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A constrained and non-smooth hydrothermal problem

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ABSTRACT

This paper addresses a hydrothermal problem that simultaneously considers non-regular Lagrangian and non-holonomic inequality constraints, obtaining a necessary minimum condition. It is further shown that the discontinuity of the lagrangian does not translate as discontinuity in the derivative of the solution. Finally, a solution algorithm is developed and applied to an example.

1. Introduction

This paper deals with the optimization of hydrothermal problems. In a previous paper [1], we considered a hydrothermal system with one hydro-plant and m thermal power plants that had been substituted by their thermal equivalent and addressed the problem of minimizing the cost of fuel $F(P)$ during the optimization interval $[0,T]$

$$F(P)=\int_0^T \Psi(P(t))\mathrm{d}t, \tag{1.1}$$

$$P(t)+H(t,z(t),z'(t))=P_d(t)\quad \forall t\in[0,T], \tag{1.2}$$

$$z(0)=0,\quad z(T)=b, \tag{1.3}$$

where Ψ is the function of thermal cost of the thermal equivalent and $P(t)$ is the power generated by said plant.

The following must also be verified: the equilibrium equation of active power (1.2), and the boundary conditions (1.3), where $P_d(t)$ is the power demand, $H(t,z(t),z'(t))$ is the power contributed to the system at the instant t by the hydro-plant, $z(t)$ being the volume that is discharged up to the instant t by the plant, $z'(t)$ the rate of water discharge of the plant at the instant t, and b the volume of water that must be discharged during the entire optimization interval.

In this paper, we likewise considered constraints for the admissible generated power

$$P(t)\geqslant 0;\quad H(t,z(t),z'(t))\geqslant 0.$$

The mathematical problem (P_1) was stated in the following terms:

$$\min_{z\in\Theta_1}F(z)=\min_{z\in\Theta_1}\int_0^T \Psi[P_d(t)-H(t,z(t),z'(t))]\mathrm{d}t=\min_{z\in\Theta_1}\int_0^T L(t,z(t),z'(t))\mathrm{d}t,$$

$$\Theta_1=\{z\in\widehat{C}^1[0,T]|z(0)=0,\ z(T)=b,\ 0\leqslant H(t,z(t),z'(t))\leqslant P_d(t)\ \forall t\in[0,T]\},$$

where $(\widehat{C}^1)$ is the set of piecewise C^1 functions.

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The problem (P_1) was formulated within the framework of optimal control [2–7] and

$$\mathbb{Y}_q(t) := -L_{z'}(t, q(t), q'(t)) \cdot \exp\left[-\int_0^t \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))} ds\right], \tag{1.4}$$

was called *the coordination function of $q \in \Theta_1$*, obtaining the following result:

Theorem 1. *If q is a solution of (P_1), then $\exists K \in \mathbb{R}^+$ such that*

$$\mathbb{Y}_q(t) \text{ is } \begin{cases} \leqslant K & \text{if } H(t, q(t), q'(t)) = 0, \\ = K & \text{if } 0 < H(t, q(t), q'(t)) < P_d(t), \\ \geqslant K & \text{if } H(t, q(t), q'(t)) = P_d(t). \end{cases}$$

In another previous paper [8], a problem of hydrothermal optimization with pumped-storage plants was addressed, though without considering constraints for the admissible generated power. In this kind of problem, the derivative of H with respect to z' (H'_z) presents discontinuity at $z' = 0$, which is the border between the power generation zone (positive values of z') and the pumping zone (negative values of z').

The mathematical problem (P_2) was stated in the following terms:

$$\min_{z \in \Theta_2} F(z) = \min_{z \in \Theta_2} \int_0^T \Psi[P_d(t) - H(t, z(t), z'(t))]dt = \min_{z \in \Theta_2} \int_0^T L(t, z(t), z'(t))dt,$$
$$\Theta_2 = \{z \in \widehat{C}^1[0, T] | z(0) = 0, z(T) = b\},$$

where $L(\cdot,\cdot,\cdot)$ and $L_z(\cdot,\cdot,\cdot)$ are the class C^0 and $L_{z'}(t, z, \cdot)$ is piecewise continuous ($L_{z'}(t, z, \cdot)$ is discontinuous in $z' = 0$).

Denoting by $¥_q(t)$, $q \in \Theta_2$, the function

$$¥_q(t) := -L_{z'}(t, q(t), q'(t)) + \int_0^t L_z(s, q(s), q'(s))ds \tag{1.5}$$

and by $¥_q^+(t)$ and $¥_q^-(t)$ the expressions obtained when considering the lateral derivatives of L with respect to z'.

