}$.

A problem of the type H_n–T_1 could thus be solved under certain conditions if we start out from the resolution of a sequence of problems of the type H_1–T_1. The solution of the problem will be constructed as the limit of a descending sequence. If the conditions of the theorem are fulfilled, this sequence provides us with an approximation of the solution. Beginning with some admissible $\overrightarrow{Q^0} = (z_1, \dots, z_n)$, we construct a sequence via successive and iterative applications of $\Phi_1, \Phi_2, \dots, \Phi_n$. The application of every Φ_i involves solving a problem of the type H_1–T_1. If we set $\Phi = (\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_2 \circ \Phi_1)$, the solution will be

$$\lim_{k \to \infty} \Phi^k\left(\overrightarrow{Q^0}\right).$$

From the algorithmic and computational point of view, we obtain the iteration process that calculates at each stage the optimal functioning of a hydraulic power station, while the behaviour of the rest of the stations is assumed fixed.

5. A Bolza problem: water cost

In the previous sections, the optimum control problem was studied considering the initial and final instants, 0 and T, respectively, to be previously fixed and the state of the system at these instants to be given by $z(0) = 0$, $z(T) = b$. The necessary optimality conditions provided by PMP were studied for the formulated problem.

In this section, we propose to study the same problem for a different final condition: when the final instant T is given and the final state has an upper boundary: $z(T) \leqslant b$. This problem arises in practice when a cost $S[z(T)]$ is assigned to the water at the hydraulic plants. This model is more real, since at many hydraulic plants the water is also put to other uses: irrigation, domestic consumption, etc. A solution that consists in not using up the maximum amount of available water in the reservoir for the optimization interval $[0, T]$ may thus make sense.

With this approach to the problem, our objective functional in Bolza's form is

$$F(z) = \int_0^T L(t, z(t), z'(t))\,\mathrm{d}t + S[z(T)].$$

To simplify the exposition, we shall carry out our approach with the H_1–T_1 problem, its extension to the general case H_n–T_1 being immediate. Remember that the constraint $0 \leqslant H(t, z(t), z'(t)) \leqslant P_\mathrm{d}(t)$ is still present.

Once more, we present the problem considering the state variable to be $z(t)$, the control variable $u(t) = H(t, z(t), z'(t))$ and the state equation $z' = f(t, z, u)$. The optimal control problem is thus:

$$\min_{u(t)} \int_0^T L(t, z(t), u(t))\,\mathrm{d}t + S[z(T)], \quad \text{with} \begin{cases} z' = f(t, z, u), \\ z(0) = 0, \quad z(T) \leqslant b, \\ u(t) \in \Omega(t) = \{x | 0 \leqslant x \leqslant P_\mathrm{d}(t)\}, \end{cases}$$

with

$$L(t, z(t), u(t)) = \Psi(P_\mathrm{d}(t) - u(t)).$$

THEOREM 3 If $q \in \hat{C}^1$ is a solution of problem H_1–T_1, then there exists a constant $K \in \mathbb{R}^+$ such that

(i) If $0 < H(t, q(t), q'(t)) < P_\mathrm{d}(t)$ (t is not a boundary point) $\Rightarrow \mathbb{Y}_q(t) = K$.
(ii) If $H(t, q(t), q'(t)) = P_\mathrm{d}(t) \Rightarrow \mathbb{Y}_q(t) \geqslant K$.
(iii) If $H(t, q(t), q'(t)) = 0 \Rightarrow \mathbb{Y}_q(t) \leqslant K$,

and

$$K \geqslant \frac{\partial S[q(T)]}{\partial z} \frac{-\mathbb{Y}_q(T)}{L_{z'}(T, q(T), q'(T))}.$$

Proof. The demonstration of (i), (ii) and (iii) is that of Theorem 1 (the main coordination theorem).

