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A Bolza problem in hydrothermal optimization

L. Bayón *, J.M. Grau, M.M. Ruiz, P.M. Suárez

University of Oviedo, Department of Mathematics, E.U.I.T.I. Campus of Viesques, 33204 Gijón, Asturias, Spain

Abstract

This paper studies the optimization of large-scale hydrothermal power systems. For the general problem with n hydro-plants, we present an algorithm using a particular strategy related to the Gauss–Southwell method of nonlinear optimization. The algorithm offers a constructive method for producing sequences of problems with one hydro-plant. For this simple problem we use Pontryagin's minimum principle to prove a condition for the extremals of the functional. We set out our problem in terms of optimal control in continuous time, with the Bolza-type functional. Finally, we present one example employing a program developed with the "Mathematica" package and analyze the convergence of the algorithm.

Keywords: Optimal control; Pontryagin's principle; Short-term hydrothermal coordination

1. Introduction

This paper addresses the short-term hydrothermal coordination (STHC) problem for large-scale power systems. This problem plays a most important role in the safety, reliability and economic operation of electric power systems whereby the generations of hydro- and thermal plants are allocated so as to minimize total operating cost in a schedule horizon of 1 day or 1 week while satisfying various constraints on plants and a certain demand in electric power.

This is a large-scale, nonlinear problem and there is a vast bibliography describing different formulations and solution methodologies applied to the STHC problem. Dynamic programming [1,2] and mixed integer linear programming [3,4] methods have been widely used in different formulations, but these approaches require substantial simplifying assumptions to make the problem computationally tractable. Promising results have been obtained by using the Lagrangian relaxation technique to generate near optimal solutions [5,6]. The disadvantage of this approach lies in the primal solution, which is infeasible. As a result, some heuristic procedures are needed to get a feasible primal solution.

The main drawback with the majority of these methods is the difficulty of treating large-scale systems. In this paper, we propose Pontryagin's minimum principle (PMP) to solve the STHC problem. Several applications of optimal control theory (OCT) in hydrothermal optimization have been reported in the literature

* Corresponding author.
E-mail address: bayon@uniovi.es (L. Bayón).

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[7–11]. What is more, some applications of Pontryagin's minimum principle to hydrothermal systems have focused on a different problem, such as that of designing the configuration of multi-reservoir systems [12].

Conventional studies consider only the fuel cost of thermal plants, but we shall also assign a cost to the water to then go on to set out the corresponding Bolza problem. This water cost does not correspond to a real cost of the m^3, as is applied for urban, industrial or irrigation consumption, since the water that turbines a plant is neither polluted nor lost. It is a similar factor to the penalization factor employed for the pollution produced by thermal plants, the aim of which is to assure a certain water reserve. It is thus considered that the water is no longer in the reservoir, but that it continues to flow downstream and may be used for the aforementioned consumptions.

Let us assume that the hydrothermal system accounts for n hydro-plants and m thermal plants ($H_n - T_m$ problem). In order to reduce the dimension of the problem, the method proposed in this paper adopts an iterative optimization algorithm and the technique consists of the following four stages:

Stage 1. The m thermal power plants are substituted by their thermal equivalent ($H_n - T_1$ problem).
Stage 2. An efficient algorithm prompted by the Gauss–Southwell method (descent in coordinate directions with maximal gradient) is developed. The $H_n - T_1$ problem could thus be solved, under certain conditions, if we resolve of a sequence of problems with one hydro-plant ($H_1 - T_1$ problem).
Stage 3. We shall use PMP as the basis for solving the $\mathbf{H_1 - T_1}$ problem, setting out our problem in terms of optimal control in continuous time, with the Bolza-type functional.
Stage 4. The generation assigned to the thermal equivalent is distributed between the m thermal plants.

This paper focuses on *Stage* 2 and *Stage* 3, as *Stage* 1 and *Stage* 4 have been fully developed in previous studies [13,14]. The paper is organized as follows. Section 2 gives a formulation of the STHC problem. Section 3 presents the simple $\mathbf{H_1 - T_1}$ problem and establishes a condition for the stationary functions of the functional. Section 4 provides a novel optimization algorithm that leads to determination of the optimal solution of the general problem with n hydro-plants. The algorithm offers a constructive method for producing sequences of problems with one hydro-plant. Section 5 includes one real example and numerical results involving a power system in Asturias (Spain) are reported. We employ a program developed for this purpose using the "Mathematica" package. Section 6 compares the behaviour of different procedures for carrying out the coordinate descent of the n hydro-plants and analyzing the convergence. Finally Section 7 presents conclusions and future perspectives.

2. Statement of the STHC problem

In prior studies [13,14], it was proven that the problem of optimization of the fuel cost of a hydrothermal system with m thermal plants may be reduced to the study of a hydrothermal system made up of one single thermal plant, called *the thermal equivalent*. Under these conditions, we present the ($\mathbf{H_n - T_1}$) problem from the electrical engineering perspective to then go on to resolve the mathematical problem thus formulated.

