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ORIGINAL PAPER

Cyclic coordinate descent in hydrothermal nonsmooth problems

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Abstract In this paper we present an algorithm, inspired by the cyclic coordinate descent method, which allows the solution of hydrothermal optimization problems involving pumped-storage plants. The proof of the convergence of the succession generated by the algorithm was based on the use of an appropriate adaptation of Zangwill's global theorem of convergence. Finally, the algorithm proposed is implemented using the Mathematica Package and is applied to an example to illustrate the results obtained.

Keywords Optimal control · Hydrothermal coordination · Coordinate descent · Zangwill's theorem

1 Introduction

In this paper we present an algorithm, inspired by the cyclic coordinate descent method, that will allow the solution of hydrothermal optimization problems involving hydraulic pumped-storage plants. In this case, the Lagrangian of the functional is continuous but not of class C^1.

This problem is large-scale and nonlinear and there is a vast bibliography describing different formulations and solution methodologies applied to solving it: the Lagrangian relaxation technique [1], Linear programming [2] and [3], dynamic programming [4] and genetic algorithms [5] have been widely used in different formulations. The main drawback is that these approaches

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require serious simplifying assumptions to make the problem computationally tractable.

We refer the reader to [6], an excellent paper that presents a review of the literature on the various optimization methods applied to solving the short-term hydrothermal coordination (STHTC) problem. These optimization methods can be generally classified into two main groups: deterministic methods and heuristic methods. Lagrangian relaxation (LR) and dynamic programming (DP) are deterministic methods. Genetic algorithms (GAs) and other evolutionary methods are heuristic.

LR uses Lagrange multipliers for the system constraints and adds the associated terms in the objective function, thus forming the Lagrangian function. For fixed values of the multipliers, the initial large-scale problem is decomposed into one subproblem per thermal plant and one subproblem per hydroelectric system. There are two major drawbacks to the LR methods: (1) convergence of the commonly employed subgradient algorithms to the dual maximum is very slow and the solution of the subproblems may be very sensitive to variations in the multipliers; and (2) the values of the multipliers that maximize the dual function do not guarantee feasibility of the primal problem owing to the nonconvexity of the problem search space. More often than not, the primal solution is infeasible and heuristic procedures are required to obtain a feasible primal solution. The duality gap is used as a measure of the quality of the solution obtained.

Several DP methods have been used, on the other hand, to solve the STHTC problem in numerous decomposition schemes. However, the "curse of dimensionality", which is the limited ability to solve large-sized problems with large number of variables, still remains the major drawback of using DP for a realistic system with multiple river basins and cascaded hydro-plants. Finally, several evolutionary computation techniques based on evolutionary theory (among these, GAs) have been introduced and applied to power system optimization problems. Their main advantages are flexibility and the ability to obtain good quality solutions; however, these are highly affected by computer requirements and convergence characteristics.

In addition to the previous comments, the reader may check that only 12 of the 123 papers analyzed in [6] consider pumped-storage units. For these reasons we consider other approach, the optimal control theory, to deal with the stated problem.

In this paper we propose Pontryagin's Minimum Principle (PMP) to solve the hydrothermal optimization problem and provide an optimization algorithm that leads to determination of the optimal solution of the general problem with hydraulic pumped-storage plants.

In a prior study [7], it was proven that the problem of optimization of the fuel cost of a hydrothermal system with several thermal plants may be reduced to the study of a hydrothermal system made up of one single thermal plant, called the thermal equivalent.

A necessary minimum condition was established in [8] for the optimization of hydrothermal problems involving one single hydraulic pumped-storage plant, thereby considering non-regular Lagrangian and non-holonomic inequality constraints. The mathematical problem was stated in the following terms:

$$\min_{z\in\Theta} F(z) = \min_{z\in\Theta} \int_0^T \Psi\left[P_d(t) - H(t, z(t), z'(t))\right] dt = \min_{z\in\Theta} \int_0^T L(t, z(t), z'(t))dt$$

$$\Theta = \{z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b,$$

$$H_{\min} \leq H(t, z(t), z'(t)) \leq H_s(t), \forall t \in [0, T]\}$$

where Ψ is the function of thermal cost of the thermal equivalent, $P(t)$ is the power generated by said plant, $P_d(t)$ is the power demand, $H(t, z(t), z'(t))$ the function of effective hydraulic contribution, $z(0) = 0$, $z(T) = b$ the boundary conditions which will have to be fulfilled, $P_d(t) - H(t, z(t), z'(t))$ is the power generated by the thermal equivalent, $(\widehat{C}^1)$ is the set of piecewise C^1 functions, $H_{\min}$ is the maximum pumping capacity, $H_s(t) = \min\{H_{\max}, P_d(t)\}$ with $H_{\max}$ the maximum generation, $L(\cdot,\cdot,\cdot)$ and $L_z(\cdot,\cdot,\cdot)$ are the class C^0 and $L_{z'}(t, z, \cdot)$ is piecewise continuous ($L_{z'}(t, z, \cdot)$ is continuous with one single point of discontinuity at $z' = 0$.