The application of PMP to this Bolza's problem leads us to the function λ_{opt} (co-state variable) satisfying

$$\lambda'_{\mathrm{opt}}(t) = -\frac{\partial \mathbb{H}(t, q(t), u_{\mathrm{opt}}(t), \lambda_{\mathrm{opt}}(t))}{\partial z} = -\lambda_{\mathrm{opt}} f_z(t, q(t), u_{\mathrm{opt}}(t)), \tag{5.1}$$

with the final condition

$$\lambda_{\mathrm{opt}}(T) - \frac{\partial S[q(T)]}{\partial z} \geqslant 0; \quad (= 0 \text{ if } q(T) < b). \tag{5.2}$$

From (5.1) it follows that

$$\lambda_{\mathrm{opt}}(t) = \lambda_{\mathrm{opt}}(0) \exp\left[-\int_0^t f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right],$$

and substituting in (5.2)

$$\lambda_{\mathrm{opt}}(0) \exp\left[-\int_0^T f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right] - \frac{\partial S[q(T)]}{\partial z} \geqslant 0,$$

$$\begin{aligned} K = \lambda_{\mathrm{opt}}(0) &\geqslant \frac{\partial S[q(T)]}{\partial z} \exp\left[\int_0^T f_z(s, q(s), u_{\mathrm{opt}}(s))\,\mathrm{d}s\right] \\ &= \frac{\partial S[q(T)]}{\partial z} \frac{-\mathbb{Y}_q(T)}{L_{z'}(T, q(T), q'(T))}. \end{aligned} \tag{5.3}$$

□

From the computational point of view, the construction of the optimal solution can be performed with the same procedure as in the previous problem (without water cost): Varying the coordination constant K, we would search for the extremal that fulfils the second boundary condition $z(T) \leqslant b$ and (5.3).

Firstly, we search for the value of K whose associated extremal satisfies $q_K(T) = b$. If the following relation is fulfilled

$$K \geqslant \frac{\partial S[q_K(T)]}{\partial z} \frac{-\mathbb{Y}_{q_K}(T)}{L_{z'}(T, q_K(T), q_K'(T))},$$

then $q_K(t)$ is the optimal solution and all the available water, b, is consumed.

If the encountered K does not verify (5.3), the value of K that fulfils the following equality is searched for

$$K = \frac{\partial S[q_K(T)]}{\partial z} \frac{-\mathbb{Y}q_K(T)}{L_{z'}(T, q_K(T), q_K'(T))},$$

then $q_K(t)$ is the optimal solution, and the optimal final volume in this case is $q_K(T) < b$.

6. A numerical example

A computer program was written (using the 'Mathematica' package) to apply the results obtained in this paper to a real power system. In this way, we avoided the use of standard commercial solvers (such as MINOS), which present serious drawbacks for resolving a problem with so many and such complex constraints as ours.

The system consists of eight thermal plants and three hydro-plants with the configuration of the hydro-network that we shall see below. Let us see the models of different subsystems used in our study.

6.1 *Cost fuel model*

The cost function that has systematically been used is a second-order polynomial

$$\Psi_i(x) = \alpha_i + \beta_i x + \gamma_i x^2.$$

TABLE 1 *Coefficients of the thermal plants*

Plant i	α_i	β_i	γ_i	b_{ii}	$P_{\max}$
1 (Aboño 1)	1227.83	17.621	0.01325	0.000103	330
2 (Aboño 2)	743.78	20.842	0.00211	0.000072	514
3 (Soto 2)	77.72	21.277	0.00286	0.000172	240
4 (Soto 3)	1615.35	16.676	0.01659	0.000100	330
5 (Narcea 2)	2248.16	−7.984	0.17026	0.000353	150
6 (Narcea 3)	1459.44	21.569	0.01489	0.000121	330
7 (Lada 3)	1625.43	6.347	0.09803	0.000220	150
8 (Lada 4)	2155.62	17.745	0.01982	0.000097	330

It is also usual to consider the function of losses $l_i(x) = b_{ii}x^2$ (Kirchmayer's model), where b_{ii} is termed the loss coefficient. As an example, we shall use the thermal system of the company HC in Asturias (Spain), which is made up of eight thermal plants. The data of the plants are summarized in Table 1. The units for the coefficients are α_i in \$/h, β_i in \$/h·MW, γ_i in \$/h·MW^2, the maximum thermal generation $P_{\max}$ in MW and the loss coefficients b_{ii} in 1/MW.

For the fuel cost model of the equivalent thermal plant, we use the quadratic model

$$\Psi(P(t)) = \alpha_{\mathrm{eq}} + \beta_{\mathrm{eq}} P(t) + \gamma_{\mathrm{eq}} P(t)^2.$$

We construct the equivalent thermal plant as we saw in Bayón *et al.* (2002c), obtaining $\alpha_{\mathrm{eq}} = 9377.2$, $\beta_{\mathrm{eq}} = 19.2616$ and $\gamma_{\mathrm{eq}} = 0.00175314$.