Let us assume that a hydrothermal system accounts for n hydro-plants. The mapping $H : \Omega_H \to \mathbb{R}^+$

$$H(t, z_1(t), \ldots, z_i(t), \ldots, z_n(t), z_1'(t), \ldots, z_i'(t), \ldots, z_n'(t)) = H(t, \bar{\mathbf{z}}(t), \bar{\mathbf{z}}'(t))$$

is called the function of effective hydraulic contribution, and is the power contributed to the system at the instant t by the set of hydro-plants, $z_i(t)$ being the volume that is discharged up to the instant t by the ith hydro-plant, $z_i'(t)$ the rate of water discharge at the instant t by the ith hydro-plant, and $\Omega_H \subset [0, T] \times (\mathbb{R}^+)^n \times (\mathbb{R}^+)^n$ the domain of definition of H.

We say that $\bar{\mathbf{z}} = (z_1, \ldots, z_n)$ is admissible for H if z_i, belong to the class, $\widehat{C}^1[0, T]$ (the set of piecewise C^1 functions), and $(t, \bar{\mathbf{z}}(t), \bar{\mathbf{z}}'(t)) \in \Omega_H \ \forall t \in [0, T]$. The volume b_i that could be discharged up to the instant T is called the admissible volume of the ith hydro-plant. Let $\bar{\mathbf{b}} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ be the vector of admissible volumes. In a general model, with hydraulic coupling between the n hydro-plants, we call $P_{hi}(t, \bar{\mathbf{z}}(t), \bar{\mathbf{z}}'(t))$ the power generated by the ith hydro-plant.

The ($\mathbf{H_n - T_1}$) problem consists in minimizing the cost needed to satisfy a certain power demand during the optimization interval $[0, T]$. Said cost may be represented by the functional

$$F(P,\bar{\mathbf{z}}) = \int_0^T \Psi(P(t))\mathrm{d}t + S[\bar{\mathbf{z}}(T)],$$

where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is the function of cost of the thermal equivalent, $P(t)$ is the power generated by said plant, and $S[\bar{\mathbf{z}}(T)]$ is the cost assigned to the water discharged. Moreover, the following equilibrium equation of active power will have to be fulfilled

$$P(t) + H(t,\bar{\mathbf{z}}(t),\bar{\mathbf{z}}'(t)) = P_{\mathrm{d}}(t) \quad \forall t \in [0,T],$$

where $P_{\mathrm{d}}(t)$ is the power demand. With the previous statement, we only admit non-negative P and H and therefore

$$0 \leqslant H(t,\bar{\mathbf{z}}(t),\bar{\mathbf{z}}'(t)) \leqslant P_{\mathrm{d}}(t) \quad \forall t \in [0,T].$$

Taking into account the equilibrium equation, the thermal component $P(t)$ disappears and our objective functional in the Bolza form is

$$J(\bar{\mathbf{z}}) = \int_0^T L(t,\bar{\mathbf{z}}(t),\bar{\mathbf{z}}'(t))\mathrm{d}t + S[\bar{\mathbf{z}}(T)] \tag{2.1}$$

with $L(t,\bar{\mathbf{z}}(t),\bar{\mathbf{z}}'(t)) = \Psi(P_{\mathrm{d}}(t) - H(t,\bar{\mathbf{z}}(t),\bar{\mathbf{z}}'(t)))$, on the set

$$\Omega = \left\{ \bar{\mathbf{z}} \in (\widehat{C}^1[0,T])^n \middle| \begin{array}{l} z_i(0) = 0,\ z_i(T) \leqslant b_i \quad \forall i = 1,\ldots,n \\ 0 \leqslant H(t,\bar{\mathbf{z}}(t),\bar{\mathbf{z}}'(t)) \leqslant P_{\mathrm{d}}(t) \quad \forall t \in [0,T] \end{array} \right\}. \tag{2.2}$$

As we see, the final instant T is given and the final state has an upper boundary: $z_i(T) \leqslant b_i$. We begin the development, in the following section, by presenting the simple problem with one hydro-plant.

3. The $(H_1 - T_1)$ problem

In the $H_1 - T_1$ problem, we have $\bar{\mathbf{z}} = z$ and our objective functional is

$$J(z) = \int_0^T L(t,z(t),z'(t))\mathrm{d}t + S[z(T)] \tag{3.1}$$

on the set

$$\Omega = \left\{ z \in \widehat{C}^1[0,T] \middle| \begin{array}{l} z(0) = 0,\ z(T) \leqslant b \\ 0 \leqslant H(t,z(t),z'(t)) \leqslant P_{\mathrm{d}}(t) \quad \forall t \in [0,T] \end{array} \right\}. \tag{3.2}$$

Hence, the problem involves non-holonomic inequality constraints (differential inclusions). We shall focus in the present paper on the development of the applications of OCT to this STHC problem. We assume throughout the paper that the functions are sufficiently smooth and are subject to the following additional assumptions:

- Let us assume that the function of thermal cost Ψ satisfies $\Psi'(x) > 0\ \forall x \in \mathbb{R}^+$ and is thus strictly increasing. This constraint is absolutely natural: it reads more cost to more generated power. Let us assume as well that $\Psi''(x) > 0\ \forall x \in \mathbb{R}^+$ and is therefore strictly convex. The models traditionally employed meet this constraint.
- Let us assume that the hydraulic generation $H(t,z,z')$ is strictly increasing with respect to the rate of water discharge z', i.e. $H_{z'} > 0$. Let us also assume that $H(t,z,z')$ is concave with respect to z', i.e. $H_{z'z'} \leqslant 0$. The real models meet these two restrictions; the former means more power to a higher rate of water discharge.

