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The explicit solution of the profit maximization problem with box-constrained inputs

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ABSTRACT

In this paper we study the profit-maximization problem, considering maximum constraints for the general case of m-inputs and using the Cobb–Douglas model for the production function. To do so, we previously study the firm's cost minimization problem, proposing an equivalent infimal convolution problem for exponential-type functions. This study provides an analytical expression of the production cost function, which is found to be a piece-wise potential. Moreover, we prove that this solution belongs to class C^1. Using this cost function, we obtain the explicit expression of maximum profit. Finally, we illustrate the results obtained in this paper with an example.

1. Introduction

One of the most important issues for firms in the field of microeconomics [1] is the profit-maximization problem. In this paper we consider a firm that operates under perfect competition, i.e. its prices are independent of the firm's input and output decisions. Consider a firm employing a vector of inputs $\mathbf{x} \in \mathbb{R}_+^m$ to produce an output $y \in \mathbb{R}_+$, where $\mathbb{R}_+^m$, $\mathbb{R}_+$ are non-negative m- and 1-dimensional Euclidean spaces, respectively. Let $P(\mathbf{x})$ be the feasible output set for the given input vector $\mathbf{x}$ and $L(y)$ the input requirement set for a given output y. Now, the technology set [2] is defined as

$$T = \left\{(\mathbf{x}, y) \in \mathbb{R}_+^{m+1}, \mathbf{x} \in L(y), y \in P(\mathbf{x})\right\}.$$

We assume that this set satisfies the following well-known regularity properties: closedness, non-emptiness, scarcity, and no free lunch.

Only on a few occasions have additional constraints been employed in the literature; for example, an expenditure constraint is considered in [3]. Most classical studies, however, simplify resource utilization without considering constraints on input usage. In this paper we establish, for the first time, a box-constrained profit-maximization problem, considering maximum constraints for the inputs.

Generally, the profit maximization problem can be formulated in the following way: the firm chooses inputs and output in order to maximize profits π (where profits are revenue minus costs), subject to technology constraints (i.e. the relationship between inputs and output):

$$\begin{aligned} &\pi(p, \mathbf{w}) = \max_{\mathbf{x},y} py - \mathbf{wx}), \\ \text{s.t.} \quad & y = f(\mathbf{x}), \\ & (\mathbf{x}, y) \in T, \\ & 0 \leqslant x_i \leqslant M_i; \quad i = 1, \ldots, m, \end{aligned} \tag{1}$$

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where p is the price of the output, $\mathbf{w} \in \mathbb{R}^m$ are the vector prices of the inputs, M_i the maximum constraints for the inputs, and $f(\mathbf{x})$ is the production function, which is a continuous, strictly increasing and strictly quasiconcave function. In this paper we consider an m-input Cobb–Douglas production function [4], [5]:

$$y = f(\mathbf{x}) = A \prod_{i=1}^{m} x_i^{\alpha_i}.$$

There are two ways of solving this problem: (i) we can either formulate the problem maximizing over the input quantities, with plain Lagrange/Kuhn–Tucker or substituting the constraint in the objective function; or (ii) we formulate the problem using a minimum cost function and then maximize over the output quantity.

In this paper we use this short-cut-via-cost minimization. Note that if a firm is maximizing its profits and decides to offer a level of production y, it must be minimizing the cost of producing this output. Otherwise, a cheaper way of obtaining y production units would exist, which would mean that the firm is not be maximizing its profits. Namely: profit max implies cost min.

On the other hand, the profit-maximization problem has traditionally been solved by differentiating the variable x_i. Nevertheless, some authors avoid using total differentiation of first-order conditions, indicating that this gives rise to complicated equations which are difficult to handle. For example, [6] and [7] employ geometric programming to derive the maximal profit for the profit function. In the present paper we obtain the analytical and explicit formulas using the classical method of calculation.

The following are common problems than can arise: (i) The production function may not be differentiable, in which case we cannot take first-order conditions. (ii) The first-order conditions given above assume an interior solution, but we must also consider boundary solutions. (iii) A profit maximizing plan might not exist. (iv) The profit maximizing production plan might not be unique. In this paper we prove, under certain assumptions, the existence and uniqueness of the solution and that it belongs to class C^1.

The paper is organized as follows. Our box-constrained profit-maximization problem is solved in two stages: we first determine how to minimize the costs of producing each amount y and then what amount of production actually maximizes profits.

In the next section we present the box-constrained cost-minimization problem. By changing certain variables, we then transform it into a non-linear (exponential) separable programming problem [8], which we state as a constrained infimal convolution problem [9]. In Section 3, we provide a number of basic definitions and develop all the mathematical results necessary for the solution of the infimal convolution problem. Section 4 presents the optimal solution of the box-constrained cost-minimization problem. In Section 5, we obtain the optimal solution of the box-constrained profit-maximization problem to then discuss the results of a numerical example in Section 6. Finally, Section 7 summarizes the main conclusions of our research.