The problem (P_2) was formulated within the framework of non-smooth analysis [9,10], using the generalized (or Clarke's) gradient, the following result being proven:

Theorem 2. *If q is a solution of (P_2), then $\exists K \in \mathbb{R}^+$ such that*

$$\begin{cases} ¥_q^+(t) = ¥_q^-(t) = K & \text{if } q'(t) \neq 0, \\ ¥_q^+(t) \leqslant K \leqslant ¥_q^-(t) & \text{if } q'(t) = 0. \end{cases}$$

This paper merges the two previous studies, simultaneously considering non-regular Lagrangian and non-holonomic inequality constraints (differential inclusions), obtaining a necessary minimum condition. Furthermore, under certain convexity conditions, we shall establish the result (*smooth transition*) that the derivative of the minimum is continuous, presents a constancy interval, the constant being the value for which $L_{z'}(t, z, \cdot)$ presents discontinuity. Finally, we shall present a solution algorithm and shall apply it to an example.

2. Mathematical statement and resolution of the problem

In this paper, we consider a hydrothermal system with one thermal plant (the thermal equivalent [11]) and one pumped hydro-plant, which will have certain constraints in both generation and pumping for H. We shall take $H_{\min}$ (maximum pumping capacity) as the lower boundary and $H_s(t) = \min\{H_{\max}, P_d(t)\}$ ($H_{\max}$ being maximum generation) as the upper boundary.

The mathematical problem (P_3) may be stated in the following terms:

$$\min_{z \in \Theta} F(z) = \min_{z \in \Theta} \int_0^T \Psi[P_d(t) - H(t, z(t), z'(t))]dt = \min_{z \in \Theta} \int_0^T L(t, z(t), z'(t))dt,$$
$$\Theta = \{z \in \widehat{C}^1[0, T] | z(0) = 0, \ z(T) = b, \ H_{\min} \leqslant H(t, z(t), \ z'(t)) \leqslant H_s(t) \ \forall t \in [0, T]\},$$

where $L(\cdot,\cdot,\cdot)$ and $L_z(\cdot,\cdot,\cdot)$ are the class C^0 and $L_{z'}(t, z, \cdot)$ is piecewise continuous ($L_{z'}(t, z, \cdot)$ is continuous with one single point of discontinuity at $z' = 0$). We shall assume that Ψ is strictly increasing and strictly convex, that H verifies $H_{z'} > 0$, and $H_z(t, z(t), 0) = 0$, and the strictly increasing nature of $L_{z'}(t, z, \cdot)$. We shall establish the necessary minimum condition for this problem with non-regular Lagrangian and constraints on the admissible functions, employing to this end the coordination function, $\mathbb{Y}_q(t)$.

We shall denote by $\mathbb{Y}_q^+(t)$ and $\mathbb{Y}_q^-(t)$ the expressions obtained when considering in (1.4) the lateral derivatives of L and H with respect to z'. We shall prove that these functions also verify Theorem 2 in the same way as $¥_q^+(t)$ and $¥_q^-(t)$, and that for the stated problem, except in $z' = 0$ for which $L_{z'}(t, z, \cdot)$ is not continuous, Theorem 1 will continue to be valid. We thus obtain the following result:

Theorem 3. *If q is a solution of (P_3), then $\exists K \in \mathbb{R}^+$ such that:*

$$\text{(i) If } q'(t) = 0 \quad \Rightarrow \quad \mathbb{Y}_q^+(t) \leqslant K \leqslant \mathbb{Y}_q^-(t),$$

$$\text{(ii) If } q'(t) \neq 0 \quad \Rightarrow \quad \mathbb{Y}_q(t) \text{ is} \begin{cases} \leqslant K & \text{if } H(t,q(t),q'(t)) = H_{\min}, \\ = K & \text{if } H_{\min} < H(t,q(t),q'(t)) < H_s(t), \\ \geqslant K & \text{if } H(t,q(t),q'(t)) = H_s(t). \end{cases}$$

Proof. Let us assume, for convenience sake, that there is a single interval $[t_1,t_2]$ with $t_1,t_2 \in (0,T)$ where $z' = 0$ (i.e. the hydro-plant remains shut down in the interval $[t_1,t_2]$). It is obvious that $q(t) = q(t_1)\ \forall t \in [t_1,t_2]$.

Let us consider the different situations that may arise in the interval $[0,T] = [0,t_1] \cup [t_1,t_2] \cup [t_2,T]$

(a) Generation—Shut down—Generation,
(b) Generation—Shut down—Pumping,
(c) Pumping—Shut down—Generation,
(d) Pumping—Shut down—Pumping.

At $[0,t_1]$ we are in a zone of generation or pumping, where only intervals with minimum or maximum constraints may appear in the hydraulic power generated. From Theorem 1 we have that $\exists K \in \mathbb{R}^+$ such that

$$\mathbb{Y}_q(t) \text{ is } \begin{cases} \leqslant K & \text{if } H(t,q(t),q'(t)) = H_{\min}, \\ = K & \text{if } H_{\min} < H(t,q(t),q'(t)) < H_s(t), \\ \geqslant K & \text{if } H(t,q(t),q'(t)) = H_s(t). \end{cases}$$

On the other hand, at $[t_2,T]$ we are also in a zone of generation or pumping, and from Theorem 1 we have that $\exists \widetilde{K} \in \mathbb{R}^+$ such that

$$\mathbb{Y}_q(t) \text{ is } \begin{cases} \leqslant \widetilde{K} & \text{if } H(t,q(t),q'(t)) = H_{\min}, \\ = \widetilde{K} & \text{if } H_{\min} < H(t,q(t),q'(t)) < H_s(t), \\ \geqslant \widetilde{K} & \text{if } H(t,q(t),q'(t)) = H_s(t). \end{cases}$$

We shall carry out the proof of Theorem 3 for case (a).