The coordinate descent method, on the other hand, enjoys a longstanding history in convex differentiable minimization. Surprisingly, very little is known about the convergence of the iterates generated by this method. Convergence typically requires restrictive assumptions such as that the cost function has bounded level sets and is in some sense strictly convex. The problem of minimizing a strictly convex function subject to linear constraints is considered in [9]; a convex function of the Legendre type subject to linear constraints is considered in [10]; while in [11], the author considers the objective to be pseudoconvex in every pair of the coordinate blocks and regular in some natural sense.

In [12] we presented an application of the algorithm of the cyclic coordinate descent in multidimensional variational problems with constrained speed. We showed an application to a hydrothermal system with one thermal plant (the thermal equivalent) and several hydro-plants but neither hydro-plants with pumping capacity.

The present paper addresses the generalization of both problems. First, we shall prove a necessary minimum condition for the optimization of hydrothermal problems involving several hydraulic pumped-storage plants and, also we shall introduce a numerical relaxation method for its solution and prove its convergence. The proof of the convergence of the succession generated by the algorithm was based, the same that in [12], on the use of an appropriate adaptation of Zangwill's global theorem of convergence [13]. The main contribution of our work is the presentation of an algorithm which allows the solution of a very complex problem of hydrothermal optimization involving several pumped-storage plants. In this kind of problem, the Lagrangian is piecewise continuous and we conseder non-holonomic inequality constraints (differential inclusions) for the admissible generated hydro-power: $H_{\min} \leq$

$H(t, z(t), z'(t)) \leq H_s(t)$. Finally, we present the solution of a hydrothermal optimization in which the potential of the proposed algorithm is evidenced.

2 Statement of the problem

Let us consider a hydrothermal system comprised of n thermal plants and m hydro-plants, assuming, with no loss in generality, that of the m hydro-plants, the first k are of the pumped-storage type (non-regular Lagrangian). The problem consists in minimizing the cost needed to satisfy a certain power demand during the optimization interval $[0, T]$. Said cost may be represented by the functional

$$J(\mathbf{z}) = \int_0^T L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))dt \tag{1}$$

$L(\cdot, \cdot, \cdot)$ is the class $C^2\left([0, T] \times \mathbb{R}^{2m} - \bigcup_{i=1}^{k} S_i\right)$ and $L(\cdot, \cdot, \dot{\mathbf{z}})$ is the class $C^2([0, T] \times R^{2m})$, such that

$$L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = \Psi\left(P_d(t) - H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))\right)$$

over the set

$$\Theta := \{\mathbf{z} \in \left(C^1[0, T]\right)^m \;/\; \mathbf{z}(0) = \mathbf{0}, \mathbf{z}(T) = \mathbf{b},$$

$$H_{i\,\min} \leq H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) \leq H_{i\,\max}\}$$

where $\mathbf{z} = (z_1, \cdots, z_m)$ is the vector of admissible volumes, $z_i(t)$ being the volume that is discharged up to the instant t by the i-th hydro-plant, $\dot{\mathbf{z}} = (\dot{z}_1, \cdots, \dot{z}_m)$ is the vector of admissible rates, $\dot{z}_i(t)$ being the rate of water discharge at the instant t by the i-th hydro-plant and $\mathbf{b} = (b_1, \cdots, b_m)$ is the vector of admissible volumes, b_i being the volume that must be discharged up to the instant T by the i-th hydro-plant. $H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$ is the power contributed to the system at the instant t by the set of hydro-plants, $H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$ the function of effective hydraulic contribution by the i-th hydro-plant, being

$$H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = \sum_{i=1}^{m} H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$$

satisfying

$$\frac{\partial^2 H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{\partial \dot{z}_i \partial \dot{z}_j} = 0, \; (i \neq j)$$

This condition means that the performance of the i-th hydro-plant is not influenced by the rate of water of the remaining plants, although their volumes may exert an influence.