The assumptions we have made guarantee the fulfilment of the following inequalities: $L_{z'z'}(t,z,z') > 0$; $L_{z'}(t,z,z') < 0$. It is evident that in the set Ω, technical constraints of the type $P_{\mathrm{h}}(t,z(t),z\prime(t)) \leqslant P_{\mathrm{hmax}} \Rightarrow H(t,z(t),z\prime(t)) \leqslant H_{\max}$ may also be considered. To do so, it is sufficient to take the function $\min\{H_{\max},P_{\mathrm{d}}(t)\}$ as the upper limit for $H(t,z(t),z'(t))$ at any instant, and the theoretical development would be the same.

If z satisfies Euler's equation for the functional J, we have that $\forall t \in [0, T]$, Euler's equation is fulfilled:

$$L_z(t, z(t), z'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}(L_{z'}(t, z(t), z'(t))) = 0. \tag{3.3}$$

Integrating (3.3), we have the integral form of Euler's equation, known as the Du Bois–Reymond equation

$$\int_0^t L_z(s, z(s), z'(s))\mathrm{d}s - L_{z'}(t, z(t), z'(t)) = -L_{z'}(0, z(0), z'(0)) = K \in \mathbb{R}^+ \quad \forall t \in [0, T]. \tag{3.4}$$

If we divide Euler's Eq. (3.3) by $L_{z'}(t, z(t), z'(t)) < 0\ \forall t$, and integrate, we have that

$$-L_{z'}(t, z(t), z'(t)) \cdot \exp\left[-\int_0^t \frac{H_z(s, z(s), z'(s))}{H_{z'}(s, z(s), z'(s))}\mathrm{d}s\right] = -L_{z'}(0, z(0), z'(0)) = K \in \mathbb{R}^+ \quad \forall t \in [0, T]. \tag{3.5}$$

We shall call relation (3.5) the *coordination equation* for $z(t)$, and the positive constant K will be termed the *coordination constant* of the extremal. Let us now see the fundamental result, which enables us to characterize the extremals of the problem and which is also the basis for elaborating the optimization algorithm that leads to determination of the optimal solution of the hydrothermal system. We shall use the above coordination Eq. (3.5) in the development of the proof of the theorem.

In this paper, we generalize a previous study [15] in which the problem was simplified for its resolution. We now present the problem without simplifications, considering the state variable to be $z(t)$ and the control variable $u(t) = H(t, z(t), z'(t))$. Moreover, as $H_{z'} > 0$, the equation

$$u(t) - H(t, z(t), z'(t)) = 0$$

allows the state equation $z' = f(t, z, u)$ to be explicitly defined and we easily obtain $f_z = -\frac{H_z}{H_{z'}}$; $f_u = \frac{1}{H_{z'}}$. The optimal control problem is thus

$$\min_{u(t)} \int_0^T L(t, z(t), u(t))\mathrm{d}t + S[z(T)] \quad \text{with} \left\{ \begin{array}{l} z' = f(t, z, u), \\ z(0) = 0, \;\; z(T) \leqslant b, \\ u(t) \in \{x | 0 \leqslant x \leqslant P_\mathrm{d}(t)\} \end{array} \right\}.$$

with $L(t, z(t), u(t)) = \Psi(P_\mathrm{d}(t) - u(t))$. Prior to proving the theorem, we define the following function.

Definition 1. Let us term the *coordination function* of $q \in \Omega$ the function in $[0, T]$, defined as follows:

$$\mathbb{Y}_q(t) = -L_{z'}(t, q(t), q'(t)) \cdot \exp\left[-\int_0^t \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))}\mathrm{d}s\right].$$

Theorem 1 (The main coordination theorem). *If $q \in \widehat{C}^1$ is a solution of problem $(H_1 - T_1)$, then there exists a constant $K \in \mathbb{R}^+$ such that*

(i) *If $0 < H(t, q(t), q'(t)) < P_d(t)$ (t is not a boundary point)$\Rightarrow \mathbb{Y}_q(t) = K$.*
(ii) *If $H(t, q(t), q'(t)) = P_\mathrm{d}(t) \Rightarrow \mathbb{Y}_q(t) \geqslant K$.*
(iii) *If $H(t, q(t), q'(t)) = 0 \Rightarrow \mathbb{Y}_q(t) \leqslant K$ and $K \geqslant \frac{\partial S[q(T)]}{\partial z} \cdot \frac{-\mathbb{Y}_q(T)}{L_{z'}(T, q(T), q'(T))}$.*

Proof. We shall term the optimal control u_opt, which we see in our case is the function $H(t, z(t), z'(t))$, and therefore the optimal state will be $q(t)$. Let $\mathbb{H}$ be the Hamiltonian associated with the problem

$$\mathbb{H}(t, z, u, \lambda) = \Psi(P_\mathrm{d}(t) - u) + \lambda \cdot f(t, z, u).$$