In this case, taking into consideration Theorem 2, we have that $\exists K^* \in \mathbb{R}^+$ such that

$$\int_0^{t_1} L_z(\tau,q(\tau),q'(\tau))\mathrm{d}\tau - L_{z'}^+(t_1,q(t_1),0) = K^*; \quad \int_0^{t_2} L_z(\tau,q(\tau),q'(\tau))\mathrm{d}\tau - L_{z'}^+(t_2,q(t_2),0) = K^*$$

and from the fact that $H_z(t,z(t),0) = 0\ \forall t \in [t_1,t_2]$, we have that $L_z(t,z(t),0) = 0$, from which

$$\int_{t_1}^{t_2} L_z(\tau,q(\tau),q'(\tau))\mathrm{d}\tau = 0$$

and therefore it is deduced that

$$L_{z'}^+(t_1,q(t_1),0) = L_{z'}^+(t_2,q(t_2),0). \tag{2.1}$$

Furthermore, $\forall t \in [t_1,t_2], z'(t) = 0$ is a discontinuity point of $L_{z'}(t,q(t),\cdot)$ and, once more, from Theorem 2

$$\int_0^{t} L_z(\tau,q(\tau),q'(\tau))\mathrm{d}\tau - L_{z'}^+(t,q(t),0) \leqslant K^* \leqslant \int_0^{t} L_z(\tau,q(\tau),q'(\tau))\mathrm{d}\tau - L_{z'}^-(t,q(t),0)$$

and, as

$$K^* = \int_0^{t_1} L_z(\tau,q(\tau),q'(\tau))\mathrm{d}\tau - L_{z'}^+(t_1,q(t_1),0) \quad \text{and} \quad \int_{t_1}^{t} L_z(\tau,q(\tau),q'(\tau))\mathrm{d}\tau = 0,$$

we deduce that for case (a) the following expression is verified

$$L_{z'}^-(t,q(t),0) \leqslant L_{z'}^+(t_1,q(t_1),0) \leqslant L_{z'}^+(t,q(t),0). \tag{2.2}$$

Besides, from Theorem 1, we have for t_1 that

$$K = -L_{z'}^+(t_1,q(t_1),0) \cdot \exp\left[-\int_0^{t_1} \frac{H_z(s,q(s),q'(s))}{H_{z'}^+(s,q(s),q'(s))}\mathrm{d}s\right]. \tag{2.3}$$

Since $\forall t \in [t_1,t_2]$

$$\int_{t_1}^{t} \frac{H_z(s,q(s),q'(s))}{H_{z'}^+(s,q(s),q'(s))}\mathrm{d}s = 0, \tag{2.4}$$

we have that $\forall t \in [t_1, t_2]$

$$\int_0^t \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s = \int_0^{t_1} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s = \int_0^{t_1} \frac{H_z(s,q(s),q'(s))}{H^-_{z'}(s,q(s),q'(s))}\mathrm{d}s = \int_0^{t} \frac{H_z(s,q(s),q'(s))}{H^-_{z'}(s,q(s),q'(s))}\mathrm{d}s.$$

For all $t \in [t_1, t_2]$, from (2.2) it is verified that

$$\begin{aligned} L^-_{z'}(t,q(t),0)\cdot \exp\left[-\int_0^t \frac{H_z(s,q(s),q'(s))}{H^-_{z'}(s,q(s),q'(s))}\mathrm{d}s\right] &\leqslant L^+_{z'}(t_1,q(t_1),0)\cdot \exp\left[-\int_0^t \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right] \\ &\leqslant L^+_{z'}(t,q(t),0)\cdot \exp\left[-\int_0^t \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right]. \end{aligned}$$

From (2.3) we have that

$$L^-_{z'}(t,q(t),0)\cdot \exp\left[-\int_0^t \frac{H_z(s,q(s),q'(s))}{H^-_{z'}(s,q(s),q'(s))}\mathrm{d}s\right] \leqslant -K \leqslant L^+_{z'}(t,q(t),0)\cdot \exp\left[-\int_0^t \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right].$$

Besides, as $\forall t \in [t_1, t_2]$, $q'(t) = 0$

$$L^-_{z'}(t,q(t),q'(t))\cdot \exp\left[-\int_0^t \frac{H_z(s,q(s),q'(s))}{H^-_{z'}(s,q(s),q'(s))}\mathrm{d}s\right] \leqslant -K \leqslant L^+_{z'}(t,q(t),q'(t))\cdot \exp\left[-\int_0^t \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right]$$

and therefore

$$\mathbb{Y}^+_q(t) \leqslant K \leqslant \mathbb{Y}^-_q(t)$$

and we obtain (i) in Theorem 3.