Furthermore, S_i, for every $i \in \{1, \ldots, k\}$, is the set of points where

$$L_{\dot{z}_i}(t, \mathbf{z}, \dot{z}_1, \ldots, \dot{z}_{i-1}, \cdot, \dot{z}_{i+1}, \ldots, \dot{z}_m)$$

presents its only discontinuity (in $\dot{z}_i = 0$, the stoppage zone of the i-th hydro-plant). That is,

$$S_i := \left\{(t, \mathbf{z}, \dot{z}_1, \ldots, \dot{z}_{i-1}, 0, \dot{z}_{i+1}, \ldots, \dot{z}_m) \in [0, T] \times \mathbb{R}^{2m}\right\}$$

We shall assume that the admissible rates $\dot{z}_i(t)$ are bounded, Ψ is strictly increasing and strictly convex, the strictly increasing nature of

$$L_{\dot{z}_i}(t, \mathbf{z}, \dot{z}_1, \ldots, \dot{z}_{i-1}, \cdot, \dot{z}_{i+1}, \ldots, \dot{z}_m)$$

that $\frac{\partial H_i(t,\mathbf{z},\dot{\mathbf{z}})}{\partial \dot{z}_i} > 0, \forall i = 1, ..., m$, and that if $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$, being $\mathbf{q} = (q_1, \ldots, q_m)$, then for every $i = 1, ..., m$,

$$H_{i\,\min} \neq H_i(t, \mathbf{q}(t), \dot{q}_1(t), \ldots, \dot{q}_{i-1}(t), 0, \dot{q}_{i+1}(t), \ldots, \dot{q}_m(t)) \neq H_{i\,\max}$$

and

$$\frac{\partial H_i(t, \mathbf{z}(t), \dot{z}_1(t), \ldots, \dot{z}_{i-1}(t), 0, \dot{z}_{i+1}(t), \ldots, \dot{z}_m(t))}{\partial z_i} = 0$$

These conditions mean respectively that at the moments of maximum generation or pumping, no discontinuities of $L_{\dot{z}_i}$ are produced and that, at the points of discontinuity of $L_{\dot{z}_i}$, there is no variation in the generation-pumping function with respect to the volume.

We consider Θ equipped with the topology induced by the norm

$$||\mathbf{p}||^* := \max\{||\mathbf{p}||_\infty, ||\dot{\mathbf{p}}||_\infty\} = \max\{\max_{i=1,\ldots,m} ||p_i||_\infty, \max_{i=1,\ldots,m} ||\dot{p}_i||_\infty\}$$

3 Minimum necessary condition

At this point, we shall test a result that will allow us to characterize the minimum candidates of the proposed problem. We define the following function.

Definition 1 If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i, \forall t$, we define the "i-th coordination function" of $\mathbf{q} \in \Theta$ in $[0, T]$ as

$$\mathbb{Y}^i_{\mathbf{q}}(t) = -L_{\dot{z}_i}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \cdot \exp\left[-\int_0^t \frac{H_{z_i}(s, \mathbf{q}(s), \dot{\mathbf{q}}(s))}{H_{\dot{z}_i}(s, \mathbf{q}(s), \dot{\mathbf{q}}(s))} ds\right]$$

We denote by $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^+(t)$ and $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^-(t)$ the expressions obtained when considering the lateral derivatives with respect to $\dot{z}$. The fundamental result is the following.

Theorem 1 *If $\mathbf{q} \in \Theta$ is solution of the problem (1), then there exists $\{C_i\}_{i=1}^m \subset \mathbb{R}^+$ satisfying:*

(i) *If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$ then $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^+(t) \leq C_i \leq \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-(t)$*

(ii) *If* $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i$ *then*

$$\mathbb{Y}^i_{\mathbf{q}}(t) \text{ is } \begin{cases} \leq C_i \text{ if } H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\min} \\ = C_i \text{ if } H_{i\min} < H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) < H_{i\max} \\ \geq C_i \text{ if } H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\max} \end{cases}$$

Proof Considering the study carried out in [8] and that Pontryagin's Principle is verified for each one of the components $i = 1, \cdots m$, the remaining components being fixed, we may conclude that for every $i = 1, \cdots m$, there exists $C_i \in R^+$ satisfying the thesis of the theorem. □

4 Definition of the descent algorithm

The solution algorithm that we shall present is based on the resolution of a problem with m hydro-plants, subsequent to solving a succession of problems with one single hydro-plant. Let $\mathbf{q} \in \Theta$. Let

$$L^i_{\mathbf{q}}(t, z_i, \dot{z}_i) := L(q_1(t), \cdots, q_{i-1}(t), z_i, q_{i+1}(t), \cdots, q_m(t), \dot{q}_1(t), \cdots, \dot{z}_i, \cdots, \dot{q}_m(t))$$

and the functional $J^i_{\mathbf{q}} : \Theta^i_{\mathbf{q}} \longrightarrow \mathbb{R}$,

$$J^i_{\mathbf{q}}(z_i) := J(q_1, \cdots, q_{i-1}, z_i, q_{i+1}, \cdots, q_m) = \int_0^T L^i_{\mathbf{q}}(t, z_i(t), \dot{z}_i(t))dt$$

where

$$\Theta^i_{\mathbf{q}} := \{z \in C^1[0, T] \,/\, z(0) = 0, z(T) = b_i,$$
$$H_{i\min} \leq H_i(t, q_1(t), .., q_{i-1}(t), z, q_{i+1}(t), .., q_n(t), \dot{q}_1(t), .., \dot{z}, .., \dot{q}_m(t)) \leq H_{i\max}\}$$