In virtue of PMP, there exists a $\widehat{C}^1$ function λ_opt (co-state variable) that satisfies the two following conditions:

$$\lambda'_\mathrm{opt}(t) = -\frac{\partial \mathbb{H}(t, q(t), u_\mathrm{opt}(t), \lambda_\mathrm{opt}(t))}{\partial z} = -\lambda_\mathrm{opt}(t) \cdot f_z(t, q(t), u_\mathrm{opt}(t)), \tag{3.6}$$

$$\mathbb{H}(t, q(t), u_\mathrm{opt}(t), \lambda_\mathrm{opt}(t)) \leqslant \mathbb{H}(t, q(t), u, \lambda_\mathrm{opt}(t)); \quad \forall u, \;\; 0 \leqslant u \leqslant P_\mathrm{d}(t). \tag{3.7}$$

From (3.6), it follows that

$$\lambda_{\text{opt}}(t)=\lambda_{\text{opt}}(0)\cdot\exp\left[-\int_0^t f_z(s,q(s),u_{\text{opt}}(s))\mathrm{d}s\right]. \tag{3.8}$$

From (3.7), it follows that for each t, $u_{\text{opt}}(t)$ minimizes the function

$$\mathbb{F}(u)=\Psi(P_{\mathrm{d}}(t)-u)+\lambda_{\text{opt}}(t)\cdot f(t,q(t),u)\quad \text{on } \{u|0\leqslant u\leqslant P_{\mathrm{d}}(t)\}.$$

Hence, in accordance with the Kuhn–Tucker Theorem, for each t there exists two real non-negative numbers, α and β, such that $u_{\text{opt}}(t)$ is a critical point of

$$\mathbb{F}^*(u)=\Psi(P_{\mathrm{d}}(t)-u)+\lambda_{\text{opt}}(t)\cdot f(t,q(t),u)+\alpha\cdot(-u)+\beta\cdot(u-P_{\mathrm{d}}(t))$$

it being verified that if $H(t,q(t),q'(t))>0$, then $\alpha=0$ and if $H(t,q(t),q'(t))-P_{\mathrm{d}}(t)<0$, then $\beta=0$. We hence have

$$\mathbb{F}^{*\prime}(u_{\text{opt}}(t))=-\Psi'(P_{\mathrm{d}}(t)-u_{\text{opt}}(t))+\lambda_{\text{opt}}(t)\cdot f_u(t,q(t),u_{\text{opt}}(t))-\alpha+\beta=0$$

and the three following cases:

Case 1: $0<u_{\text{opt}}(t)=H(t,q(t),q'(t))<P_{\mathrm{d}}(t)$. In this case, $\alpha=\beta=0$ and hence

$$\Psi'(P_{\mathrm{d}}(t)-u_{\text{opt}}(t))=\lambda_{\text{opt}}(t)\cdot f_u(t,q(t),u_{\text{opt}}(t)).$$

From (3.8) we have

$$\Psi'(P_{\mathrm{d}}(t)-u_{\text{opt}}(t))=f_u(t,q(t),u_{\text{opt}}(t))\cdot\lambda_{\text{opt}}(0)\exp\left[-\int_0^t f_z(s,q(s),u_{\text{opt}}(s))\mathrm{d}s\right]\frac{\Psi'(P_{\mathrm{d}}(t)-u_{\text{opt}}(t))}{f_u(t,q(t),u_{\text{opt}}(t))}$$
$$\cdot\exp\left[\int_0^t f_z(s,q(s),u_{\text{opt}}(s))\mathrm{d}s\right]=\lambda_{\text{opt}}(0).$$

Bearing in mind that $\frac{\Psi'}{f_u}=\Psi'\cdot H_{z'}=-L_{z'}$ and $f_z=-\frac{H_z}{H_{z'}}$, the following relation is fulfilled:

$$-L_{z'}(t,q(t),q'(t))\cdot\exp\left[-\int_0^t\frac{H_z(s,q(s),q'(s))}{H_{z'}(s,q(s),q'(s))}\mathrm{d}s\right]=\lambda_{\text{opt}}(0)\Rightarrow\mathbb{Y}_q(t)=K.$$

Case 2: $u_{\text{opt}}(t)=H(t,q(t),q'(t))=P_{\mathrm{d}}(t)$, then $\beta\geqslant 0$ and $\alpha=0$. By analogous reasoning, we have $\mathbb{Y}_q(t)\geqslant K$.

Case 3: $u_{\text{opt}}(t)=H(t,q(t),q'(t))=0$, then $\alpha\geqslant 0$ and $\beta=0$. By analogous reasoning, we have $\mathbb{Y}_q(t)\leqslant K$.