To obtain (ii) in Theorem 3 we must prove that K in $[0, t_1]$ is the same that $\widetilde{K}$ in $[t_2, T]$. We have for t_2 that

$$\widetilde{K} = -L^+_{z'}(t_2,q(t_2),0)\cdot \exp\left[-\int_0^{t_2} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right].$$

From (2.3) and (2.1) we have that

$$K = -L^+_{z'}(t_1,q(t_1),0)\cdot \exp\left[-\int_0^{t_1} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right] = -L^+_{z'}(t_2,q(t_2),0)\cdot \exp\left[-\int_0^{t_1} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right]$$

and from (2.4)

$$\int_0^{t_2} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s = \int_0^{t_1} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s + \int_{t_1}^{t_2} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s = \int_0^{t_1} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s,$$

so, we have that

$$K = -L^+_{z'}(t_2,q(t_2),0)\cdot \exp\left[-\int_0^{t_2} \frac{H_z(s,q(s),q'(s))}{H^+_{z'}(s,q(s),q'(s))}\mathrm{d}s\right] = \widetilde{K}$$

and the case (a) is proven.

The below expressions are obtained by analogous reasoning:

In case (b) it is verified that

$$L^+_{z'}(t_1,q(t_1),0) = L^-_{z'}(t_2,q(t_2),0), \tag{2.5}$$

$$L^-_{z'}(t,q(t),0) \leqslant L^+_{z'}(t_1,q(t_1),0) \leqslant L^+_{z'}(t,q(t),0). \tag{2.6}$$

In case (c) it is verified that

$$L^-_{z'}(t_1,q(t_1),0) = L^+_{z'}(t_2,q(t_2),0), \tag{2.7}$$

$$L^-_{z'}(t,q(t),0) \leqslant L^-_{z'}(t_1,q(t_1),0) \leqslant L^+_{z'}(t,q(t),0). \tag{2.8}$$

In case (d) it is verified that

$$L^-_{z'}(t_1,q(t_1),0) = L^-_{z'}(t_2,q(t_2),0), \tag{2.9}$$

$$L^-_{z'}(t,q(t),0) \leqslant L^-_{z'}(t_1,q(t_1),0) \leqslant L^+_{z'}(t,q(t),0). \tag{2.10}$$

The proof for the remaining cases would be analogous, employing (2.5) and (2.6) in case (b), (2.7) and (2.8) in case (c) and (2.9) and (2.10) in case (d).

If the shut down interval contains 0 or T, bearing in mind that

$$\int_{t_1}^{t_2} L_z(\tau, q(\tau), q'(\tau))\mathrm{d}\tau = 0 \quad \text{and} \quad \int_{t_1}^{t_2} \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))}\mathrm{d}s = 0$$

the proof will be similar. $\square$

3. Smooth transition

In this section, we present a qualitative aspect of the solution of the problem (P_2). We prove that, under certain conditions, the discontinuity of the derivative of the Lagrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verified that the derivative of the extremal where the minimum is reached presents an interval of constancy, the constant being the value for which $L_{z'}(t, z, \cdot)$ presents discontinuity. The character C^1 of the solution is thus guaranteed.

Definition 1. Let us take $t_0 \in (0, T)$ and $\varepsilon > 0$. We consider the auxiliary function $h_\varepsilon^{t_0}$ defined on $[0, T]$

$$h_\varepsilon^{t_0}(t) = \begin{cases} 0 & \text{if } t \in [0, t_0 - \varepsilon] \cup [t_0 + \varepsilon, T], \\ (t - t_0 + \varepsilon) & \text{if } t \in [t_0 - \varepsilon, t_0], \\ -(t - t_0 - \varepsilon) & \text{if } t \in [t_0, t_0 + \varepsilon]. \end{cases}$$

Notice that $h_\varepsilon^{t_0} \in \widehat{C}^1[0, T]$, $0 \leqslant h_\varepsilon^{t_0}(t) \leqslant \varepsilon \ \forall t \in [0, T]$, and

$$(h_\varepsilon^{t_0})'(t) = \begin{cases} 0 & \text{if } t \in [0, t_0 - \varepsilon) \cup (t_0 + \varepsilon, T], \\ 1 & \text{if } t \in (t_0 - \varepsilon, t_0), \\ -1 & \text{if } t \in (t_0, t_0 + \varepsilon). \end{cases}$$

Theorem 4. *Let $L(\cdot,\cdot,\cdot)$ be the Lagrangian of the functional F in the conditions stated above, and let us assume that the function $L_{z'}(t_0, z(t_0), \cdot)$ is strictly increasing (decreasing) and discontinuous in 0 If q is minimum (maximum) for F, then: (i) t_0 is not an isolated point of a change in the sign of q', (ii) $q' \equiv 0$ in some interval that contains t_0 and (iii) q' is continuous in t_0.*

Proof

(i) We'll proceed by contradiction.
Let $q \in \Theta_2$ be a minimum of F, and let us first assume that q' is negative to the left of t_0 and positive to the right of t_0. That is, let us assume that for $t_0 \in (0, T)$ there exist $\varepsilon > 0$ such that

$$q'(t) < 0 \quad \forall t \in (t_0 - \varepsilon, t_0); \qquad q'(t) > 0 \quad \forall t \in (t_0, t_0 + \varepsilon).$$