6.2 *Hydro-network model*

The hydro-network is assumed to have several chains of hydro-plants on different rivers as well as hydraulically isolated plants. We assume that the rate of discharge at the upstream plant affects the behaviour at the downstream plants. We say that the hydraulic system has hydraulic coupling. We consider that the transmission losses for the hydro-plant are also expressed by Kirchmayer's model. We use a variable-head model and the i-th hydro-plant's active power generation $P_{\mathrm{h}i}$ is given by

$$P_{\mathrm{h}i}(t) = A_i(t)z'_i(t) - B_i z'_i(t)[z_i(t) - \mathrm{Coup}_i(t)],$$

where $A_i(t)$ and B_i are the coefficients

$$A_i(t) = \frac{1}{G_i} B_{y_i}(S_{0i} + t i_i), \quad B_i = \frac{B_{y_i}}{G_i},$$

and $\mathrm{Coup}_i(t)$ represents the hydraulic coupling between plants. The parameters that appear in this formula are the efficiency G, the natural inflow i, the initial volume S_0 and the coefficient B_y, a parameter that depends on the geometry of the reservoir.

In the variable-head models, the term $-B_i z'_i(t)[z_i(t) - \mathrm{Coup}_i(t)]$ represents the negative influence of the consumed volume, and reflects the fact that consuming water lowers the effective height and hence the performance of the station. So, the function of effective hydraulic generation is

$$H_i(t) = P_{\mathrm{h}i}(t) - b_{ii}(P_{\mathrm{h}i}(t))^2,$$

TABLE 2 *Hydro-plant coefficients*

Plant i	G	b	b_{ii}	i	S_0	B_y	$H_{\max}$	ν
1 (Salime)	519840	11×10^6	0.000166	133200	239.5×10^6	4.34079×10^{-7}	120	0.003
2 (Tanes)	337542	8×10^6	0.000154	21600	25.3×10^6	3.06555×10^{-6}	200	0.004
3 (La Barca)	363950	5×10^6	0.000364	111600	25.2×10^6	2.61709×10^{-6}	60	0.003

where b_{ii} is the loss coefficient. As an example, we shall use the hydro-system of the company HC, which is made up of three variable-head hydro-plants. The plants in this system belong to different rivers, i.e.

$$\mathrm{Coup}_1(t) = 0, \quad \mathrm{Coup}_2(t) = 0, \quad \mathrm{Coup}_3(t) = 0.$$

Furthermore, we shall consider a linear model for the associated water cost

$$S[z(T)] = \nu z(T),$$

where ν is a water conversion factor, which accounts for the unit conversion from cubic metres to dollars.

The data of the hydro-plants are summarized in Table 2. The units for the coefficients of the hydro-plants are the efficiency G in $\mathrm{m}^4/\mathrm{h}{\cdot}\mathrm{MW}$, the constraint on the volume b in m^3, the loss coefficients b_{ii} in 1/MW, the natural inflow i in m^3/h, the initial volume S_0 in m^3, the coefficients B_y in m^{-2} and the maximum effective hydraulic generation $H_{\max}$ in MW.

6.3 *Solution*

We consider a short-term hydrothermal scheduling (24 h) with an optimization interval [0, 24] and we consider a discretization of 96 subintervals. The optimal power for the hydro-plants are shown in Fig. 1 and the system's power demand and the optimal power for the equivalent thermal plant in Fig. 2.

As can be seen in Fig. 1, the power generated by Plant 1 is limited by its technical maximum $H_{\max} = 120$. The same occurs in Plant 3, whose generated power is limited by its technical maximum $H_{\max} = 60$, while Plant 2 (with little available water b) does not reach this value. The three plants present intervals (of greater or lesser amplitude) in which they are switched off, coinciding with the trough in power demand.

Plant 1 has a considerably sized reservoir (see its S_0), compared to Plants 2 and 3, which are smaller. This is why the influence of the volume is greater in these last two plants, 2 and 3 (see their B_y). The different behaviours observed in these two plants with similar characteristics are due to the fact that Plant 3, with a large natural inflow i, consumes all its available water at the end of the interval so as to be able to make full use of this natural inflow i. Plant 2, in contrast, with a small natural inflow i, consumes its available water proportionally throughout the whole interval.