Otherwise, the application of PMP to this Bolza problem leads us to the function λ_{opt} (3.8) having to satisfy the final condition

$$\lambda_{\text{opt}}(T)-\frac{\partial S[q(T)]}{\partial z}\geqslant 0;\quad (=0 \text{ if } q(T)<b).$$

Therefore

$$\lambda_{\text{opt}}(0)\cdot\exp\left[-\int_0^T f_z(s,q(s),u_{\text{opt}}(s))\mathrm{d}s\right]-\frac{\partial S[q(T)]}{\partial z}\geqslant 0,$$

$$K=\lambda_{\text{opt}}(0)\geqslant\frac{\partial S[q(T)]}{\partial z}\cdot\exp\left[\int_0^T f_z(s,q(s),u_{\text{opt}}(s))\mathrm{d}s\right]=\frac{\partial S[q(T)]}{\partial z}\cdot\frac{-\mathbb{Y}_q(T)}{L_{z'}(T,q(T),q'(T))}.\quad\square \tag{3.9}$$

We note that with the hypothesis $L_{z'z'}(t,z,z')>0$ the solution may also be guaranteed to be of class C^1 (see [16]). In the next section, we consider once more the general problem $H_{\mathbf{n}}-T_{\mathbf{1}}$ with n hydro-plants.

4. The optimization algorithm

The problem of optimization of a hydrothermal system that involves n hydro-plants is highly complicated because the associated variational problem is related to solving a boundary-value problem for a system of differential equations. In this section, we present an algorithm of its numerical resolution using a particular strategy related to the Gauss–Southwell method of coordinate descent [17]. With this method, a problem of the type $H_\mathbf{n} - T_\mathbf{1}$ could be solved, under certain conditions, if we start out from the resolution of a sequence of problems of the type $H_\mathbf{1} - T_\mathbf{1}$.

Let the function $G:\mathbb{R}^n \to \mathbb{R}$, $G \in C^1(\mathbb{R}^n)$, and $\bar{\mathbf{x}} = (x_1, \ldots, x_j, \ldots, x_n)$. The idea of the coordinate descent method is to use the coordinate axes as descent directions. The method sequentially searches for the minimum of G in all the directions $\bar{\mathbf{e}}_j$. Descent with respect to the x_j coordinate means that $G(x_1, \ldots, x_j, \ldots, x_n)$ is minimized with respect to x_j, while the rest remain fixed.

There exists a number of different selection strategies for the coordinates. However, we are specifically interested in the *Gauss–Southwell*-type selection scheme, which selects the coordinate that has the largest absolute value in the gradient vector. Instead of doing steps in the direction of the negative gradient as in standard *gradient descent* methods, only the variable that has the largest gradient component is changed. Now we adapt the finite-dimensional version of this algorithm to our functional (2.1) on (2.2).

The algorithm for the $H_n - T_1$ problem carries out several iterations and at each kth iteration calculates n stages, one for each hydro-plant. At each stage, it calculates the optimal functioning of a hydro-plant, while the behaviour of the rest of the plants is assumed fixed. For every $\bar{\mathbf{q}} = (q_1, \ldots, q_n) \in \Omega$, we consider the functional $J^i_\mathbf{q}$ defined by

$$J^i_{\bar{\mathbf{q}}}(z_i) = \int_0^T \Psi(P_\mathrm{d}(t) - H^i_{\bar{\mathbf{q}}}(t, z_i(t), z_i'(t)))\mathrm{d}t, \quad \text{with}$$
$$H^i_{\bar{\mathbf{q}}}(t, z_i, z_i') = H(t, q_1, \ldots, q_{i-1}, z_i, q_{i+1}, \ldots, q_n, q_1', \ldots, q_{i-1}', z_i', q_{i+1}', \ldots, q_n')$$

where $H^i_{\bar{\mathbf{q}}}$ represents the power generated by the hydraulic system as a function of the rate of water discharge and the volume turbined by the ith plant, under the assumption that the rest of the plants behave in a definite way. We call the ith minimizing mapping the mapping $\phi_i:\Omega \to \Omega$, defined in the following way: for every $\bar{\mathbf{q}} \in \Omega$

$$\phi_i(q_1, \ldots, q_i, \ldots, q_n) = (q_1, \ldots, q^*, \ldots, q_n),$$

where q^* minimizes $J^i_{\bar{\mathbf{q}}}$. Beginning with some admissible $\bar{\mathbf{q}}^0 = (q_1^0, \ldots, q_n^0)$, we construct a sequence of $\bar{\mathbf{q}}^k$ via successive applications of $\{\phi_i\}_{i=1}^n$. In order to select the "coordinate" q_i at which we carry out the descent at each stage, instead of calculating the coordinate that has the largest absolute value in the gradient vector, which now does not make sense, we consider the function $\mathbb{Y}^i_{\bar{\mathbf{q}}}(t)$, and the set of instants $\chi^i_{\bar{\mathbf{q}}}$ where the solution is a free extremal:

$$\mathbb{Y}^i_{\bar{\mathbf{q}}}(t) = -L_{z_i'}(t, \bar{\mathbf{q}}(t), \bar{\mathbf{q}}'(t)) \cdot \exp\left[-\int_0^t \frac{H_{z_i}(s, \bar{\mathbf{q}}(s), \bar{\mathbf{q}}'(s))}{H_{z_i'}(s, \bar{\mathbf{q}}(s), \bar{\mathbf{q}}'(s))}\mathrm{d}s\right],$$
$$\chi^i_{\bar{\mathbf{q}}} = \left\{ t \in [0,T] \,\middle|\, \begin{matrix} 0 < P_{\mathrm{h}i}(t, \bar{\mathbf{q}}(t), \bar{\mathbf{q}}'(t)) < P_{\mathrm{h}i\max} \\ 0 < H(t, \bar{\mathbf{q}}(t), \bar{\mathbf{q}}'(t)) < P_\mathrm{d}(t) \end{matrix} \right\}.$$