2. Cost-minimization problem

In this section we first present the classic firm's cost-minimization problem. This problem can be expressed as follows: produce a given output y, and choose inputs to minimize its cost:

$$\begin{aligned} & c(\mathbf{w}, y) = \min_{\mathbf{x} \geqslant \mathbf{0}} \mathbf{wx}, \\ \text{s.t.} \quad & f(\mathbf{x}) = y, \end{aligned} \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^m$ are the inputs and $\mathbf{w} \in \mathbb{R}^m$ are the factor prices. There are a number of different ways to mathematically express how inputs are transformed into output. In this paper we consider the general Cobb–Douglas production function

$$y = f(x) = A \prod_{i=1}^{m} x_i^{\alpha_i}$$

and we shall usually measure units so that the total factor productivity $A = 1$. The sum of α_i determines the returns to scale.

The formulas for the corresponding cost function $c(\mathbf{w}, y)$ are well known [10] when the production function follows the Cobb–Douglas model:

$$c(\mathbf{w}, y) = \alpha y^{\frac{1}{\alpha}} \prod_{i=1}^{m} \left(\frac{w_i}{\alpha_i} \right)^{\frac{\alpha_i}{\alpha}}, \quad \text{with } \alpha = \sum_{i=1}^{m} \alpha_i.$$

These formulas, which can be obtained simply using the Lagrange multipliers method, present the drawback that they are not applicable when upper limit constraints are considered for the different inputs.

In this paper we establish the analytical expression for the cost function $c(\mathbf{w}, y)$ using the Cobb–Douglas model, considering maximum constraints for the inputs. Our cost-minimization problem will be:

$$\begin{aligned} & c(\mathbf{w},y)=\min\sum_{i=1}^{m}w_i x_i, \\ \text{s.t.}\quad & y=\prod_{i=1}^{m}x_i^{\alpha_i}, \\ & 0\leqslant x_i\leqslant M_i;\quad i=1,\ldots,m. \end{aligned} \tag{3}$$

Problems of this kind, with box constraints, become complicated in the presence of boundary solutions. There is a vast array of software packages for numerically solving nonlinear optimization problems [11]. These methods only obtain an approximate solution for specific values of the output y, but do not provide the analytical expression of the cost function $c(\mathbf{w},y)$. It is thus not possible to know the marginal cost expression $\partial c(\mathbf{w},y)/\partial y$ needed to solve the profit maximization problem.

We shall address this problem in an exact way in this paper, transforming it into a non-linear (exponential) separable programming problem, which we state as a constrained infimal convolution problem. Taking into account the following changes in the variables:

$$\begin{aligned} &\ln y=q, \\ &\alpha_i\ln x_i=z_i,\quad i=1,\ldots,m, \end{aligned}$$

the cost-minimization problem (3) is equivalent to the infimal convolution problem:

$$\begin{aligned} & \tilde{c}(\mathbf{w},q)=\min\sum_{i=1}^{m}w_i e^{\frac{1}{\alpha_i}z_i}, \\ \text{s.t.}\quad & \sum_{i=1}^{m}z_i=q, \\ & -\infty<z_i\leqslant\alpha_i\ln M_i=P_i^{\max};\quad i=1,\ldots,m. \end{aligned} \tag{4}$$

The function $\tilde{c}(\mathbf{w},\cdot)$ is, in fact, the infimal convolution of the exponential functions

$$F_i(z_i)\ :=\ w_i e^{\frac{1}{\alpha_i}z_i}.$$

The case of quadratic F_i functions is well known and has been studied by the authors in [12] within the framework of hydrothermal optimization. However, the same kind of study is unknown for exponential functions. In this paper we develop the necessary mathematical tools to justify the proposed method for solving the stated problem.

3. Infimal convolution problem

Let us calculate the infimal convolution of the convex functions $F_i(z_i)$ considering their domain to be constrained to $(-\infty,P_i^{\max}]$. Let us assume throughout the paper, without loss of generality, that:

$$F_i'\left(P_i^{\max}\right)\leqslant F_{i+1}'\left(P_{i+1}^{\max}\right),\quad \forall i=1\ldots m. \tag{5}$$

Let the function $F:(-\infty,P_1^{\max}]\times\cdots\times(-\infty,P_m^{\max}]\to\mathbb{R}$ given by:

$$F(z_1,\ldots,z_m)\ :=\ \sum_{i=1}^{m}F_i(z_i).$$

Let C_q be the set:

$$C_q\ :=\ \left\{(z_1,\cdots,z_m)\in\left(-\infty,P_1^{\max}\right]\times\cdots\times\left(-\infty,P_m^{\max}\right]\Big/\sum_{i=1}^{m}z_i=q\right\}.$$

The infimal convolution of $\{F_i\}_{i=1}^{m}$ is

$$(F_1\odot\cdots\odot F_m)(q)\ :=\ \min_{C_q}\sum_{i=1}^{m}F_i(z_i).$$

Let us now see the definitions of the elements that are present in our problem.

