International Journal of Computer Mathematics
Vol. 88, No. 9, June 2011, 1949–1959

An algorithm for bang–bang control of fixed-head hydroplants

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(*Received 31 August 2009; revised version received 10 March 2010; second revision received 1 June 2010; accepted 12 June 2010*)

This paper deals with the optimal control (OC) problem that arise when a hydraulic system with fixed-head hydroplants is considered. In the frame of a deregulated electricity market, the resulting Hamiltonian for such OC problems is linear in the control variable and results in an optimal singular/bang–bang control policy. To avoid difficulties associated with the computation of optimal singular/bang–bang controls, an efficient and simple optimization algorithm is proposed. The computational technique is illustrated on one example.

Keywords: optimal control; singular/bang–bang problems; hydroplants

2000 AMS Subject Classification: 49J30

1. Introduction

The computation of optimal singular/bang–bang controls is of particular interest to researchers because of the difficulty in obtaining the optimal solution. Several engineering control problems, such as the chemical reactor start-up or hydrothermal optimization problems, are known to have optimal singular/bang–bang controls. This paper deals with the optimal control (OC) problem that arises when addressing the new short-term problems that are faced by a generation company in a deregulated electricity market. Our model of the spot market explicitly represents the price of electricity as a known exogenous variable and we consider a system with fixed-head hydroplants. These plants, with a large capacity reservoir, are the most important in the electricity market. The resulting Hamiltonian for such OC problems, H, is linear in the control variable, u, and results in an optimal singular/bang–bang control policy.

In general, the application of Pontryagin's maximum principle (PMP) is not well suited for computing singular control problems as it fails to yield a unique value for the control. Different methods for determining OCs with a possibly singular part have already been developed.

A popular approach introduced by Jacobson *et al.* [11] has been used by a number of researchers including Edgar and Lapidus [7,8] and more recently by Chen and Huang [3]. This method involves solving the singular/bang–bang OC problem as the limit of a series of non-singular problems. It is

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ISSN 0020-7160 print/ISSN 1029-0265 online

DOI: 10.1080/00207160.2010.502934
http://www.informaworld.com

important to establish the limitations of these perturbation-based methods for practical problems. In fact, the convergence criterion described in [11] requires that the perturbation parameter, ε, be sufficiently small; however, numerical difficulties result when ε approaches a zero limit. The reader is referred to [3,5] and [6] for further details. The procedure may therefore not be a suitable technique for certain types of problems.

Other studies have assumed *a priori* knowledge of the number and location of the singular subarcs. For example, Maurer *et al.* [14] presented a numerical scheme for computing optimal bang–bang controls. They assume that every component of the OC is bang–bang and that there are only a known finite number of switching times (in the Rayleigh problem, specifically three). Hence, these type of studies are less general.

In the present paper, a major effort has been made to develop a general, computationally efficient algorithm for a wide class of OC problems with the final state and the final time fixed. We propose a simple and efficient optimization algorithm that avoids all the difficulties that the aforementioned methods present. The algorithm combines optimality conditions with the shooting method to develop the optimal solution.

Our method needs no prior knowledge of the number and location of the bang–bang arcs. Neither does it handle any parameter (such as, for example, discretization or a penalization factor) that has an influence on convergence or the precision of the result. As it is specifically designed for problems of this type, we shall see that it is much faster and more reliable than commercial solvers that address any type of general OC problem.

Our method has a very wide-ranging field of application. In this paper, the algorithm has been illustrated by means of the hydraulic system optimization problem. We have chosen this hydraulic framework as it is a very important problem within the fields of both applied mathematics and electrical engineering. We also underline the fact that bang–bang problems have barely been tackled due to the computational complexities they involve.

The paper is organized as follows. In Section 2, we present the mathematical environment of our work: the singular OC problem with control appearing linearly. In Section 3, we present the mathematical models of our fixed-head hydroplant. In Section 4, we formulate our optimization problem and prove that singular controls can be excluded. In Section 5, we describe the algorithm that provides the structure of bang–bang arcs. The results of the application of the method to a numerical example are presented in Section 6. Finally, the main conclusions of our research are summarized in Section 7.