We give the name *imbalance* in the ith plant at $\bar{\mathbf{q}}$ to the positive number

$$\delta^i_{\bar{\mathbf{q}}} = \max_{t \in \chi^i_{\bar{\mathbf{q}}}} \mathbb{Y}^i_{\bar{\mathbf{q}}}(t) - \min_{t \in \chi^i_{\bar{\mathbf{q}}}} \mathbb{Y}^i_{\bar{\mathbf{q}}}(t).$$

The algorithm, at each stage of the kth iteration, it selects the coordinate ith with largest $\delta^i_{\bar{\mathbf{q}}}$. If we set

$$\Phi_{\sigma_k} = (\phi_{\sigma_k(n)} \circ \phi_{\sigma_k(n-1)} \circ \cdots \circ \phi_{\sigma_k(2)} \circ \phi_{\sigma_k(1)}),$$

$\sigma_k$being the permutation that at the kth iteration establishes the above mentioned order, and

$$\bar{\mathbf{q}}^k = \Phi_{\sigma_k}(\bar{\mathbf{q}}^{k-1})$$

the algorithm will search

$$\lim_{k\to\infty} \bar{\mathbf{q}}^k.$$

The application of every ϕ_i involves solving a problem of the type $H_1 - T_1$; the optimal functioning of one hydro-plant being calculated at each stage in the following way. The problem will consist in finding for each K the function q_K that satisfies $q_K(0) = 0$, the conditions (i), (ii) and (iii) of the main coordination theorem, and from among these functions, the one that gives rise to an admissible function ($q_K(T) \leqslant b$) and

$$K \geqslant \frac{\partial S[q_K(T)]}{\partial z} \cdot \frac{-\mathbb{Y}_{q_K}(T)}{L_{z'}(T, q_K(T), q_K'(T))}. \tag{4.1}$$

Step 1: From the computational point of view, the construction of q_K can be performed with the use of a discretized version of the coordination Eq. (3.5). In general, the construction of q_K' cannot be carried out all at once over the entire interval $[0, T]$. The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs and boundary arcs until completing the interval $[0, T]$. If the values obtained for z and z' do not obey the constraints, we force the solution q_K to belong to the boundary until the moment when the conditions of leaving the domain (established in the main coordination theorem) are fulfilled.

Step 2: Varying the coordination constant K, we would search for the extremal that fulfils the second boundary condition $q_K(T) \leqslant b$ and (4.1). Firstly, we search for the value of K whose associated extremal satisfies $q_K(T) = b$. The procedure is similar to the shooting method used to resolve second-order differential equations with boundary conditions. Effectively, we may consider the function $\varphi(K) := q_K(T)$ and calculate the root of $\varphi(K) - b = 0$, which may be realized approximately using elemental procedures.

Step 3: If (4.1) is fulfilled, then $q_K(t)$ is the optimal solution and all the available water, b, is consumed. If the encountered K does not verify (4.1), the value of K that fulfills the equality in (4.1) is searched for once more using the shooting method, and then $q_K(t)$ is the optimal solution, and the optimal final volume in this case is $q_K(T) < b$.

5. Application to a hydrothermal problem

A program that resolves the optimization problem was elaborated using the Mathematica package and was then applied to one example of a hydrothermal system made up of eight thermal plants and one hydro-plant of variable head. We consider the functional (3.1). We shall use the thermal system of Asturias (Spain), which is made up of eight thermal plants. The cost function Ψ_i that has been used is a quadratic model

$$\Psi_i(x) = \alpha_i + \beta_i x + \gamma_i x^2,$$

and we consider Kirchmayer's model for the transmission losses: $l_i(x) = b_{ii} \cdot x^2$, where b_{ii} is termed the loss coefficient. The data of the plants is summarized in Table 1.

The units for the coefficients are α_i in (\$/h), β_i in (\$/h MW), γ_i in (\$/h MW^2), and the loss coefficients b_{ii} in (1/MW). We construct the equivalent thermal plant as we saw in [13,14], obtaining $\alpha_{eq} = 9438.13$; $\beta_{eq} = 19.1762$; $\gamma_{eq} = 0.00178282$.