Definition 1. Let us call the function $\Psi_i:(-\infty,\sum_{j=1}^{m}P_j^{\max}]\to(-\infty,P_i^{\max}]$ the ith distribution function, defined by

$$\Psi_i(q)=z_i,\quad \forall i=1,\ldots,m,$$

where $(z_1,\ldots,z_m)$ is the unique minimum of F on the set C_q, i.e.:

$$\sum_{i=1}^{m}\Psi_i(q)=q\quad\text{and}\quad\sum_{i=1}^{m}F_i(\Psi_i(q))=(F_1\odot\cdots\odot F_m)(q).$$

The following lemma guarantees that if z_i reaches its maximum value, all those z_k for which the derivative of F_k at its maximum value is less than or equal to the derivative corresponding to F_i will likewise have already reached their maximum.

Lemma 1. *If the function F reaches at $(a_1,\ldots,a_m)$ the minimum on the set C_q, and for a certain $i \in \{1,\ldots,m\},\ a_i = P_i^{\max}$, then:*

$$\forall k \in \{1,\ldots,m\}/F'_k(P_k^{\max}) \leqslant F'_i(P_i^{\max}) \Rightarrow a_k = P_k^{\max}.$$

Proof. We shall argue by contradiction. Let us assume that for a certain $i \in \{1,\ldots,m\},\ a_i = P_i^{\max}$ and that $a_j < P_j^{\max}$, being

$$F'_j\left(P_j^{\max}\right) \leqslant F'_i(P_i^{\max}).$$

Consider the function

$$g(\varepsilon) = F(a_1,\ldots,a_j+\varepsilon,\ldots,a_i-\varepsilon,\ldots,a_m) - F(a_1,\ldots,a_m),$$

$$g(\varepsilon) = F_j(a_j+\varepsilon) + F_i(a_i-\varepsilon) - F_j(a_j) - F_i(a_i).$$

It is clear that if $(a_1,\ldots,a_m) \in C_q$, then $(a_1,\ldots,a_j+\varepsilon,\ldots,a_i-\varepsilon,\ldots,a_m) \in C_q$ for

$$0 \leqslant \varepsilon < P_j^{\max} - a_j.$$

Let us show the existence of an ε such that $g(\varepsilon) < 0$, which contradicts the fact that F has a minimum in $(a_1,\ldots,a_m)$ within C_q. We have that g is continuous and derivable at zero with $g(0) = 0$; therefore it suffices to observe that $g'(0) < 0$. In fact,

$$g'(\varepsilon) = F'_j(a_j+\varepsilon) - F'_i(a_i-\varepsilon) = F'_j(a_j+\varepsilon) - F'_i(P_i^{\max}-\varepsilon),$$
$$g'(0) = F'_j(a_j) - F'_i(P_i^{\max}) < F'_j\left(P_j^{\max}\right) - F'_i(P_i^{\max}) \leqslant 0. \qquad \square$$

The following lemma establishes the order of the points at which the variables reach their maximum value.

Lemma 2. *The parameters*

$$\theta_k := \sum_{i=k}^{m} \frac{\alpha_i}{\alpha_k} P_k^{\max} + \sum_{i=k}^{m} \ln\left(\frac{\alpha_i w_k}{\alpha_k w_i}\right)^{\alpha_i} + \sum_{i=1}^{k-1} P_i^{\max},$$

satisfy

$$\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_m = \sum_{i=1}^{m} P_i^{\max}.$$

Proof

$$\theta_k = \sum_{i=k}^{m} \frac{\alpha_i}{\alpha_k} P_k^{\max} + \sum_{i=k}^{m} \ln\left(\frac{\alpha_i w_k}{\alpha_k w_i}\right)^{\alpha_i} + \sum_{i=1}^{k-1} P_i^{\max} = \sum_{i=k+1}^{m} \alpha_i\left[\ln\frac{w_k}{\alpha_k} + \frac{P_k^{\max}}{\alpha_k}\right] + \sum_{i=k+1}^{m} \ln\left(\frac{\alpha_i}{w_i}\right)^{\alpha_i} + \sum_{i=1}^{k} P_i^{\max} \leqslant \sum_{i=k+1}^{m} \alpha_i\left[\ln\frac{w_{k+1}}{\alpha_{k+1}} + \frac{P_{k+1}^{\max}}{\alpha_{k+1}}\right]$$
$$+ \sum_{i=k+1}^{m} \ln\left(\frac{\alpha_i}{w_i}\right)^{\alpha_i} + \sum_{i=1}^{k} P_i^{\max} = \sum_{i=k+1}^{m} \frac{\alpha_i}{\alpha_{k+1}} P_{k+1}^{\max} + \sum_{i=k+1}^{m} \ln\left(\frac{\alpha_i w_{k+1}}{\alpha_{k+1} w_i}\right)^{\alpha_i} + \sum_{i=1}^{k} P_i^{\max} = \theta_{k+1}. \qquad \square$$

The following theorem establishes a necessary and sufficient condition to obtain the interior solution.