2. General statement of the singular OC problem

Let us assume a system given by: a state $x(t) \in \mathbb{R}^n$ at time $t \in [0, T]$, a control $u(t) \in U(t) \subset \mathbb{R}^m$, where u is piecewise continuous and $U(t)$ is compact for every $t \in [0, T]$, a state equation $x'(t) = f(t, x(t), u(t))$ almost everywhere, an initial condition $x(0) = x_0$ and a final condition $x(T) \in Z \neq \varnothing$, where $[0, T]$ is fixed, and the scalar functions g and L with a suitable domain. The following problem is called the Bolza problem **(P)**:

Find an admissible pair (x, u) on $[0, T]$ *such that the functional*

$$J(u) = g(x(T)) + \int_0^T L(t, x(t), u(t))\, \mathrm{d}t$$

becomes maximal. If $g \equiv 0$, we call **(P)** a Lagrange problem, while **(P)** is called a Mayer problem if $L \equiv 0$. We define the Hamiltonian:

$$H(u, x, \lambda, t) := L(t, x, u) + \lambda^T f(t, x, u),$$

where $\lambda \in \mathbb{R}^n$. We assume that every f_i $(i = 1, \ldots, n)$ is continuous in (t, x, u), that the derivatives $(\partial/\partial t) f_i$ and $\nabla_x f_i$ exist and are continuous in (t, x, u) for every i. Furthermore, we assume that $g \in C^1$ and that **(P)** has a solution (x^*, u^*) with $Z = \mathbb{R}^n$. The following theorem [17] is often very useful in solving Bolza problems.

THEOREM 1 (PMP) *Under the above hypothesis, there thus exists an absolutely continuous function* $\lambda : [0, T] \to \mathbb{R}^n$ *with the following properties*:

(a) $x' = H_\lambda$ and $\lambda' = -H_x$ along (x^*, u^*).
(b) $H(u^*(t), x^*(t), \lambda(t), t) = \max\{H(u, x^*(t), \lambda(t), t) | u \subset U(t)\}$ for every $t \in [0, T]$.
(c) $\lambda \neq 0$ on $[0, T]$.
(d) $\lambda(T)\, \mathrm{d}x(T) - \mathrm{d}g = 0$ (transversality condition).

In the usual case, the optimality condition

$$H(u^*(t), x^*(t), \lambda(t), t) = \max\{H(u, x^*(t), \lambda(t), t) | u \subset U(t)\} \tag{1}$$

is used to solve for the extremal control in terms of the state and adjoint (x, λ). Normally, the optimality condition is imposed as $H_u = 0$ and this system of equations is solved for the control vector $u(t)$.

We now consider the case of scalar control appearing linearly (H_{uu} is singular):

$$\begin{aligned}
&\max \int_0^T [f_1(t, x) + u f_2(t, x)]\, \mathrm{d}t \\
&x' = g_1(t, x) + u g_2(t, x); \quad x(0) = x_0 \\
&u_{\min} \leq u(t) \leq u_{\max}.
\end{aligned}$$

The variational Hamiltonian is linear in u and can be written as

$$H(u, x, \lambda, t) := f_1(t, x) + \lambda g_1(t, x) + [f_2(t, x) + \lambda g_2(t, x)]u.$$

The optimality condition (maximize H w.r.t. u) leads to

$$u^*(t) = \begin{cases} u_{\max} & \text{if } f_2(t, x) + \lambda g_2(t, x) > 0 \\ u_{\text{sing}} & \text{if } f_2(t, x) + \lambda g_2(t, x) = 0 \\ u_{\min} & \text{if } f_2(t, x) + \lambda g_2(t, x) < 0 \end{cases}$$

and u^* is undetermined if $\Phi(x, \lambda) \equiv H_u = f_2(t, x) + \lambda g_2(t, x) = 0$. The function Φ is called the switching function. If $\Phi(x^*(t), \lambda(t)) = 0$ only at isolated time points, then the OC switches between its upper and lower bounds, which is said to be a bang–bang type control (i.e. the problem is not singular). The times when the OC switches from $u_{\max}$ to $u_{\min}$ or vice versa are called switch times.