And finally, we shall use the hydro-plant of *Salime* in Asturias (Spain). We use a variable-head model and the hydro-plant's active power generation P_h (variable head) is a function of $z(t)$ and $z'(t)$

$$P_h(t, z(t), z'(t)) := A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t), \quad A(t) := \frac{B_y}{G}(S_0 + t \cdot i); \quad B = \frac{B_y}{G}.$$

In variable-head models, the term $-B \cdot z(t) \cdot z'(t)$ represents the negative influence of the consumed volume and reflects the fact that consuming water lowers the effective height and hence the performance of the plant. We consider that the transmission losses for the hydro-plant are also expressed by Kirchmayer's model (where b_{ll} is the loss coefficient). Hence, the function of effective hydraulic generation is

$$H(t, z(t), z'(t)) := P_h(t, z(t), z'(t)) - b_{ll} P_h^2(t, z(t), z'(t)).$$

Table 1
Coefficients of the thermal plants

Plant i	α_i	β_i	γ_i	b_{ii}
1 (*Aboño* 1)	1227.83	17.621	0.01325	0.000103
2 (*Aboño* 2)	743.78	20.842	0.00211	0.000072
3 (*Soto* 2)	77.72	21.277	0.00286	0.000172
4 (*Soto* 3)	1615.35	16.676	0.01659	0.000100
5 (*Narcea* 2)	2248.16	−7.984	0.17026	0.000353
6 (*Narcea* 3)	1459.44	21.569	0.01489	0.000121
7 (*Lada* 3)	1625.43	6.347	0.09803	0.000220
8 (*Lada* 4)	2155.62	17.745	0.01982	0.000097

Furthermore, we shall consider a linear model for the associated water cost $S[z(T)] = v \cdot z(T)$, where v is a water conversion factor that accounts for the unit conversion from (m^3) to (\$).

The data of the hydro-plant (*Salime*) is summarized in Table 2. The units for the coefficients of the hydro-plant are the efficiency G in (m^4/h MW), the constraint on the volume b in (m^3), the loss coefficient b_{ll} in (1/MW), the natural inflow i in (m^3/h), the initial volume S_0 in (m^3), the coefficient B_y (a parameter that depends on the geometry of the tanks) in (m^{-2}) and the maximum hydraulic generation $P_{\mathrm{h\,max}}$ in (MW). Finally, for the water cost we present two cases: (a) $v = 0.00375$ (\$/m^3) and (b) $v = 0.00475$ (\$/m^3).

We consider a short-term hydrothermal scheduling (24 h) with an optimization interval [0,24] and we consider a discretization of 96 subintervals. The optimal power for the hydro-plant, $P_{\mathrm{h}}(t)$, for two cases (a) and (b) is shown in Fig. 1, and the system's power demand, $P_{\mathrm{d}}(t)$, and the optimal power for the equivalent thermal plant, $P_{\mathrm{th}}(t)$, in case (a) in Fig. 2.

As can be seen in Fig. 1, for $v = 0.00375$ the hydro-plant consume all the available water, and the power generated by the hydro-plant is limited by its technical maximum $P_{\mathrm{h\,max}} = 120$. However, with $v = 0.00475$ the hydro-plant does not consume the available 11×10^6 m^3, but only 7.71017×10^6 m^3. The algorithm shows a rapid convergence to the optimal solution. The secant method was used to calculate the approximate value of K for which $q_K(T) - b = 0$. For example, in case (a) with 13 iterations: $|q_K(T) - b| < 10^{-3}$ (m^3) for $K = 4803.89860288 \times 10^{-6}$ and the time required by the program was 15 s on a personal computer (Pentium IV/2 GHz).

Table 2
Hydro-plant coefficients

G	b	b_{ll}	i	S_0	B_y	$P_{\mathrm{h\,max}}$
519840	11 10^6	0.000166	133200	239.5 10^6	4.34079 10^{-7}	120

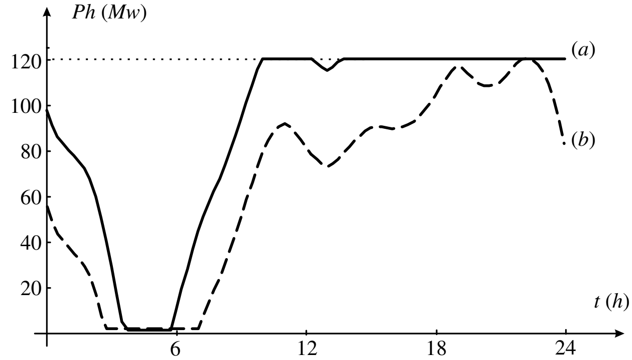

Fig. 1. Optimal hydro-power $P_{\mathrm{h}}(t)$.

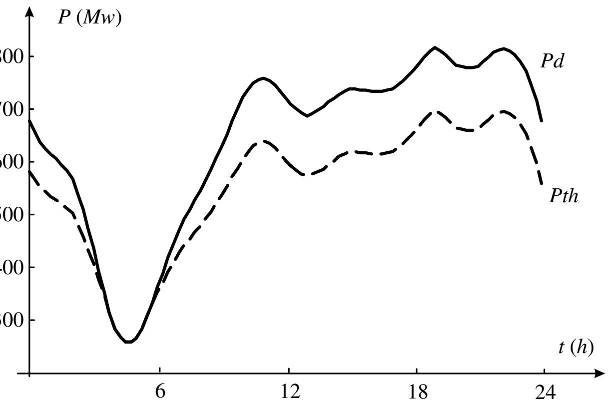

Fig. 2. Optimal thermal power $P_{\mathrm{th}}(t)$ and $P_{\mathrm{d}}(t)$.