Finally, we highlight the cost $S[z(T)] = \nu z(T)$ that we have associated with the water in the three plants. This term does not correspond to a real cost of the cubic metres, as is applied for urban, industrial or irrigation consumption, since the water that turbines a plant is neither polluted nor lost. It is a factor similar to the penalization factor employed for the pollution produced by thermal power plants, the aim of which is to assure a certain water reserve. It is thus considered that the water is no longer in the reservoir, but that it continues to flow downstream and may be used for the aforementioned consumptions (urban, industrial and irrigation). With this interpretation, and the considered ν, it is seen that Plants 1 and 3 consume all the available water, whilst Plant 2 does not consume the available $8 \times 10^6\ \mathrm{m}^3$, but only $7.21685 \times 10^6\ \mathrm{m}^3$.

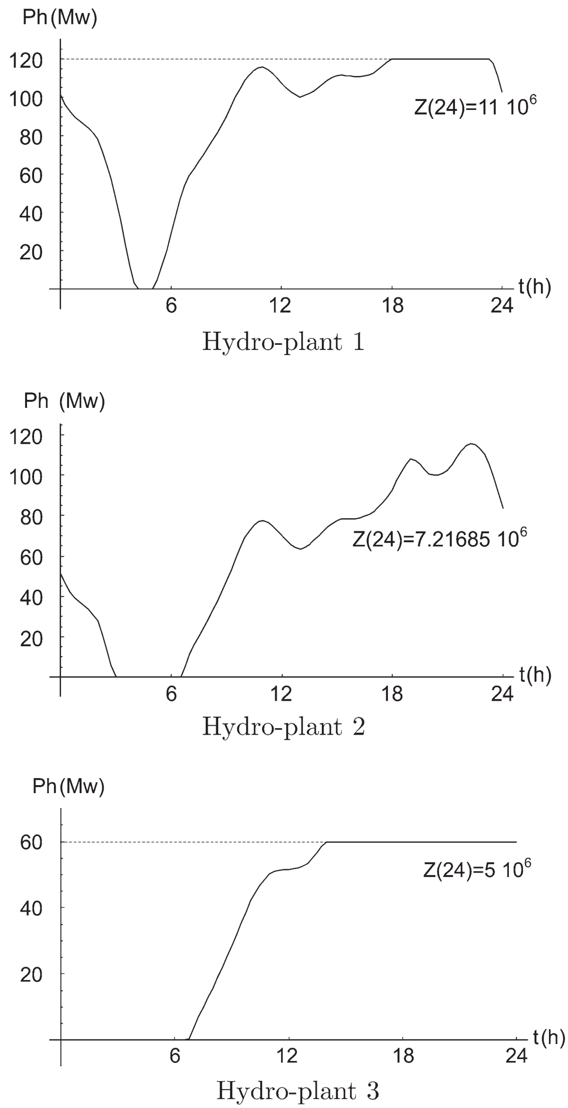

FIG. 1. Optimal hydro-power.

The algorithm shows a rapid convergence to the optimal solution. In the example, it turned out to be sufficient to perform four iterations to obtain the prescribed error (10^{-7}). The variation of relative error in absolute value with iterations is shown in Fig. 3. The time required by the program was 2 min on a personal computer (Pentium IV/2 GHz).

7. Conclusions and contributions

This paper describes a method for scheduling large-scale hydrothermal power systems based on PMP. We have developed a simple theory that resolves the problem of minimization of a functional $F(z)$

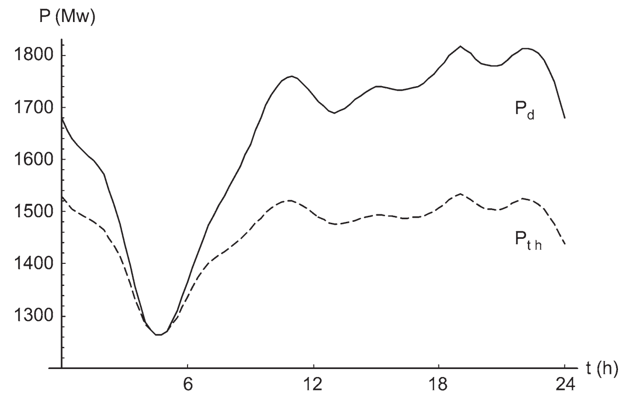

FIG. 2. Power demand $P_{\rm d}(t)$ and optimal thermal power $P_{\rm th}(t)$.