If $\Phi(x^*(t), \lambda(t)) = 0$ for every t in some subinterval $[t', t'']$ of $[0, T]$, then the original problem is called a singular control problem and the corresponding trajectory for $[t', t'']$, a singular arc. The case when Φ vanishes over an interval is more troublesome, because the optimality condition is vacuous, since $H(u, x^*(t), \lambda(t), t)$ is independent of u. In the singular case, PMP yields no information on the extremal (or stationary) control.

In order to find the control on a singular arc, we use the fact that H_u remains zero along the whole arc. Hence, all the time derivatives are zero along such an arc. By successive differentiation of the switching function, one of the time derivatives may contain the control u, in which case u can be obtained as a function of x and λ. The next result [12] is important.

PROPOSITION 1 *If H_u is successively differentiated with respect to time, then u cannot first appear in an odd-order derivative.*

As u first appears in an even-order derivative, we denote this by $(\mathrm{d}^{2q}(H_u))/\mathrm{d}t^{2q}$ and q is the order of the singular arc. An important theorem [12] is the necessary condition for a singular arc to be optimal: the generalized Legendre–Clebsch (GLC) condition.

THEOREM 2 (GLC condition) *If $x^*(t), u^*(t)$ are optimal on a singular arc, then, for scalar u,*

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{\mathrm{d}^{2q}(H_u)}{\mathrm{d}t^{2q}}\right] \leq 0.$$

3. Hydroplant performance models

Conventional hydroplants are classified as run-of-river plants and storage plants. Run-of-river plants have little storage capacity and use water as it becomes available. The water not utilized is spilled. Storage plants are associated with reservoirs that have significant storage capacity. During periods of low power requirements, water can be stored and then released when demand is high.

A basic physically based relationship between the active power generated P (in MW) by a hydroplant and the rate of water discharge, q (in m^3/s), and the effective head, h (in m), is given by

$$P = 0.0085qh\eta(q, h),$$

where η is a function of q and h. A variety of models have been proposed in the literature [9, 13] due to the diversity of plant types and their characteristics (Table 1). The appropriate choice of mathematical models for representing the physical system is a crucial aspect when addressing any optimization problem. In this paper, we consider the approximation presented by El-Hawary and Christensen [9] to be the most appropriate on account of its precision and flexibility.

The Glimn–Kirchmayer model gives the variation of the rate of discharge as a bi-quadratic function of h and P. K is a constant of proportionality and the parameters of the model are assumed to be obtainable. The problem of estimating these parameters is treated in [10]. A more generalized form of the model is that of Hildebrand, in which L and K are usually taken to be 2. In Hamilton–Lamont's model, the equation is modified through division by the head h. The proportionality constant K is not required and a cubic term in P is added. All of these models can be interpreted as a consequence of Taylor expansion for a function of several variables. Finally, the Arvanitidis–Rosing model uses an exponential variation, in which S is reservoir storage.

Table 1. Hydroplant models.

Glimn–Kirchmayer	$q = K\psi(h)\phi(P)$	$\psi(h) = \alpha h^2 + \beta h + \gamma$ $\phi(P) = xP^2 + yP + z$
Hildebrand	$q = \sum_{i=0}^{L}\sum_{j=0}^{K} C_{ij}P^i h^j$	
Hamilton–Lamont	$q = \psi(h)(\phi(P))/h$	$\psi(h) = \alpha h^2 + \beta h + \gamma$ $\phi(P) = xP^3 + yP + z$
Arvanitidis–Rosing	$P = qh(\beta - \mathrm{e}^{-\alpha S})$	

El-Hawary's model. In this model the output power P (MW) is given by

$$P = \frac{qh}{G},$$

where G is the efficiency ($\mathrm{m}^4/\mathrm{h\,MW}$). For the sake of simplicity, we assume the rate of water spillage and the penstock head losses to be negligible. Thus, we have $h = y - y_{\mathrm{T}}$, where y is the forebay elevation and y_{T} the tailrace elevation. In most cases, a typical linear variation between y_{T} and the discharge, q, and a typical linear elevation–storage curve may be assumed as follows:

$$y(t) = [y_0 + B_y s(t)] - [y_{T_0} + B_T q(t)],$$

where $s(t)$ is the reservoir storage and the coefficients B_y (in m^{-2}) and B_t (in $\mathrm{h\,m}^{-2}$) are parameters that depend on the geometry of the reservoir. The reservoir's dynamics may be suitably described by the equation