The strict growth of $L_{z'}$, as well as its discontinuity, implies that

$$L_{z'}(t, q(t), q'(t)) < L_{z'}^-(t_0, q(t_0), 0) < L_{z'}^+(t_0, q(t_0), 0) < L_{z'}(\tilde{t}, q(\tilde{t}), q'(\tilde{t})) \quad \forall t \in (t_0 - \varepsilon, t_0) \quad \forall \tilde{t} \in (t_0, t_0 + \varepsilon).$$

Bearing in mind that $\forall t \in [0, T]$, $0 \leqslant h_\varepsilon^{t_0}(t) \leqslant \varepsilon$, it is evident that we may choose the previous ε sufficiently small for the following inequality to be verified:

$$\sup_{t\in(t_0-\varepsilon,t_0)} [L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))] < \inf_{t\in(t_0,t_0+\varepsilon)} [L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))],$$

from which the following inequalities are deduced:

$$\begin{aligned} I_1 &= \int_{t_0-\varepsilon}^{t_0} [L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))]\mathrm{d}t \leqslant \varepsilon \cdot \sup_{t\in(t_0-\varepsilon,t_0)} \left[L_{z'}(t, q(t), q'(t)) + h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))\right] \\ &< \varepsilon \cdot \inf_{t\in(t_0,t_0+\varepsilon)} \left[L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))\right] \leqslant \int_{t_0}^{t_0+\varepsilon} [L_{z'}(t, q(t), q'(t)) - h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t))]\mathrm{d}t = I_2. \end{aligned}$$

Let us now take into account that

$$h_\varepsilon^{t_0}(t) = 0 \quad \forall t \in [0, t_0 - \varepsilon] \cup [t_0 + \varepsilon, T]; \qquad (h_\varepsilon^{t_0})'(t) = 0 \quad \forall t \in [0, t_0 - \varepsilon) \cup (t_0 + \varepsilon, T],$$

then

$$\delta^+ F(q, h_\varepsilon^{t_0}) := \lim_{x\to 0^+} \frac{F(q + x h_\varepsilon^{t_0}) - F(q)}{x} = \int_{t_0-\varepsilon}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) + (h_\varepsilon^{t_0})'(t) \cdot L_{z'}(t, q(t), q'(t))]\mathrm{d}t$$

and hence

$$\delta^+F(q,h_\varepsilon^{t_0}) = \int_{t_0-\varepsilon}^{t_0} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + 1\cdot L_{z'}(t,q(t),q'(t))]\mathrm{d}t + \int_{t_0}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + (-1)\cdot L_{z'}(t,q(t),q'(t))]\mathrm{d}t,$$

we have that

$$\delta^+F(q,h_\varepsilon^{t_0}) = \int_{t_0-\varepsilon}^{t_0} [L_{z'}(t,q(t),q'(t)) + h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]\mathrm{d}t - \int_{t_0}^{t_0+\varepsilon} [L_{z'}(t,q(t),q'(t)) - h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]\mathrm{d}t = I_1 - I_2 < 0,$$

which contradicts the assumption that q is a minimum of F.If q' were positive to the left of t_0 and negative to the right of t_0, the proof would be analogous, taking $\delta^+F(q,-h_\varepsilon^{t_0})$.

(ii) Follows from (i).

(iii) We'll proceed by contradiction. Let us assume that $q'(t_0^-) < q'(t_0^+)$ (if we assume that $q'(t_0^-) > q'(t_0^+)$, the proof will be analogous). Bearing in mind (i) and (ii), q' is discontinuous in t_0 only in the following cases:

(a) $q'(t_0^-) < 0;\quad q'(t_0^+) = 0,$

(b) $q'(t_0^-) = 0;\quad q'(t_0^+) > 0.$

For (a), in view of (i), there will exist an $\varepsilon > 0$ such that $q'(x) = 0$ at $[t_0, t_0+\varepsilon]$. We may choose ε such that $q'(x) < 0$ at $[t_0-\varepsilon, t_0)$. In this case

$$\delta^+F(q,h_\varepsilon^{t_0}) = \int_{t_0-\varepsilon}^{t_0} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + (h_\varepsilon^{t_0})'(t)\cdot L_{z'}(t,q(t),q'(t))]\mathrm{d}t + \int_{t_0}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + (h_\varepsilon^{t_0})'(t)\cdot L_{z'}^-(t,q(t),q'(t))]\mathrm{d}t$$

and, by identical reasoning to that used in (i), we shall have that $\delta^+F(q,h_\varepsilon^{t_0}) < 0$, which once more means a contradiction of the fact that q is a minimum of F.