Definition 2 We define the i-th minimizing map as the map $\Phi_i : \Theta \longrightarrow \Theta$ that satisfies for every $\mathbf{q} = (q_1, \ldots, q_i, ..., q_m) \in \Theta$

$$\Phi_i(q_1, \ldots, q_i, \ldots, q_m) = (q_1, \ldots, q^*, \ldots, q_m)$$

where

$$J^i_{\mathbf{q}}(q^*) < J^i_{\mathbf{q}}(z_i), \;\; \forall z_i \in \Theta^i_{\mathbf{q}} - \{q^*\}$$

We shall denote by Φ the map associated with the descent algorithm, which will be the composition of the i-th minimizing map:

$$\Phi := \Phi_m \circ \cdots \circ \Phi_1$$

In every k-th iteration of the algorithm, "the m hydro-plants will have been minimized" through the i-th minimizing applications in the established order, thus obtaining the new, admissible element, q_k,

$$\mathbf{q}_k = \Phi(\mathbf{q}_{k-1}) = (\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_2 \circ \phi_1)(\mathbf{q}_{k-1})$$

The limit of this descending succession will be provided by the sought after minimum. We denote $\mathbf{q}_i^* = (q_1, \ldots, q_i^*, \ldots, q_m)$ and $\dot{\mathbf{q}}_i^* = (\dot{q}_1, \cdots, \dot{q}_i^*, \cdots, \dot{q}_m)$. The following proposition is verified.

Proposition 1 *If* $\mathbf{q} \in \Theta$, *then* $\Phi_i(\mathbf{q}) = \mathbf{q}_i^*$ *is of class* C^1 *and there exists* $\{C_i\}_{i=1}^m \subset \mathbb{R}^+$ *satisfying:*

(i) *If* $\dot{q}_i^*(t)$ *is a point of discontinuity of* $\left(L_{\mathbf{q}}^i\right)_{\dot{z}_i}(t, q_i^*(t), \cdot)$,

$$\left(\mathbb{Y}^i_{\Phi_i(\mathbf{q})}\right)^+(t) \leq C_i \leq \left(\mathbb{Y}^i_{\Phi_i(\mathbf{q})}\right)^-(t)$$

(ii) *If* $\left(L_{\mathbf{q}}^i\right)_{\dot{z}_i}(t, q_i^*(t), \cdot)$ *is continuous in* $\dot{q}_i^*(t)$,

$$\mathbb{Y}^i_{\Phi_i(\mathbf{q})}(t) \text{ is } \begin{cases} \leq C_i \text{ if } H_i(t, \mathbf{q}_i^*(t), \dot{\mathbf{q}}_i^*(t)) = H_{i\,\min} \\ = C_i \text{ if } H_{i\,\min} < H_i(t, \mathbf{q}_i^*(t), \dot{\mathbf{q}}_i^*(t)) < H_{i\,\max} \\ \geq C_i \text{ if } H_i(t, \mathbf{q}_i^*(t), \dot{\mathbf{q}}_i^*(t)) = H_{i\,\max} \end{cases}$$

Proof Considering that q_i^* minimizes the functional $J_{\mathbf{q}}^i$, and the results obtained in [14] for those points where $\left(L_{\mathbf{q}}^i\right)_{\dot{z}_i}$ is continuous and in [8] for those points of discontinuity of $\left(L_{\mathbf{q}}^i\right)_{\dot{z}_i}$ (existence of plants with pumping capacity), we may guarantee that the minimum of the proposed problem belongs to C^1. Therefore, $\Phi_i(\mathbf{q})$ is of class C^1.

Moreover, as q_i^* minimizes the functional $J_{\mathbf{q}}^i$, following analagous reasoning to that of Theorem 1, we may conclude that there exists $\{C_i\}_{i=1}^m \subset R^+$ satisfying the thesis of Proposition 1. □

Approximate construction of $\phi_i(\mathbf{q})$ The implementation of the minimizing map $\phi_i(\mathbf{q})$ is performed by means of the approximate solution of the problem resulting from setting the "components" different to *i*.

The properties of $\phi_i(\mathbf{q})$, expressed in Proposition 1, allows us to undertake its approximate calculation using similar numerical methods to those used to solve differential equations in combination with an appropriate adaptation of the classical shooting method. This is achieved by implementing a discretized version of Theorem 1 in a manner similar to the one used in the soft case [12].