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = i(t) - q(t) \longrightarrow s(t) = S_0 + it - Q(t)$$

being i the natural inflow (i.e., in general, assumed constant), $Q(t)$ being the volume discharged up to the instant t by the plant and S_0 the initial volume. So, we have that

$$P(t, Q(t), q(t)) := A(t)q(t) - BQ(t)q(t) - Cq^2(t)$$
$$A(t) = \frac{(y_0 - y_{T_0}) + B_y(S_0 + it)}{G}, \quad B = \frac{B_y}{G}, \quad C = \frac{B_T}{G}. \tag{2}$$

This is a *variable-head* model and the hydroplant's hydraulic generation, P, is a function of $Q(t)$ and $q(t)$. According to El-Hawary's model, power output is a function of discharge and the head. For a large capacity reservoir, it is practical to assume that the effective head is constant over the optimization interval. Here the *fixed-head* hydroplant model is defined and P is represented by the linear equation:

$$P(t) = \frac{(y_0 - y_{T_0}) + B_y(S_0)}{G} q(t) = Aq(t). \tag{3}$$

4. Structure of the solution of the optimization problem

For convenience of formulation, in this section, we introduce this new notation: $q(t) \equiv z'(t)$; $Q(t) \equiv z(t)$. Let $P(t, z(t), z'(t))$ be the function of the hydroplant's hydraulic generation, where $z(t)$ is the volume that is discharged up to the instant t by the plant, and $z'(t)$ the rate of water discharge of the plant at the instant t. If we assume that b is the volume of water that must be discharged during the entire optimization interval $[0, T]$, the following boundary conditions will have to be fulfilled:

$$z(0) = 0, \quad z(T) = b.$$

Throughout the paper, we assume that $P(t, z, z')$: $[0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$; that is, we shall only admit non-negative volumes, $z(t)$, and rates of water discharge, $z'(t)$ (pumped-storage plants will be not considered). Besides the previous statement, we consider $z'(t)$ to be bounded by technical constraints

$$q_{\min} \leq z'(t) \leq q_{\max}, \quad \forall t \in [0, T].$$

No transmission losses will be considered in our study; this is a crucial aspect when addressing the optimization problem from a centralized viewpoint. From the perspective of a generation company and within the framework of the new electricity market, said losses are not relevant, as power generators currently receive payment for all the energy they generate in power plant bars.

This study constitutes a modification of previous papers by the authors [1,2], where a variable-head model (2) was considered. When the term $-C \cdot q^2(t)$ is considered, the Hamiltonian is not linear in u and the control is not singular/bang–bang. The Hamiltonian is also not linear in u when transmission losses are considered using the classic Kirchmayer model: $P_L = BP(t)^2$; P_L being the losses.

In our problem, the objective function is given by revenue during the optimization interval $[0, T]$

$$F(z) = \int_0^T L(t, z(t), z'(t))\,\mathrm{d}t = \int_0^T \pi(t) P(t, z(t), z'(t))\,\mathrm{d}t.$$

Revenue is obtained by multiplying the hydraulic production of the hydroplant by the clearing price $\pi(t)$ at each hour t. Our model of the spot market explicitly represents the price of electricity as a known exogenous variable. Here the fixed-head hydroplant model (3) for P is used. In keeping with the previous statement, our objective functional in continuous time form is

$$\max_z F(z) = \max_z \int_0^T \pi(t) A z'(t)\,\mathrm{d}t$$

$$\text{on } \Omega = \{z \in \hat{C}^1[0, T] | z(0) = 0, z(T) = b;\ q_{\min} \le z'(t) \le q_{\max}, \forall t \in [0, T]\},$$

where $\hat{C}^1$ is the set of piecewise C^1 functions. A standard Lagrange-type OC problem of type (2) can be mathematically formulated as follows:

$$\max \int_0^T A\pi(t) u\,\mathrm{d}t = \max \int_0^T f(t) u\,\mathrm{d}t$$

$$z' = u, \quad z(0) = 0, \quad z(T) = b$$

$$u_{\min} \le u(t) \le u_{\max}.$$

With the aim of obtaining a solution numerically, we first attempt to determine the structure of the solution; that is, the sequence of the bang–bang and the singular parts. **W**e define the Hamiltonian:

$$H(u, x, \lambda, t) := f(t)u + \lambda u = [f(t) + \lambda]u.$$

The switching function is $\Phi(x, \lambda) \equiv H_u = f(t) + \lambda$. The optimality condition (1) leads to

$$u^*(t) = \begin{cases} u_{\max} & \text{if } f(t) + \lambda > 0, \\ u_{\text{sing}} & \text{if } f(t) + \lambda = 0, \\ u_{\min} & \text{if } f(t) + \lambda < 0. \end{cases} \tag{4}$$