Finally, for (b), in view of (i), there will exist an $\varepsilon > 0$ such that $q'(x) = 0$ at $[t_0-\varepsilon, t_0]$. We may choose ε such that $q'(x) < 0$ at $(t_0, t_0+\varepsilon]$. In this case

$$\delta^+F(q,-h_\varepsilon^{t_0}) = \int_{t_0-\varepsilon}^{t_0} [-h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) - (h_\varepsilon^{t_0})'(t)\cdot L_{z'}(t,q(t),q'(t))]\mathrm{d}t + \int_{t_0}^{t_0+\varepsilon} [-h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) - (h_\varepsilon^{t_0})'(t)\cdot L_{z'}^+(t,q(t),q'(t))]\mathrm{d}t,$$

where, by identical reasoning to that used in section (i), we shall once more have the contradiction

$$\delta^+F(q,-h_\varepsilon^{t_0}) < 0. \qquad \square$$

Note that this result has a very clear interpretation in terms of pumping plants: under optimum operating conditions, pumping plants never switch brusquely from generating power to pumping water or vice versa, but rather carry out a smooth transition, remaining inactive during a certain period of time.

4. Optimization algorithm

From the computational point of view, the construction of q_K can be performed with the use of a discretized version of Theorem 3. The problem will consist in finding for each K the function q_K that satisfies conditions (i) and (ii) of Theorem 3, and from among these functions, an admissible function $q_K \in \Theta$. In general, the construction of q_K cannot be carried out all at once over the entire interval $[0,T]$. The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs, until completing the interval $[0,T]$, where

- $H_{\min} < H(t,q(t),q'(t)) < H_s(t)$ (free extremal arcs), or
- $q'(t) = 0$ (the hydro-plant is on shut down), or
- $H(t,q(t),q'(t)) = H_s(t)$ (the hydro-plant generates all the demanded power or its technical maximum), or
- $H(t,q(t),q'(t)) = H_{\min}$ (the hydro-plant is functioning at its maximum pumping power).

If the values obtained for q and q' do not obey the constraints, we force q_K to belong to the boundary until the moment when the conditions of leaving the domain (established in Theorem 3) are fulfilled.

We will denote by m the rate of water discharge at the instant $t = 0$ that is needed for the hydro-plant to reach its maximum pumping capacity: $H(0,0,m) = H_{\min}$ and we will denote by M the rate of water discharge at the instant $t = 0$ that is needed for the hydro-plant to reach its maximum generating capacity, i.e. $H(0,0,M) = H_s(0)$. We also set

$$K_m = -L_{z'}(0,0,m) \quad \text{and} \quad K_M = -L_{z'}(0,0,M)$$

as the respective coordination constants for these initial rates. We observe that $\forall x \in (m, M)$ (with the hypothesis $L_{z'z'}(t,z,z') > 0$), we have that

$$K_M < -L_{z'}(0,0,x) < K_m.$$

To construct the solution, we proceed by the stages shown below:

Stage I: Concatenation of extremal arcs

For each K, we construct q_K.
First arc: Given K, we distinguish the following cases:

(i) If $-L^+_{z'}(0,0,0) \leqslant K \leqslant -L^-_{z'}(0,0,0)$ (shut down zone), we set $q'_K(t) = 0$ in the maximal interval $[0, t_1]$, where $\forall t \in [0, t_1]$, satisfying

$$\mathbb{Y}^+_{q_K}(t) \leqslant K \leqslant \mathbb{Y}^-_{q_K}(t).$$

(ii) If $-L^+_{z'}(0,0,0) > K$ (hydro-generation zone) and $K_M < K$, there exists a positive solution $q'_K(0)$ for the equation $-L_{z'}(0,0,q'_K(0)) = K$. In this case, we construct an interior arc of the extremal, $q_K(t)$, which satisfies Euler's equation in its maximal domain $[0, t_1]$ (with $q_K(0) = 0$), where $\forall t \in [0, t_1]$, satisfying $q'_K(t) > 0$ and

$$K = \mathbb{Y}_{q_K}(t).$$

(iii) If $-L^+_{z'}(0,0,0) > K$ (hydro-generation zone) and $K \leqslant K_M$, we set $q_K(t) = w(t)$, the solution of the differential equation $H(t, w(t), w'(t)) = H_s(t)$ with $w(0) = 0$ in the maximal interval $[0, t_1]$, where $\forall t \in [0, t_1]$ it is verified that

$$K \leqslant \mathbb{Y}_\omega(t).$$

(iv) If $-L^-_{z'}(0,0,0) < K$ (pumping zone) and $K < K_m$, there exists a negative solution $q'_K(0)$ for the equation $-L_{z'}(0,0,q'_K(0)) = K$. In this case, we construct an interior arc of the extremal, $q_K(t)$, which satisfies Euler's equation in its maximal domain $[0, t_1]$ (with $q_K(0) = 0$), where $\forall t \in [0, t_1]$, satisfying $q'_K(t) < 0$ and

$$K = \mathbb{Y}_{q_K}(t).$$

(v) If $-L^-_{z'}(0,0,0) < K$ (pumping zone) and $K \geqslant K_m$, we set $q_K(t) = w(t)$, the solution of the differential equation $H(t, w(t), w'(t)) = H_{\min}$ with $w(0) = 0$ in the maximal interval $[0, t_1]$, where $\forall t \in [0, t_1]$ it is verified that

$$K \geqslant \mathbb{Y}_\omega(t).$$

ith Arc: There are two possibilities:

(A) If $[t_{i-1}, t_i]$ is a maximal interval of shut down or boundary interval of hydro-plant, i.e. $q'_K(t) = 0$ or $H(t, q_K(t), q'_K(t)) = H_s(t)$ or $H(t, q_K(t), q'_K(t)) = H_{\min}$ in said interval, we consider the maximal interval $[t_i, t_{i+1}]$ such that $\forall t \in [t_i, t_{i+1}]$

$$K = -L_{z'}(t, \omega(t), \omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q'_K(s))}{H_{z'}(s, q_K(s), q'_K(s))} \mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))} \mathrm{d}s\right],$$

$\omega(t)$ being an interior arc of the extremal, with $\omega(t_i) = q_K(t_i)$, which satisfies Euler's equation in its maximal domain $[t_i, t_{i+1}]$ and the above equation. Now, we set $q_K(t) = \omega(t)$ $\forall t \in [t_i, t_{i+1}]$.

(B) If q_K has an interior arc in $[t_{i-1}, t_i]$, there are three possibilities:

(i) If $H(t_i, q_K(t_i), q'_K(t_i)) = H_{\min}$, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K \geqslant -L_{z'}(t, \omega(t), \omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q'_K(s))}{H_{z'}(s, q_K(s), q'_K(s))} \mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))} \mathrm{d}s\right],$$

$\omega(t)$ being a solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = H_{\min} \quad \text{with } \omega(t_i) = q_K(t_i).$$

If this is the case, we set $q_K(t) = \omega(t)$ $\forall t \in [t_i, t_{i+1}]$.

(ii) If $H(t_i, q_K(t_i), q'_K(t_i)) = H_s(t_i)$, we consider the maximal interval $[t_i, t_{i+1}]$ such that $\forall t \in [t_i, t_{i+1}]$

$$K \leqslant -L_{z'}(t, \omega(t), \omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q'_K(s))}{H_{z'}(s, q_K(s), q'_K(s))} \mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))} \mathrm{d}s\right],$$

$\omega(t)$ being a solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = H_s(t) \quad \text{with } \omega(t_i) = q_K(t_i).$$

If this is the case, we set $q_K(t) = \omega(t)$, $\forall t \in [t_i, t_{i+1}]$.

(iii) In another case, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$-L^{+}_{z'}(t,\omega(t),\omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s,q_K(s),q'_K(s))}{H_{z'}(s,q_K(s),q'_K(s))}\mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s,\omega(s),\omega'(t))}{H^{+}_{z'}(s,\omega(s),\omega'(t))}\mathrm{d}s\right] \geqslant K,$$

$$K \leqslant -L^{-}_{z'}(t,\omega(t),\omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s,q_K(s),q'_K(s))}{H_{z'}(s,q_K(s),q'_K(s))}\mathrm{d}s - \int_{t_i}^{t} \frac{H_z(s,\omega(s),\omega'(t))}{H^{-}_{z'}(s,\omega(s),\omega'(t))}\mathrm{d}s\right]$$

with $\omega(t_i) = q_K(t_i)$, which satisfies the above equation and $\omega'(t) = 0$ in its maximal domain $[t_i, t_{i+1}]$. Now, we set $q_K(t) = \omega(t), \forall t \in [t_i, t_{i+1}]$.

Stage II: Calculation of K

We determine K, such that $q_K \in \Theta_b$.

Varying K, we would search for the extremal that fulfils the second boundary condition $q_K(T) = b$. The procedure is similar to the shooting method used to resolve second-order differential equations with boundary conditions.

5. Application to a hydrothermal problem

A program that resolves the optimization problem was elaborated using the Mathematica package and was then applied to one example of hydrothermal system made up of the thermal equivalent plant [11] and a hydraulic pumped-storage plant.

We use the quadratic model: $\Psi(x) := c_1 + c_2x + c_3x^2$, for the fuel cost model of the equivalent thermal plant, where the values for the coefficients are: $c_1 = 10696.1$ (\$/h); $c_2 = 16.5477$ (\$/h Mw): $c_3 = 0.00329982$ (\$/h Mw^2).

The power production H of the hydro-plant (variable head) is a function of $z(t)$ and $z'(t)$ and its power consumption during pumping is a lineal function of the amount of water pumped $(M \cdot z'(t))$. Hence the function H is defined piecewise as

$$H(t,z(t),z'(t)) := \begin{cases} A(t)\cdot z'(t) - B\cdot z(t)\cdot z'(t) & \text{if } z'(t) > 0, \\ M\cdot z'(t) & \text{if } z'(t) \leqslant 0, \end{cases}$$

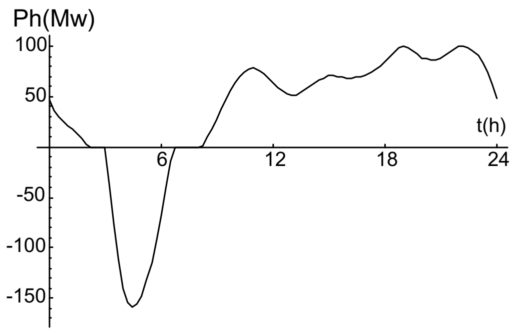

Fig. 1. Optimal solution with M_1.