Given $\mathbf{q} = (q_1, \ldots, q_m) \in \Theta$, we shall consider, for each $C \in \mathbb{R}$,

$$\mathbf{q}_C^i = (q_1, \cdots, q_{i-1}, Q_C, q_{i+1}, \cdots, q_m)$$

satisfying the conditions (i) and (ii) of Theorem 1 and

$$Q_C(0) = 0$$

That is, Q_C minimizes the functional $J_{\mathbf{q}}^i$ within the set:

$$\{z \in C^1[0, T] \;/\; z(0) = 0, z(T) = Q_C(T),$$
$$H_{i\,\min} \leq H_i(t, q_1(t), .., q_{i-1}(t), z, q_{i+1}(t), .., q_n(t), \dot{q}_1(t), .., \dot{z}, .., \dot{q}_m(t)) \leq H_{i\,\max}\}$$

We shall undertake two processes of approximation:

(PHASE 1) Approximate construction of Q_C (the adapted Euler method).

The approximate construction of Q_C, which we shall call $\widetilde{Q}_C$, is carried out by means of polygonals (Euler's method) considering the triple recurring sequence (X_n, Y_n, I_n) with

$$n = 0, \cdots, N,\ h = \frac{T}{N},\ t_n = h \cdot n$$

which represents the following approximations:

$$\begin{aligned}
&Q_C(t_n) \approx \widetilde{Q}_C(t_n) := X_n \\
&\dot{Q}_C(t_n) \approx \widetilde{\dot{Q}}_C(t_n) := Y_n \\
&Q_C(t) \approx \widetilde{Q}_C(t) := X_{n-1} + (t - t_{n-1}) \cdot Y_{n-1} \text{ in } [t_{n-1}, t_n], n > 0 \\
&Let\ \widetilde{\mathbf{q}}^i_C = (q_1, \cdots, q_{i-1}, \widetilde{Q}_C, q_{i+1}, \cdots, q_m) \\
&\exp\left(-\int_0^{t_n} \frac{H_{z_i}(s, \mathbf{q}^i_C(s), \dot{\mathbf{q}}^i_C(s))}{H_{\dot{z}_i}(s, \mathbf{q}^i_C(s), \dot{\mathbf{q}}^i_C(s))} ds\right) \approx I_n := \exp\left(-\int_0^{t_n} \frac{H_{z_i}(s, \widetilde{\mathbf{q}}^i_C(s), \widetilde{\dot{\mathbf{q}}}^i_C(s))}{H_{\dot{z}_i}(s, \widetilde{\mathbf{q}}^i_C(s), \widetilde{\dot{\mathbf{q}}}^i_C(s))} ds\right)
\end{aligned}$$

and which obeys the following relation of recurrence:

$$X_0 = 0;\ I_0 = 1$$

$$\text{Let } \chi^+ \text{ such that } - I_n * \frac{\partial L^{i+}_{\mathbf{q}}(t_n, X_n, \chi^+)}{\partial \dot{z}_i} = C$$

$$\text{Let } \chi^- \text{ such that } - I_n * \frac{\partial L^{i-}_{\mathbf{q}}(t_n, X_n, \chi^-)}{\partial \dot{z}_i} = C$$

Let:

$$\begin{aligned}
\mathbf{X}_n &= (q_1(t_n), \cdots, q_{i-1}(t_n), X_n, q_{i+1}(t_n), \cdots, q_m(t_n)) \\
\mathbf{Y}_n &= (\dot{q}_1(t_n), \cdots, \dot{q}_{i-1}(t_n), Y_n, \dot{q}_{i+1}(t_n), \cdots, \dot{q}_m(t_n)) \\
\chi^+ &= (\dot{q}_1(t_n), \cdots, \dot{q}_{i-1}(t_n), \chi^+, \dot{q}_{i+1}(t_n), \cdots, \dot{q}_m(t_n)) \\
\chi^- &= (\dot{q}_1(t_n), \cdots, \dot{q}_{i-1}(t_n), \chi^-, \dot{q}_{i+1}(t_n), \cdots, \dot{q}_m(t_n))
\end{aligned}$$

$$Y_n \text{ is such that } \begin{cases}
H_i(t_n, \mathbf{X}_n, \mathbf{Y}_n) = H_{i\ \max} & \text{if } \quad H_i(t_n, \mathbf{X}_n, \chi^+) \geq H_{i\ \max} \\
H_i(t_n, \mathbf{X}_n, \mathbf{Y}_n) = H_{i\ \min} & \text{if } \quad H_i(t_n, \mathbf{X}_n, \chi^-) \leq H_{i\ \min} \\
\chi^+ & \text{if } 0 < H_i(t_n, \mathbf{X}_n, \chi^+) < H_{i\ \max} \\
\chi^- & \text{if } H_{i\ \min} < H_i(t_n, \mathbf{X}_n, \chi^-) < 0 \\
0 & \text{otherwise}
\end{cases}$$