On the other hand, the co-state equation of PMP allows us to obtain

$$\lambda' = -H_z = 0 \longrightarrow \lambda = \lambda_0(\text{cte}). \tag{5}$$

To find the control on a singular arc, we use the fact that H_u remains zero along the whole arc. By differentiation of the switching function, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_u = \frac{\mathrm{d}}{\mathrm{d}t}[f(t) + \lambda] = f'(t) = A\pi'(t) = 0.$$

$$\vdots$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} H_u = A\pi^{(n)}(t) = 0.$$

We can see that in the successive derivatives of H_u w.r.t. t does not appear the control u. We have only derivatives of the clearing price $\pi(t)$. The presence of singular arcs in the solution are thus ruled out.

5. Algorithm for the bang–bang solution

Having ruled out the presence of singular arcs, we now determine the bang–bang segments and the boundary on which the solution is situated. To obtain the optimal solution, we apply Equations (4) and (5), obtaining

$$u^*(t) = \begin{cases} u_{\max} & \text{if } f(t) > -\lambda_0, \\ u_{\min} & \text{if } f(t) < -\lambda_0. \end{cases} \tag{6}$$

The algorithm that leads to the optimal solution (6) comprises the following steps:

(i) First, $f(t)$ must be interpolated to obtain a continuous function. Note that in real electricity markets, the clearing price $\pi(t)$ is only known at each hour ($t = 1, 2, \ldots, 24$). In this paper, we have used linear interpolation with good results.

(ii) Second, we have to determine the switch times: $t_1, t_2, \ldots$ These instants are calculated solving the equation

$$f(t) = -\lambda.$$

(iii) Third, the optimal value λ_0 must be determined in order for

$$z_\lambda(T) = \sum_{i=1}^{N_s} \delta_i q_{\max} + \left(T - \sum_{i=1}^{N_s} \delta_i\right) q_{\min} = b,$$

δ_i being the duration of the ith bang–bang segment in the upper bound $u_{\max}$, N_s the number of such segments and $z_\lambda(T)$ the final volume obtained for each λ. Figure 1 illustrates the proposed method.

(iv) To calculate an approximate value of λ_0, we propose an iterative method (such as, for example, bisection or the secant method) using this condition

$$\text{Error} = |z_\lambda(T) - b| < \text{tol}$$

to finalize the algorithm (Figure 2). As we shall see in the next section, the secant method has provided satisfactory results using these initial values:

$$\lambda^{\min} = \min f(t); \quad \lambda^{\max} = \max f(t).$$

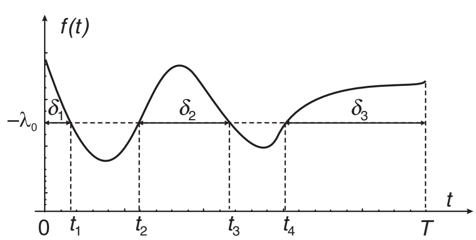

Figure 1. Illustration of the method.

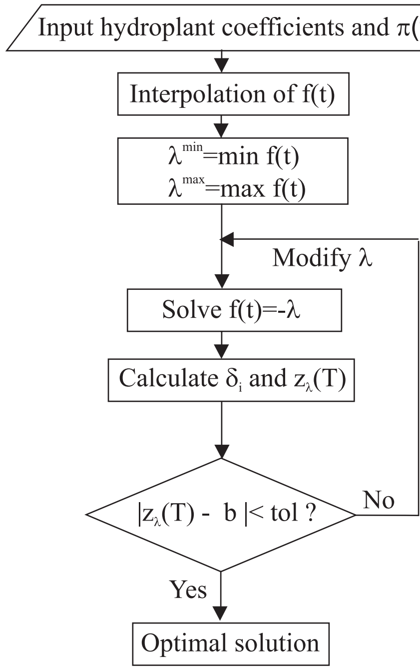

Figure 2. Computational flow of the proposed algorithm.