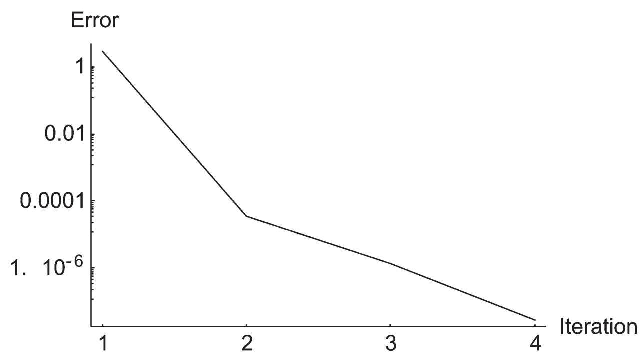

FIG. 3. Convergence to the optimal solution.

within the set of piecewise C^1 functions that satisfy boundary conditions and non-holonomic inequality constraints. We have established a necessary condition for the stationary functions of the functional, setting out our problem in terms of optimal control in continuous time, with the Lagrange-type functional. This theorem allows us to elaborate the optimization algorithm that leads to the determination of the optimal solution of the hydrothermal system. The problem has also been generalized assigning a cost to the water and solving the resulting Bolza's problem. This demonstrates that this technique allows the cost functional to be modified and the solution to different problems to be obtained in a simple way. Finally, we have presented an example employing the algorithm developed for this purpose with the 'Mathematica' package. The developed program is very simple and easy to use and the algorithm obtained with this study should be extensible to a large set of hydrothermal problems.

From the engineering perspective, one of the main contributions of this paper is that the implemented algorithm is independent of the models used for both thermal and hydraulic power plants, in contrast to the majority of methods in this field, which use concrete models. What is more, we have obtained a very simple method that enables us to find an optimal solution in the presence of inequality constraints and which requires very little computational effort.

From the mathematical point of view, we have also obtained notable results. The main contribution of this paper is a property of the extremals in variational problems with non-holonomic constraints. The said property permits the solution to be constructed by means of a method inspired by the shooting method that is much simpler than those employed to date for resolving this type of problem.

The algorithm presents a series of advantages. First of all, one does not have to start out from specially selected initial values in order to run the method. Moreover, it shows a rapid convergence to the optimal solution, and it can be run in a relatively short time due to the simplicity of the operations to be performed in this method.

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Appendix A

Step 1 (The first arc)

(i) If $K \geqslant K_m$, we set $q_K(t) = \omega(t)$, the solution of the differential equation $H(t, \omega(t), \omega'(t)) = 0$ with $\omega(0) = 0$ in the maximal interval $[0, t_1]$, where $K \geqslant \mathbb{Y}_\omega(t)$. (The thermal power station generates all the power demanded in $[0, t_1]$.)

(ii) If $K \leqslant K_M$, we set $q_K(t) = \omega(t)$, the solution of the differential equation $H(t, \omega(t), \omega'(t)) = P_{\mathrm{d}}(t)$ with $\omega(0) = 0$ in the maximal interval $[0, t_1]$, where $K \leqslant \mathbb{Y}_\omega(t)$. (The hydraulic power station generates all the power demanded in $[0, t_1]$.)

(iii) $K_M < K < K_m$ ($\exists x$ such that $K = -L_{z'}(0, 0, x)$). Now q_K will be the arc of the interior extremal (with $q_K(0) = 0$) which satisfies Euler's equation in its maximal domain $[0, t_1]$ and therefore the coordination equation $K = \mathbb{Y}_{q_K}(t)$.

i-th Step (i-th arc)

(A) If q_K has an interior arc in $[t_{i-1}, t_i]$, there are two possibilities:

(I) If $H(t_i, q_K(t_i), q_K'(t_i)) = 0$, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K \geqslant -L_{z'}(t, \omega(t), \omega'(t)) \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q_K'(s))}{H_{z'}(s, q_K(s), q_K'(s))}\,\mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))}\,\mathrm{d}s\right],$$

$\omega(t)$ being a solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = 0, \qquad \text{with } \omega(t_i) = q_K(t_i).$$

If this is the case, we set $q_K(t) = \omega(t), \forall t \in [t_i, t_{i+1}]$.