$$X_{n+1} = X_n + h \cdot Y_n$$

$$I_{n+1} = I_n * \exp\left(-\int_{t_n}^{t_{n+1}} \frac{H_{z_i}(s, \mathbf{X}_{n+1}, \mathbf{Y}_n)}{H_{\dot{z}_i}(s, \mathbf{X}_{n+1}, \mathbf{Y}_n)} ds\right)$$

(PHASE 2) For every *i-th* component, construction of a sequence $\{C_j\}_{j\in\mathbb{N}}$ such that $Q_{C_j}(T)$ converges to b_i (the adapted shooting method).

In these terms, the objective is to construct $C^* = \lim_{j\to\infty} C_j$ such that $Q_{C^*}(T) = b_i$, in which case:

$$\phi_i(\mathbf{q}) = (q_1, \ldots, q_{i-1}, Q_{C^*}, q_{i+1,} \ldots, q_m)$$

Of course, C^* cannot be computed exactly, but given a certain tolerance Tol, one should work with an approximation, for which

$$\left|Q_{C_j}(T) - b_i\right| < Tol$$

5 Convergence of the algorithm

We now go on to present a topological version of the global convergence theorem of descent algorithms with more general hypotheses that do not affect the correctness of the demonstration given in [13] by Zangwill; specifically, the continuity of the descending function is substituted by sequential continuity and the compactness by relative sequential compactness.

Theorem 2 (Global convergence, generalized version) *Let Φ be a map on the topological space (X, τ) and $x_0 \in X$. Let us assume that the recurrence sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by*

$$x_{n+1} = \Phi(x_n)$$

verifies the following:

(i) *$\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{K} \subset X$, where $\mathbb{K}$ is relatively sequentially compact.*
(ii) *There exists a sequentially continuous function $F : (\mathbb{X}, \tau) \to (\mathbb{R}, |\quad|)$ satisfying:*

$$\Phi(x) \neq x \Longrightarrow F(\Phi(x)) < F(x)$$

(iii) *Φ is sequentially continuous in X.*

Hence, every convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to a fixed point on Φ.

Proof This is identical to the proof presented by Zangwill for the global convergence theorem of descent algorithms. It need only be pointed out that the transformation Φ associated with the algorithm is pointwise and that the solution set is that of fixed points on Φ. □

Although in measurable topological spaces the sequential character of compactness and continuity is irrelevant, we shall maintain this terminology so as to facilitate the exposition.

We now base the demonstration of the convergence of the proposed algorithm on Theorem 2. First, let us see a series of preparatory results, the demonstrations of which we shall omit in some cases on account of their being simple or well known.

Lemma 1 *Let (X, τ) be a topological space with $K \subset X$ relatively sequentially compact. If the sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$ verifies that all its convergent subsequences have the same limit, then $\{x_n\}_{n\in\mathbb{N}}$ converges to this same limit.*

Lemma 2 *Given a family of functions*

$$\mathbb{F} = \{\mathbf{f}_\lambda = (f_{\lambda,1}, \ldots, f_{\lambda,m})\}_{\lambda\in I} \subset \widehat{C}^1[0, T]^m$$

if the family of its derivatives $\{\dot{\mathbf{f}}_\lambda\}_{\lambda\in\mathbb{I}}$ is uniformly bounded, then $\mathbb{F}$ is equicontinuous.

Lemma 3 *Let $\Omega \subset [0, T] \times \mathbb{R}^{2m}$ and $L : \Omega \longrightarrow \mathbb{R}$ locally Lipschitz. For each $\mathbf{h} \in C^1([0, T], \mathbb{R}^m)$ such that $(t, \mathbf{h}(t), \dot{\mathbf{h}}(t)) \in \Omega\ \forall t \in [0, T]$ we define*

$$L_{\mathbf{h}}(t) := L(t, \mathbf{h}(t), \dot{\mathbf{h}}(t)) \text{ and } W_h(t) := \int_0^t L(s, \mathbf{h}(s), \dot{\mathbf{h}}(s))ds$$

If $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ converges to $\mathbf{q}$ in $(\Theta, ||\ \ ||^*)$ then $\{L_{\mathbf{q}_n}\}$ converges uniformly (c.u.) to $L_{\mathbf{q}}$ and $\{W_{\mathbf{q}_n}\}$ converges pointwise (c.p.) to $W_{\mathbf{q}}$.