6. Example

A program was written using the Mathematica package to apply the results obtained in this paper to an example of a hydraulic system made up of one fixed-head hydroplant. The hydroplant data are summarized in Table 2.

We shall also consider the technical constraints: $q_{\min} = 0$ and $q_{\max} = 3.94258 \times 10^6$ (m^3/h), which correspond, respectively, to $P_{\min} = 0$ and $P_{\max} = 100$ (MW). With these coefficients, the hydraulic model is $P(t) = 0.0000253641q(t)$.

In this paper, we focus on the problem that a generation company faces when preparing its offers for the day-ahead market. Thus, the classic optimization interval of $T = 24$ h was considered. The clearing price $\pi(t)$ (euros/h MW) corresponding to 1 day was taken from the Spanish electricity market [4]. The known values of $\pi(t) : t = 1, 2, \ldots, 24$ were linearly interpolated (Figure 3).

The solution may be constructed in a simple way by taking into account the above algorithm. In this example, we have: $f(t) = 0.0000253641\pi(t)$, $\lambda^{\min} = \min f(t) = 0.00139528$ and $\lambda^{\max} = \max f(t) = 0.00279005$.

The secant method was used to calculate the approximate value of λ for which

$$\text{Error} = |z_\lambda(T) - b| < \text{tol}$$

Table 2. Hydroplant coefficients.

G(m^4/h MW)	b(m^3)	S_0(m^3)	y_0(m)	y_{T_0} (m)	B_y (m^{-2})
319,840	45×10^6	2.395×10^8	6.18166	5	2.89386×10^{-8}

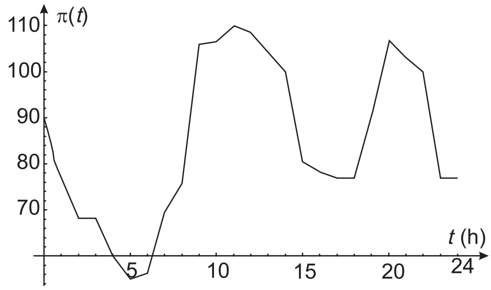

Figure 3. Clearing price $\pi(t)$.

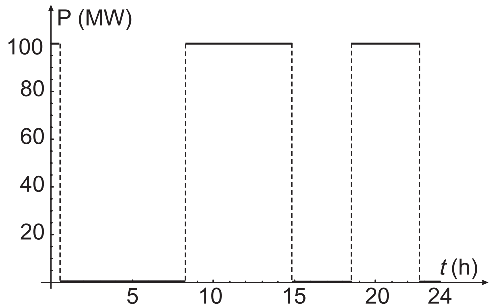

Figure 4. Optimal hydropower $P(t)$.

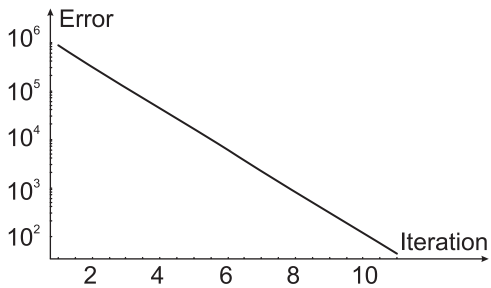

Figure 5. Convergence of the algorithm.

with tol $= 50$ (m^3). The optimal value obtained is $\lambda_0 = 0.002107617885177008$ and the switch times are: $t_1 = 0.528346$, $t_2 = 8.24259$, $t_3 = 14.8669$, $t_4 = 18.4717$, $t_5 = 22.7328$. Figure 4 presents the optimal hydropower, P. The profits from the optimal solution are 130,908 euros.

The algorithm runs very quickly (Figure 5). In the example, 11 iterations were needed and the CPU time required by the program was 0.188 s on a personal computer (Pentium IV/2 GHz).

7. Comparison with PROPT

In this section, we perform a comparison with a well-known solver: PROPT [15,16] a Matlab Optimal Control Software, and compare the results with those in the paper. PROPT is built on top of the source transformation package TomSym in the TOMLAB Base Module. PROPT aims to encompass all areas of OC, including the bang–bang control.