Theorem 1. *The function F attains its minimum value on the set C_q at the point $(a_1,\ldots,a_m) \in \overset{\circ}{C}_q$ iff*

$$q < \sum_{i=1}^{m} \frac{\alpha_i}{\alpha_1} P_1^{\max} + \sum_{i=1}^{m} \ln\left(\frac{\alpha_i w_1}{\alpha_1 w_i}\right)^{\alpha_i} = \theta_1.$$

Proof. NecessityIf $(a_1,\ldots,a_m)$ is an interior point where F attains its minimum value, it is a point of relative minimum of F on the set

$$\left\{(z_1,\ldots,z_m) \in (-\infty, P_1^{\max}) \times \ldots \times (-\infty, P_m^{\max}) \,\middle|\, \sum_{i=1}^{m} z_i = q\right\}.$$

It follows that for some $\lambda \in \mathbb{R}$, $(a_1, \ldots, a_m)$ is a critical point of

$$F^*(z_1, \ldots, z_m) = F(z_1, \ldots, z_m) - \lambda(z_1 + \cdots + z_m - q).$$

Using the Lagrange multipliers method, we have that

$$\left.\begin{array}{l} \frac{w_1}{\alpha_1} e^{\frac{z_1}{\alpha_1}} - \lambda = 0 \\ \frac{w_2}{\alpha_2} e^{\frac{z_2}{\alpha_2}} - \lambda = 0 \\ \vdots \\ \frac{w_m}{\alpha_m} e^{\frac{z_m}{\alpha_m}} - \lambda = 0 \\ z_1 + z_2 + \cdots + z_m = q \end{array}\right\} \Rightarrow \left.\begin{array}{l} z_1 = \alpha_1 \left[\ln \lambda - \ln \frac{w_1}{\alpha_1}\right] \\ z_2 = \alpha_2 \left[\ln \lambda - \ln \frac{w_2}{\alpha_2}\right] \\ \vdots \\ z_m = \alpha_m \left[\ln \lambda - \ln \frac{w_m}{\alpha_m}\right] \\ z_1 + z_2 + \cdots + z_m = q. \end{array}\right\}$$

Hence

$$\ln \lambda = \frac{1}{\sum_m^{i=1} \alpha_i} q + \frac{1}{\sum_m^{i=1} \alpha_i} \sum_{i=1}^m \ln \left(\frac{w_i}{\alpha_i}\right)^{\alpha_i}$$

and

$$\lambda = \left[e^q \prod_{i=1}^m \left(\frac{w_i}{\alpha_i}\right)^{\alpha_i}\right]^{\frac{1}{\sum_m^{i=1} \alpha_i}}.$$

Let us consider $\Psi_k(q)$ to be a function of the unknown z_k

$$\Psi_k(q) = z_k = \frac{\alpha_k}{\sum_m^{i=1} \alpha_i} \left[q + \sum_{i=1}^m \ln \left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i}\right],$$

$$\Psi_k(q) = \frac{\alpha_k}{\sum_m^{i=1} \alpha_i} \left[q + \sum_{i=1}^m \ln \left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i}\right] = P_k^{\max},$$

$$\iff \sum_{i=1}^m \alpha_i \frac{P_k^{\max}}{\alpha_k} + \sum_{i=1}^m \ln \left(\frac{\alpha_i w_k}{\alpha_k wi}\right)^{\alpha_i} = q.$$

Letting

$$\Delta_k := \sum_{i=1}^m \alpha_i \frac{P_k^{\max}}{\alpha_k} + \sum_{i=1}^m \ln \left(\frac{\alpha_i w_k}{\alpha_k wi}\right)^{\alpha_i}$$

and, bearing in mind (5), we see that

$$\theta_1 = \Delta_1 \leqslant \Delta_2 \leqslant \cdots \leqslant \Delta_m.$$

It is evident that for every k, the solution $\Psi_k(q)$ is strictly increasing as a function of q. Thus,

$$q \geqslant \theta_1 \Rightarrow \Psi_1(q) = a_1 \geqslant \Psi_1(\theta_1) = P_1^{\max}$$

or, conversely,

$$\Psi_1(q) = a_1 < P_1^{\max} \Rightarrow q < \theta_1.$$

(Sufficiency). Since C_q is compact, the minimum of F clearly exists. Let us now consider

$$(a_1, \ldots, a_m) = (\Psi_1(q), \ldots, \Psi_m(q)),$$

a critical point of the convex functional

$$F^*(z_1, \ldots, z_m) = F(z_1, \ldots, z_m) - \lambda(z_1 + \cdots + z_m - q),$$

where

$$\lambda = \left[e^q \prod_{i=1}^m \left(\frac{w_i}{\alpha_i}\right)^{\alpha_i}\right]^{\frac{1}{\sum_{i=1}^m \alpha_i}}.$$

We have that $(a_1, \ldots, a_m)$ delivers the minimum value to F^* and, hence, it is also the minimum of F under the constraint

$$\left\{(z_1, \ldots, z_m) \middle| \sum_{i=1}^m z_i = q\right\}.$$

Moreover, it is evident that for every k, the solution $\Psi_k(q)$ is strictly increasing as a function of q. Thus,

$$q < \theta_1 \Rightarrow q < \Delta_k, \quad \forall k = 1,\ldots,m \Rightarrow \Psi_k(q) = a_k < P_k^{\max}, \quad \forall k = 1,\ldots,m,$$

so that $(a_1,\ldots,a_m) \in \overset{\circ}{C}_q$. $\square$

Having proven the above results, we are now in a position to obtain the distribution functions:

Theorem 2. *For every $k = 1,\ldots,m$, the kth distribution function is*

$$\Psi_k(q) = \begin{cases} \frac{\alpha_k}{\sum_{m}^{i=1}\alpha_i}\left[q + \sum\limits_{i=1}^{m} \ln\left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i}\right] & \text{if} \quad q < \theta_1 \\ \frac{\alpha_k}{\sum_{i=j+1}^{m}\alpha_i}\left[q + \sum\limits_{i=j+1}^{m} \ln\left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i} - \sum\limits_{i=1}^{j} P_i^{\max}\right] & \text{if} \quad \theta_j \leqslant q < \theta_{j+1} \leqslant \theta_k \\ P_k^{\max} & \text{if} \quad q \geqslant \theta_k \end{cases}$$

with the coefficients:

$$\theta_k = \sum_{i=k}^{m} \frac{\alpha_i}{\alpha_k} P_k^{\max} + \sum_{i=k}^{m} \ln\left(\frac{\alpha_i w_k}{\alpha_k w_i}\right)^{\alpha_i} + \sum_{i=1}^{k-1} P_i^{\max}.$$

Proof. In view of Theorem 1, if $q < \theta_1$, then the distribution functions $\Psi_k(q) < P_k^{\max}$ for all k and it remains to derive the expression for z_k. If $\theta_1 \leqslant q < \theta_2$, then the minimum of $\sum_{i=1}^{m} F_i(z_i)$ cannot be attained in the interior. According to Lemma 1, at least $z_1 = P_1^{\max}$. Thus, $\Psi_1(q) = P_1^{\max}$.

The same argument applies to the remaining problem of dimension $m - 1$.

$$\Psi_k(q) = \frac{\alpha_k}{\sum_{i=2}^{m}\alpha_i}\left[q - P_1^{\max} + \sum_{i=2}^{m} \ln\left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i}\right].$$

If $\theta_2 \leqslant q < \theta_3$, then $\Psi_1(q) = P_1^{\max}$ and, arguing as above, $\Psi_2(q) = P_2^{\max}$, and for $k > 2$, we have that

$$\Psi_k(q) = \frac{\alpha_k}{\sum_{i=3}^{m}\alpha_i}\left[q - P_1^{\max} - P_2^{\max} + \sum_{i=3}^{m} \ln\left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i}\right].$$

Finally, repeating the argument once again, we have that if $\theta_j \leqslant q < \theta_{j+1}$, then the kth distribution function is equal to $P_k^{\max}$ if $q \geqslant \theta_k$, and if $\theta_k > q$ ($k = 1,\ldots,j + 1$),

$$\Psi_k(q) = \frac{\alpha_k}{\sum\limits_{i=j+1}^{m}\alpha_i}\left[q - \sum_{i=1}^{j} P_i^{\max} + \sum_{i=j+1}^{m} \ln\left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i}\right] \qquad \square$$

We shall also prove that the infimal convolution of the functions $\{F_i\}_{i=1}^{m}$ belongs to class C^1. Let us see the following lemma first.

Lemma 3. *Let $\{F_i\}_{i=1}^{2} \subset C^1(\mathbb{R})$ be two convex functions satisfying $F_1'(M_1) \leqslant F_2'(M_2)$. Let us consider*

$$(F_1 \odot F_2)(\xi) := \min_D\{F_1(x) + F_2(y)\},$$
$$\textit{with} \quad D = \{(x,y) \in (-\infty, M_1] \times (-\infty, M_2] | x + y = \xi\}.$$

Then

$$(F_1 \odot F_2) \in C^1(-\infty, M_1 + M_2].$$

Proof. Let $\hat{g}_1$ and $\hat{g}_2$ be the functions of class C^1 that satisfy the following equality

$$\min_{x+y=\xi}\{F_1(x) + F_2(y)\} = F_1(\hat{g}_1(\xi)) + F_2(\hat{g}_2(\xi))$$

with $\hat{g}_1(\xi) + \hat{g}_2(\xi) = \xi$ and $F_1'(\hat{g}_1(\xi)) = F_2'(\hat{g}_2(\xi)) \forall \xi \in \mathbb{R}$.

We now have that the infimal convolution of the functions F_i constrained to their respective domains $(-\infty, M_i]$ will be

$$(F_1 \odot F_2)(\xi) = F_1(g_1(\xi)) + F_2(g_2(\xi))$$

with

$$g_2(\xi) := \xi - g_1(\xi),$$
$$g_1(\xi) := \begin{cases} M_1 \text{ if } \hat{g}_1(\xi) > M_1, \\ \hat{g}_1(\xi) \text{ if } \hat{g}_1(\xi) \leqslant M_1. \end{cases}$$

Let δ be such that $\hat{g}_1(\delta) = M_1$ (note that $F_1'(M_1) = F_2'(\hat{g}_2(\delta)) = F_2'(\delta - M_1)$)

$$(F_1 \odot F_2)(\xi) = \begin{cases} F_1(g_1(\xi)) + F_2(g_2(\xi)) & \text{if } \xi \leqslant \delta, \\ F_1(M_1) + F_2(\xi - M_1) & \text{if } \delta < \xi \leqslant M_1 + M_2. \end{cases}$$