(II) If $H(t_i, q_K(t_i), q_K'(t_i)) = P_{\mathrm{d}}(t_i)$, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K \leqslant -L_{z'}(t, \omega(t), \omega'(t)) \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q_K'(s))}{H_{z'}(s, q_K(s), q_K'(s))}\,\mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))}\,\mathrm{d}s\right],$$

$\omega(t)$ being a solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = P_{\mathrm{d}}(t), \qquad \text{with } \omega(t_i) = q_K(t_i).$$

If this is the case, we set $q_K(t) = \omega(t), \forall t \in [t_i, t_{i+1}]$.

(B) If $[t_{i-1}, t_i]$ is the boundary interval, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K = -L_{z'}(t, \omega(t), \omega'(t)) \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q_K'(s))}{H_{z'}(s, q_K(s), q_K'(s))}\,\mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))}\,\mathrm{d}s\right],$$

$\omega(t)$ being an interior arc of the extremal, with $\omega(t_i) = q_K(t_i)$, which satisfies Euler's equation in its maximal domain $[t_i, t_{i+1}]$ and therefore satisfies the coordination equation. Now, we set $q_K(t) = \omega(t), \forall (t) \in [t_i, t_{i+1}]$.

Appendix B

DEFINITION 1 For every $\vec{Q} = (q_1, \ldots, q_n) \in \Omega$, we consider

$$H_i^{\vec{Q}}(t, z_i, z'_i) = H(t, q_1, \ldots, q_{i-1}, z_i, q_{i+1}, \ldots, q_n, q'_1, \ldots, q'_{i-1}, z'_i, q'_{i+1}, \ldots, q'_n)$$

and the functional $J_i^{\vec{Q}}$ defined over Ω_i by

$$J_i^{\vec{Q}}(z_i) = \int_0^T \Psi(P_\mathrm{d}(t) - H_i^{\vec{Q}}(t, z_i(t), z'_t(t)))\, \mathrm{d}t.$$

$H_i^{\vec{Q}}$ represents the power generated by the hydraulic system as a function of the rate of water discharge and the turbined volume by the i-th plant, under the assumption that the rest of the stations behave in a definite way.

DEFINITION 2 We call the i-th minimizing mapping, the mapping $\Phi_i \colon \Omega \to \Omega$, defined in the following way: For every $\vec{Q} = (q_1, \ldots, q_n) \in \Omega$

$$\Phi_i(q_1, \ldots, q_i, \ldots, q_n) = (q_1, \ldots, q^*, \ldots, q_n),$$

where $J(q_1, \ldots, q^*, \ldots q_n) < J(q_1, \ldots, X, \ldots, q_n),\ \forall X \in \Omega b_i - \{q^*\}$.

The following is evident: If J is convex in Ω, then

$$\Phi_i(\vec{Q}) = \vec{Q}, \quad \forall i = 1, \ldots, n \Leftrightarrow \vec{Q} \text{ is a solution of } \wp\{J, \vec{b}\},$$

and if J is convex in Ω, the minimizing mappings satisfy

$$\Phi_i(\vec{Q}) = \vec{Q} \Leftrightarrow J(\Phi_i(\vec{Q})) = J(\vec{Q}).$$

DEFINITION 3 Let us call a descending sequence of $\wp\{J, \vec{b}\}$ every sequence $\{\vec{Q}_{i,k}\} \subset \Omega$ with $(i, k) \in \{1, \ldots, n\} \times \mathbb{N}$, defined in the following recurrent manner:

$$\begin{aligned}
\vec{Q}_{i,1} &= \overrightarrow{Q^0} \in \Omega, \quad \forall i = 1, \ldots, n,\\
\vec{Q}_{i+1,k} &= \Phi_{i+1}(\vec{Q}_{i,k}), \quad \forall i = 1, \ldots, n-1, \quad \forall k \in \mathbb{N},\\
\vec{Q}_{1,k+1} &= \Phi_1(\vec{Q}_{n,k}), \quad \forall k \in \mathbb{N}.
\end{aligned}$$

DEFINITION 4 We call the i-th descending subsequence of $\{\vec{Q}_{i,k}\}$ the sequence resulting from fixing i: $S_k^i = \vec{Q}_{i,k}$.