PROPT uses pseudospectral collocation methods to solve OC problems. This means that the solution takes the form of a polynomial. It is very important to note that a solution computed by

PROPT only satisfies the ordinary differential equation and constraints in the specified collocation points. There is no guarantee that the solution holds between these points! This constitutes a serious drawback of this solver. The default choice is to use n Gauss points as collocation points, although the user can specify any set of points to use. A common way of testing the integrity of a solution is to rerun the computation using twice as many collocation points. If nothing changes, then there were probably sufficient points in the first computation.

In this section, we present a test and choose the same hydroplant as in the previous section. For the sake of clarity in the comparison, we have chosen a fictitious clearing price, $\pi(t)$, that takes the following values: 90 (euros/h MW) in even hours and 70 (euros/h MW) in odd hours. Figure 6(a) shows the optimal solution obtained with our algorithm and Figure 6(b) the optimal solution obtained with PROPT and $n = 150$. Owing to the regularity that the price function, $\pi(t)$, presents in this example, the switch times obtained with our algorithm also present the following regularity: $t_1 = 0.47558$, $t_2 = 1.52442$, $t_3 = 2.47558$, $t_4 = 3.52442$, $t_5 = 4.47558$, $t_6 = 5.52442, \ldots, t_{23} = 22.47558$, $t_{24} = 23.52442$.

Table 3 affords a very interesting comparison between our algorithm and PROPT. We use two metrics to determine efficiency (CPU time and the number of iterations). We also present the final solution obtained (i.e. the profit) and the first switching time. As can be seen, our algorithm obtains the solution in a very short time. In contrast, PROPT needs a large number of iterations and hence CPU time in order to progressively approach the real solution of the problem. Furthermore, for higher values of n, PROPT presents problems of convergence.

Summing up, our algorithm has the following advantages:

- There is no need to carry out any kind of prior estimation of the number of switch times.
- There is no need to use an increasing number of collocation points to ensure the goodness of the solution.
- It is fast, even when compared with commercial packages that address more general problems.

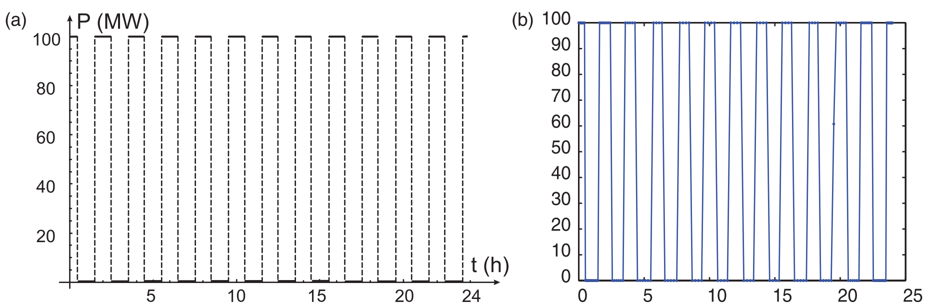

Figure 6. Optimal solution. (a) Solution with our algorithm. (b) Solution with PROPT.

Table 3. Comparison.

	Our algorithm	PROPT $n = 100$	PROPT $n = 150$	PROPT $n = 300$	PROPT $n = 600$
Iterations	3	111	160	324	649
CPU time (s)	0.015	0.0468	0.0937	0.5781	5.750
Solution (euros)	97,296.5	96,867.5	97,093.4	97,250.4	97,283.2
t_1	0.47558	0.5531	0.5054	0.4733	0.4749

We may therefore consider our algorithm to be an advantageous tool with respect to commercial packages of general application.

8. Conclusions

This paper presents a novel method for developing optimal bang–bang control for a wide class of problems with the final state and the final time fixed. We have proven that singular controls do not exist and a simple and very efficient algorithm has been developed.

We have compared our method with a well-known commercial solver and have proven that it presents numerous advantages with respect to the said package: speed, precision, reliability and, as already stated, no prior estimation of the solution is needed nor it is necessary to use specific parameters of the algorithm unrelated to the problem. Although we have presented a hydraulic example, it should be noted that our method may be applied to other problems with the same characteristics.

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