In $(-\infty, \delta)$, the function $(F_1 \odot F_2)$ obviously belongs to class C^1 and also in $(\delta, M_1 + M_2]$, since $(F_1 \odot F_2)(\xi) = F_1(M_1) + F_2(\xi - M_1)$. The only conflicting point is δ. Let us study the continuity of $(F_1 \odot F_2)$ in δ:

$$(F_1 \odot F_2)(\delta-) = F_1(\hat{g}_1(\delta)) + F_2(\delta - \hat{g}_1(\delta)) = F_1(M_1) + F_2(\delta - M_1) = (F_1 \odot F_2)(\delta+).$$

Let us likewise study the continuity of the derivative in δ:

$$(F_1 \odot F_2)'(\delta-) = F_1'(\hat{g}_1(\delta))\hat{g}_1'(\delta) + F_2'(\delta - \hat{g}_1(\delta))(1 - \hat{g}_1'(\delta)) = F_1'(\hat{g}_1(\delta)) = F_1'(M_1),$$
$$(F_1 \odot F_2)'(\delta+) = F_2'(\delta - M_1) = F_1'(M_1).$$

Therefore, $(F_1 \odot F_2) \in C^1$. $\square$

Theorem 3. *Let $\{F_i\}_{i=1}^m \subset C^1(\mathbb{R})$. Let us consider*

$$(F_1 \odot F_2 \odot \cdots \odot F_m)(\xi) := \min_D \sum_{i=1}^m F_i(x_i),$$
$$\text{with} \quad D = \left\{ (x_1, \ldots x_m) \in \prod_{i=1}^m (-\infty, M_i] \,\middle|\, \sum_{i=1}^m x_i = \xi \right\}.$$

Then

$$(F_1 \odot F_2 \odot \cdots \odot F_m) \in C^1\left(-\infty, \sum_{i=1}^m M_i\right].$$

Proof. It suffices to reason by induction, bearing in mind, due to the associativity of $\odot$, that
$(F_1 \odot F_2 \odot \cdots \odot F_m) = (F_1 \odot \cdots \odot F_{m-1}) \odot F_m$. $\square$

We may now also obtain the analytical expression of $(F_1 \odot F_2 \odot \cdots \odot F_m)$.

Theorem 4. *The infimal convolution of the exponential functions $F_i(z_i)$ is an exponential (plus constant) piecewise function:*

$$(F_1 \odot F_2 \odot \cdots \odot F_m)(q) = \begin{cases} \tilde{w}_1 e^{\frac{q}{\tilde{\alpha}_1}} & \text{if} \quad q < \theta_1 \\ \tilde{\mu}_k + \tilde{w}_k e^{\frac{q}{\tilde{\alpha}_k}} & \text{if} \quad \theta_{k-1} \leqslant q < \theta_k \end{cases}$$

with the coefficients:

$$\tilde{\mu}_k = \sum_{i=1}^{k-1} w_i e^{\frac{P_i^{\max}}{\alpha_i}}; \quad \tilde{\alpha}_k = \sum\nolimits_{i=k}^m \alpha_i,$$

$$\tilde{w}_k = \exp\left[\left(-\sum_{i=1}^{k-1} P_i^{\max}\right) \Big/ \tilde{\alpha}_k\right] \left[\tilde{\alpha}_k \prod_{j=k}^m \left(\frac{w_j}{\alpha_j}\right)^{\frac{\alpha_j}{\tilde{\alpha}_k}}\right].$$

Moreover, it belongs to class C^1.

Proof. From Theorem 3, it is evident that

$$(F_1 \odot F_2 \odot \cdots \odot F_m) \in C^1\left(-\infty, \sum_{i=1}^m P_i^{\max}\right].$$

Furthermore, the infimal convolution expression for the exponential functions $\{F_i\}_{i=1}^m$ is easily obtained simply taking into account the definition of $(F_1 \odot F_2 \odot \cdots \odot F_m)$, Theorem 2 and the fact that $\Psi_i(q) = z_i$, $\forall i = 1,\ldots,m$. $\square$

4. Solution of the cost-minimization problem

Considering the fact that $c(\mathbf{w}, y) = \tilde{c}(\mathbf{w}, \ln y) = (F_1 \odot F_2 \odot \cdots \odot F_m)(\ln y)$, the following theorem is verified:

Theorem 5. *The conditional demand function for the kth input is:*

$$x_k(\mathbf{w}, y) = \begin{cases} \prod_{i=1}^{m} \left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\frac{\alpha_i}{\tilde{\alpha}_1}} \cdot y^{\frac{1}{\tilde{\alpha}_1}} & \text{if } y < e^{\theta_1}, \\ \exp\left[\left(-\sum_{i=1}^{j} P_i^{\max}\right)/\tilde{\alpha}_{j+1}\right] \cdot & \\ \cdot \prod_{i=j+1}^{m} \left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\frac{\alpha_i}{\tilde{\alpha}_{j+1}}} \cdot y^{\frac{1}{\tilde{\alpha}_{j+1}}} & \text{if } e^{\theta_j} \leqslant y < e^{\theta_{j+1}} \leqslant e^{\theta_k}, \\ e^{\frac{P_k^{\max}}{\alpha_k}} & \text{if } y \geqslant e^{\theta_k} \end{cases}$$

and the cost function is a piecewise potential (plus constant):

$$c(\mathbf{w}, y) = \begin{cases} \tilde{w}_1 y^{\frac{1}{\tilde{\alpha}_1}} & \text{if} \quad y < e^{\theta_1}, \\ \tilde{\mu}_k + \tilde{w}_k y^{\frac{1}{\tilde{\alpha}_k}} & \text{if} \quad e^{\theta_{k-1}} \leqslant y < e^{\theta_k}, \end{cases}$$

where $\tilde{\mu}_k, \tilde{\alpha}_k$ and $\tilde{w}_k$ are the coefficients defined in Theorem 4.

Proof. Taking into consideration the changes in the variable

$$\ln y = q,$$
$$z_k = \alpha_k \ln x_k = \Psi_k(q)$$

and Theorem 2, we obtain the expression of the conditional demand function for the kth input, $x_k(\mathbf{w},y)$. Similarly, as $c(\mathbf{w}, y = \tilde{c}(\mathbf{w}, \ln y)$, from Theorem 3 we obtain the cost function expression $c(\mathbf{w},y)$. $\square$

5. Solution of the profit-maximization problem

Having calculated the cost function $C(y) := c(\mathbf{w},y)$ and having established its character, C^1, the profit-maximization problem:

$$\pi(p, \mathbf{w}) = \max_y(py - c(\mathbf{w}, y)) = \max_y(py - C(y))$$

translates into the determination of the optimum level of output y^* for which the marginal cost coincides with the price p. Naturally, this consideration is only valid for output levels for which the marginal cost is increasing ($C(y)$ convex).

$$p = C'(y^*) \wedge \text{ convexity of } C \Rightarrow \pi(p, \mathbf{w}) = py^* - C(y^*).$$

Bearing in mind that the cost function is piecewise potential, the correct calculation of the output level requires prior investigation of the interval $[e^{\theta_{k-1}}, e^{\theta_k}]$ for which:

$$C'(e^{\theta_{k-1}}) \leqslant p \leqslant C'(e^{\theta_k}).$$

This question is trivial, as we already have the analytical expression of $C(y)$.

6. Example

We shall now present a profit maximization problem with a Cobb–Douglas type production function which, without considering technical constraints for the inputs, would be totally unreal, as it would present increasing returns to scale at all levels of production (and hence a concave cost function). By considering inputs to be limited, the problem becomes totally real and the resulting cost function presents a region of concavity (increasing returns to scale) and another of convexity (decreasing returns to scale) where the solution to the problem is to be found: the level of production at which the marginal cost and output price coincide.

We consider the following example:

$$\pi(p,\mathbf{w}) = \max_{\mathbf{x},y}(py - \mathbf{wx}),$$
$$\text{s.t.}\quad y = \prod_{i=1}^{m} x_i^{\alpha_i},$$
$$0 \leqslant x_i \leqslant M_i;\quad i = 1,\ldots,m$$

with $m = 20$ inputs, output price $p = 20$, and with the data presented in Table 1.

Table 1
Example data.

i	1	2	3	4	5	6	7	8	9	10
α_i	0.12	0.25	0.1	0.11	0.13	0.14	0.24	0.22	0.15	0.19
M_i	1	2	1.5	3	2.4	3.9	3	1	1.9	1
w_i	1.1	2	3.2	6.1	4	1.7	5	4.2	2.9	2
i	11	12	13	14	15	16	17	18	19	20
α_i	0.15	0.09	0.18	0.05	0.17	0.08	0.16	0.21	0.1	0.15
M_i	2	3	2.5	3	1	2.8	2	1.4	2	3
w_i	6	1.1	3	4.1	5	2.8	1	3	2	4.5

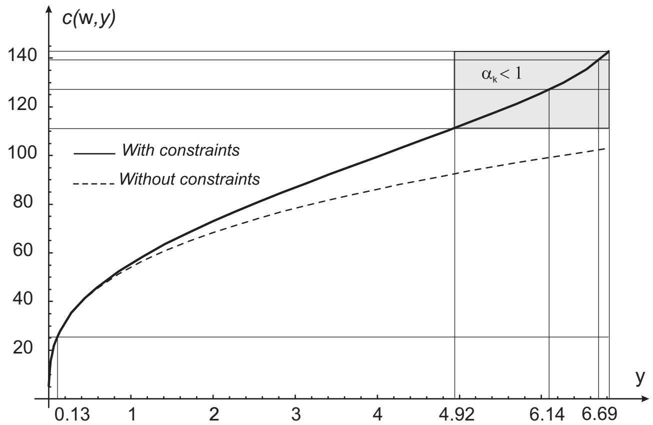

Fig. 1. The cost function.