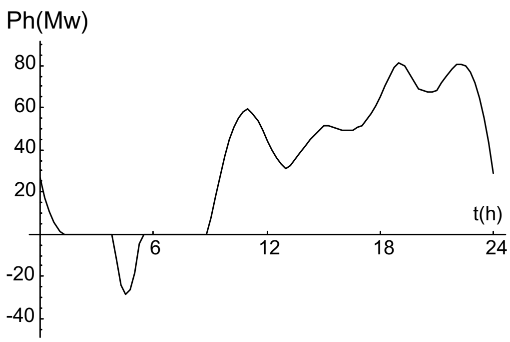

Fig. 2. Optimal solution with M_2.

where $A(t) = \frac{B_y}{G}(S_0 + t \cdot i), B = \frac{B_y}{G}$. The values for the coefficients of the hydro-plant are: the efficiency $G = 526,315$ m^4/h Mw), the restriction on the volume $b = 141.6 \times 10^5$ m^3, the natural inflow $i = 101.952 \times 10^5$ m^3/h, the initial volume $S_0 = 203.904 \times 10^9$ m^3 and the coefficient $B_y = 149.5 \times 10^{-12}$ m^{-2} (a parameter that depends on the geometry of the tanks). M (h Mw/m^3) is the factor of water-conversion of the pumped-storage plant and we consider two cases: (1) $M_1 = (1,04)A(0)$ and (2) $M_2 = (1,10)A(0)$.

We consider a short-term hydrothermal scheduling (24 h) with an optimization interval [0,24] and we consider a discretization of 96 subintervals. Figs. 1 and 2 present the optimum solution.

It can be seen how the interval of smooth transition varies when considering two different values of M.

The secant method was used to calculate the approximate value of K for which $q_K(T) - b = 0$. The algorithm shows a rapid convergence to the optimal solution. For example, for M_2, in four iterations: $|q_K(T) - b| < 10^{-2} \quad (m^3)$ for $K = 0.001307118235071412$. The time required by the program was 27 s on a personal computer (Pentium IV/2 GHz).

6. Conclusions and future perspectives

The main contribution of this paper is that simultaneously considers non-regular Lagrangian and non-holonomic inequality constraints for the optimization of hydrothermal problems, unifying two theories that have been addressed independently by the authors until now. Furthermore, we have established the result called smooth transition, i.e. that the derivative of the minimum presents a constancy interval. This behavior had been observed in pumping plants in the examples solved computationally and has now been proven theoretically.

As far as future perspectives are concerned, it would be most interesting to apply this method to models in which the discontinuity of the Lagrangian is not produced at $z' = 0$, but rather to solutions of a differential equation of the form

$$z' = f(t, z).$$

References

[1] L. Bayón, J.M. Grau, M.M. Ruiz, P.M. Suárez, A Bolza problem in hydrothermal optimization, in: T.E. Simos, G. Maroulis (Eds.), Proceedings of the ICCMSE 2004, Lecture Series on Computer and Computational Sciences, vol. I, 2004, pp. 57–60.
[2] E. Lee, L. Markus, Foundations of Optimal Control Theory, John Wiley & Sons, New York, 1967.
[3] M.E. El-Hawary, G.S. Christensen, Optimal Economic Operation of Electric Power Systems, Academic Press, New York, 1979.
[4] M. Papageorgio, Optimal multireservoir network control by the discrete maximum principle, Water Resour. Res. 21 (12) (1985) 1824–1830.
[5] G.S. Christensen, M.E. El-Hawary, S.A. Soliman, Optimal Control Applications in Electric Power Systems, Plenum Press, New York, 1987.
[6] J.M. Pursimo, H.K. Antila, M.K. Vilkko, P.A. Lautala, A short-term scheduling for a hydropower plant chain, Int. J. Electr. Power Energy Syst. 20 (8) (1998) 525–532.
[7] H. Mousavi, A.S. Ramamurthy, Multi-reservoir design using Pontryagin principle, Adv. Water Res. 25 (6) (2002) 677–687.
[8] L. Bayón, J.M. Grau, M.M. Ruiz, P.M. Suárez, Nonsmooth optimization of hydrothermal problems, in: E. Brandas, J. Vigo-Aguiar (Eds.), Proceedings of the CMMSE 2004, vol. I, 2004, pp. 116–125.
[9] F.H. Clarke, Optimization and Nonsmooth Analysis, John Wiley & Sons, New York, 1983.
[10] P.D. Loewen, R.T. Rockafellar, New necessary conditions for the generalized problem of Bolza, SIAM J. Contr. Optim. 34 (5) (1996) 1496–1511.
[11] L. Bayón, J.M. Grau, P.M. Suárez, A new formulation of the equivalent thermal in optimization of hydrothermal systems, Math. Prob. Eng. 8 (3) (2002) 181–196.