6. Convergence of the algorithm

The analysis of the convergence of the minimizing sequence $\{\bar{\mathbf{q}}^k\}$, like the verification of the minimizing character of its limit, is a nontrivial problem of functional analysis that exceeds the goals of this paper. It is, however, simple to justify the convergence of the algorithm in a finite number of steps, simply by considering the following solution set:

$$\left\{\bar{\mathbf{q}}|\exists\sigma \in \sum_n \text{ such that } F[\bar{\mathbf{q}}] - F[\Phi_\sigma(\bar{\mathbf{q}})] < \varepsilon\right\},$$

i.e. the set of admissible elements over which, after one iteration of the algorithm, the functional has not decreased more than ε It need only be borne in mind that the value of the functional is lower limited by zero and hence an infinite sequence of descents greater than ε cannot occur.

Finally we compare the behaviour of different procedures to carry out the coordinate descent of the n hydro-plants and analyze the convergence. In a previous paper [18] addressing a similar problem, the authors consider an algorithm based on the *cyclic coordinate descent* (CCD) method. The coordinate selection defined in this method is slightly different from Gauss–Southwell selection. The CCD method minimizes a function $G(x_1, \ldots, x_j, \ldots, x_n)$ cyclically with respect to the coordinate variables. That is, first x_1 is searched, then x_2, etc. This method is generally attractive because of its easy implementation, but its convergence properties are generally poorer than Gauss–Southwell descent.

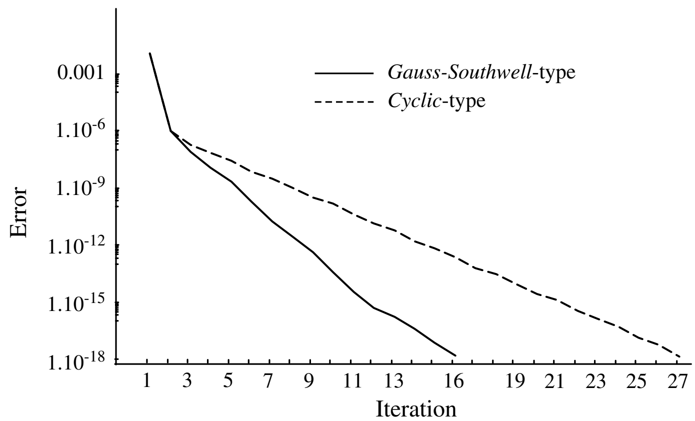

Fig. 3. Convergence with 10 hydro-plants.

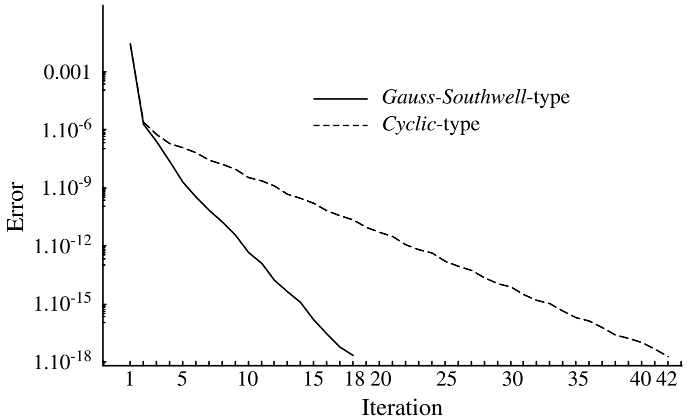

Fig. 4. Convergence with 20 hydro-plants.

To verify this statement, two tests were conducted considering in both the same eight thermal plants from the above example and 10 and 20 hydro-plants respectively with the same variable-head model as in the previous example. Figs. 3 and 4 present the obtained results. We can see how the Gauss–Southwell-type method presents a much more rapid convergence than its CCD-type counterpart, this effect being much more pronounced as the number of plants increases.

It is most important, in fact, to highlight that when considering twice the number of plants, the Gauss–Southwell-type method only requires two more iterations (from 16 to 18) to achieve the established tolerance. This makes said method an ideal tool for working with large-scale systems.

7. Conclusions, contributions and future perspectives

This paper describes a method for coordinating large-scale hydrothermal power systems based on Pontryagin's minimum principle. We have developed a simple theory that resolves the problem of minimization of a functional within the set of piecewise C^1 functions that satisfy boundary conditions and non-holonomic inequality constraints. We have seen that the treatment of the constraints of the problem using this new approach is very simple. The problem has been generalized assigning a cost to the water and solving the resulting Bolza problem. We have established a condition for the stationary functions of the functional, setting out our problem in terms of optimal control in continuous time. This theorem allows us to elaborate the optimization algorithm that leads to determination of the optimal solution of the hydrothermal system. Finally, we have presented examples employing the program developed with the "Mathematica" package. Simulation results show that the proposed method has enough efficiency for practical use in terms of convergence characteristics, thus indicating a robust and efficient tool for short-term coordination.

As far as future perspectives are concerned, it would be most interesting to apply this method when the system is made up of n hydro-plants including pumped hydro-plants. In this kind of problem, the derivative of H with respect to $z'(H_{z'})$ presents discontinuity at $z' = 0$, which is the point at which a sudden change of $H_{z'}$ is produced, as it is the border between the power generation zone (positive values of z') and the pumping zone (negative values of z'). The problem could hence be formulated within the framework of nonsmooth analysis, using Clarke's gradient.

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