Table 2
The piecewise cost function.

$c(\mathbf{w},y)$	$y \in [a,b]$
$54.2167y^{0.3344}$	[0,0.1301]
$1.1 + 53.55y^{0.3484}$	[0.1301,0.1934)
$3.1 + 52.07y^{0.3731}$	[0.1934,0.3066]
$5.1 + 50.35y^{0.3968}$	[0.3066,0.5712]
$9.1 + 46.48y^{0.4405}$	[0.5712,0.8529]
$13.30 + 42.29y^{0.4878}$	[0.8529,0.9383]
$17.50 + 38.1y^{0.5435}$	[0.9383,1.9077]
$22.50 + 33.36y^{0.5988}$	[1.9077,2.7568]
$25.80 + 30.49y^{0.6329}$	[2.7568,2.7648]
$31.30 + 25.80y^{0.6993}$	[2.7648,3.123]
$35.31 + 22.60y^{0.7519}$	[3.123,3.2972]
$42.81 + 16.98y^{0.8696}$	[3.2972,3.8202]
$49.44 + 12.69y^{0.9901}$	[3.8202,3.8726]
$54.24 + 9.86y^{1.0989}$	[3.8726,4.924]
$69.24 + 3.87y^{1.4925}$	[4.924,5.5063]
$78.84 + 1.69y^{1.8519}$	[5.5063,5.7495]
$90.84 + 0.35y^{2.5641}$	[5.7495,6.0198]
$104.34 + 0.01y^{4.1667}$	[6.0198,6.1441]
$112.18 + 1.8\cdot10^{-4}y^{6.25}$	[6.1441,6.687]
$130.50 + 2.6\cdot10^{-16}y^{20.}$	[6.687,6.8191]

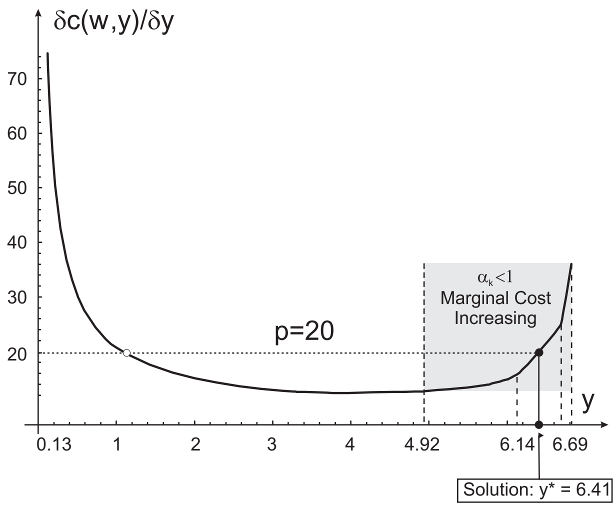

Fig. 2. The marginal cost function.

Table 3
Solution for y^* = 6.41467.

i	1	2	3	4	5	6	7	8	9	10
x_i	1	2	1.5	2.31349	2.4	3.9	3	1	1.9	1
i	11	12	13	14	15	16	17	18	19	20
x_i	2	3	2.5	1.56455	1	2.8	2	1.4	2	3

Fig. 1 shows the graph of the cost function $c(\mathbf{w},y)$, together with the resulting graph in the case of not having considered constraints for the inputs; note that both coincide in the interval $[0, e^{\theta_1} = 0.1301]$. From this point onward, not considering constraints leads to a very different cost function to the correct one. In addition, the area in which the production function presents decreasing returns to scale, and hence a convex cost function, is highlighted in grey.

The values $\{e^{\theta_k}\}_{k=1}^{20} = \{0.1301, 0.1934, 0.3066, 0.5712, 0.8529, 0.9383, 1.9077, 2.7568, 2.7648, 3.123, 3.2972, 3.8202, 3.8726, 4.924, 5.5063, 5.7495, 6.0198, 6.1441, 6.687, 6.8191\}$ constitute the different levels of output at which the parameters of the cost function expression change. These correspond to the levels at which the different inputs achieve their maximum value, which, according to the theoretical development (5), they do so in this example in the following order: $\{1, 10, 17, 2, 8, 18, 15, 12, 9, 19, 13, 6, 3, 7, 5, 11, 20, 16, 4, 14\}$. The analytical expression of the piecewise cost function is presented in Table 2, being obtained as shown in Theorem 5:

Fig. 2 shows the graph of the marginal cost function, which, as has already been established, is continuous (i.e. the cost function belongs to C^1). It can be seen that there are two points for which the marginal cost is p = 20. Naturally, however, the area represented in white does not provide the maximum value, as it is located in an area of decreasing marginal cost: the only maximum is obtained for the output value y^* = 6.41467.

Finally, in Table 3 we present the conditional demand function for the ith input.

7. Conclusions

In this paper we have established the analytical solution for the classic firm's profit maximization problem in the general case with m inputs. We have used the Cobb–Douglas model for the production function and have considered, for the first time, maximum constraints for the inputs. Our study has a number of advantages over other methods: the exact boundary solution is obtained and the method is not affected by the size of the problem.

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