Lemma 4 *If $\{\mathbf{q}_n\}_{n\in\mathbb{N}} \subset \Theta$ c.u. to $\mathbf{q}$ and $\{\dot{\mathbf{q}}_n\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded, then $\{\dot{\mathbf{q}}_n\}_{n\in\mathbb{N}}$ c.u. to $\dot{\mathbf{q}}$.*

We shall next see that if a sequence $\{\mathbf{q}_n\}$ satisfies the thesis of Theorem 1 for a certain sequence $\{C_{n,i}\}$, then its limit also satisfies said thesis for the limit of the $\{C_{n,i}\}$.

Proposition 2 *Let L in the conditions of the problem (1). If $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ c.u. to $\mathbf{q}$ in $(\Theta, ||||^*)$ it is verified that*

(i) *If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i$ then*

$$\left\{\mathbb{Y}^i_{\mathbf{q}_n}\right\}_{n\in\mathbb{N}} \text{ c.p. to } \mathbb{Y}^i_{\mathbf{q}},\ \forall i = 1, \ldots m$$

(ii) *If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$ then $\exists\{\mathbf{q}_{n_k}\}_{k\in\mathbb{N}} \subset \{\mathbf{q}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{q}_{n_s}\}_{s\in\mathbb{N}} \subset \{\mathbf{q}_n\}_{n\in\mathbb{N}}$ such that*

$$\left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_k}}\right)^+\right\}_{k\in\mathbb{N}} \text{ c.p. to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^+ \text{ and/or } \left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_s}}\right)^-\right\}_{s\in\mathbb{N}} \text{ c.p. to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-$$

Proof Considering, for each $\mathbf{z} \in\Theta$

$$\left.\begin{array}{l} \mathbb{L}^i_{\mathbf{z}}(t) := -L_{\dot{z}_i}(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) \\ \mathbb{S}^i_{\mathbf{z}}(t) := \dfrac{H_{z_i}(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{H_{\dot{z}_i}(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))} \\ \mathbb{I}^i_{\mathbf{z}}(t) := \int_0^t \mathbb{S}^i_{\mathbf{z}}(s)ds \end{array}\right\} \Longrightarrow \mathbb{Y}^i_{\mathbf{q}_n}(t) = \mathbb{L}^i_{\mathbf{q}_n}(t) \exp\left[\mathbb{I}^i_{\mathbf{q}_n}(t)\right]$$

We denote by $\left(\mathbb{Y}^i_{\mathbf{q}_n}\right)^+(t)$ and $\left(\mathbb{Y}^i_{\mathbf{q}_n}\right)^-(t)$ the expressions obtained when considering the lateral derivatives with respect to $\dot{z}$.

(i) If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i$, $L_{\dot{z}_i}$ is of class C^1 and, therefore, Lipschitz in $\bar{S}_i$. In virtue of Lemma 4, $\{\mathbb{L}^i_{\mathbf{q}_n}\}_{n\in\mathbb{N}}$ c.u. to $\mathbb{L}^i_{\mathbf{q}}$ and applying the same reasoning, as $H_{\dot{z}_i} \neq 0, \forall i = 1, ..., m$, we have that $\{\mathbb{S}^i_{\mathbf{q}_n}\}_{n\in\mathbb{N}}$ c.u. to $\mathbb{S}^i_{\mathbf{q}}$. Thus, $\{\mathbb{I}^i_{\mathbf{q}_n}\}_{n\in\mathbb{N}}$ c.p. to $\mathbb{I}^i_{\mathbf{q}}$ and

$$\left\{\mathbb{Y}^i_{\mathbf{q}_n}\right\}_{n\in\mathbb{N}} \text{ c.p. to } \mathbb{Y}^i_{\mathbf{q}}$$

(ii) If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$, $L_{\dot{z}_i}$ is discontinuous at this point and, therefore, $\dot{q}_i(t) = 0$ is a point of discontinuity of $\left(L^i_{\mathbf{q}}\right)_{\dot{z}_i}$. As $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ c.u. to $\mathbf{q}$ in $(\Theta, \|\|^*)$, it may occur that:

(a) $\forall n$, $\dot{q}_{n,i}(t) \geq 0$. It is verified that $L_{\dot{z}_i}$ in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^{i-1} \times \mathbb{R}^* \times \mathbb{R}^{n-i}$ is of class C^1; specifically, it will be locally Lipschitz and, in virtue of Lemma 3, $\left\{\left(\mathbb{L}^i_{\mathbf{q}_n}\right)^+\right\}_{n\in\mathbb{N}}$ c.u. to $\left(\mathbb{L}^i_{\mathbf{q}}\right)^+$. Likewise, considering $H^+_{\dot{z}_i} \neq 0$, $\forall i = 1, ..., m$, we have that $\left\{\left(\mathbb{S}^i_{\mathbf{q}_n}\right)^+\right\}_{n\in\mathbb{N}}$ c.u. to $\left(\mathbb{S}^i_{\mathbf{q}}\right)^+$ and, hence, that $\left\{\left(\mathbb{I}^i_{\mathbf{q}_n}\right)^+\right\}_{n\in\mathbb{N}}$ c.p. to $\left(\mathbb{I}^i_{\mathbf{q}}\right)^+$, from which we conclude that

$$\left\{\left(\mathbb{Y}^i_{\mathbf{q}_n}\right)^+\right\}_{n\in\mathbb{N}} \text{ c.p. to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^+$$

(b) $\forall n$, $\dot{q}_{n,i}(t) \leq 0$. Reasoning analogously to the above, we conclude that

$$\left\{\left(\mathbb{Y}^i_{\mathbf{q}_n}\right)^-\right\}_{n\in\mathbb{N}} \text{ c.p. to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-$$

(c) There exists $\{q_{n_k}\}_{k\in\mathbb{N}}$, $\{q_{n_s}\}_{s\in\mathbb{N}} \subset \{q_n\}_{n\in\mathbb{N}}$ such as: $\dot{q}_{n_k,i}(t) \geq 0$; $\dot{q}_{n_s,i}(t) \leq 0$. Reasoning analogously to the previous cases, we shall have that

$$\left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_k}}\right)^+\right\}_{k\in\mathbb{N}} \text{ c.p. to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^+; \ \left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_s}}\right)^-\right\}_{s\in\mathbb{N}} \text{ c.p. to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-$$

□

Corollary 1 *If* $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *converges to* $\mathbf{q}$ *in* $(\Theta, \|\|^*)$ *verifying*

(a) *If* $(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) \in S_i$, $\left(\mathbb{Y}^i_{\mathbf{q}_n}\right)^+(t) \leq C_{n,i} \leq \left(\mathbb{Y}^i_{\mathbf{q}_n}\right)^-(t)$

(b) *If* $(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) \notin S_i$,

$$\mathbb{Y}^i_{\mathbf{q}_n}(t) \textit{ is } \begin{cases} \leq C_{n,i} \textit{ if } H_i(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) = H_{i\min} \\ = C_{n,i} \textit{ if } H_{i\min} < H_i(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) < H_{i\max} \\ \geq C_{n,i} \textit{ if } H_i(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) = H_{i\max} \end{cases}$$

then the sequence $\{C_{n,i}\}_{n\in\mathbb{N}}$ *converges and, calling its limit* C_i*, it is verified that*

(i) *If* $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i$ *then*

$$\mathbb{Y}^i_{\mathbf{q}}(t) \textit{ is } \begin{cases} \leq C_i \textit{ if } H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\min} \\ = C_i \textit{ if } H_{i\min} < H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) < H_{i\max} \\ \geq C_i \textit{ if } H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\max} \end{cases}$$

(ii) *If* $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$ *then* $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^+(t) \leq C_i \leq \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-(t)$

Proof It is evident that, $\forall i = 1, \ldots, m$, the sequence $\{C_{n,i}\}_{n\in\mathbb{N}}$ converges, since otherwise $\{\mathbb{Y}^i_{\mathbf{q}_n}\}_{n\in\mathbb{N}}$ or $\left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_k}}\right)^+\right\}_{k\in\mathbb{N}}$ or $\left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_s}}\right)^-\right\}_{s\in\mathbb{N}}$ would not converge, thus contradicting Proposition 2.

(i) If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i$, we have the following possibilities:

- If $H_{i\min} < H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) < H_{i\max}$, $\exists k \in \mathbb{N}$ / $\forall n > k$, $H_{i\min} < H_i(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) < H_{i\max}$, and therefore

$$\mathbb{Y}^i_{\mathbf{q}}(t) = \lim_{n\to\infty} \mathbb{Y}^i_{\mathbf{q}_n}(t) = \lim_{n\to\infty} C_{n,i} = C_i$$

- If $H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\min}$, $\exists k \in \mathbb{N}$ / $\forall n > k$, $H_i(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) = H_{i\min}$ and therefore

$$\mathbb{Y}^i_{\mathbf{q}_n}(t) \leq C_{i,n} \Longrightarrow \mathbb{Y}^i_{\mathbf{q}}(t) \leq C_i$$

- If $H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\max}$, $\exists k \in \mathbb{N}$ / $\forall n > k$, $H_i(t, \mathbf{q}_n(t), \dot{\mathbf{q}}_n(t)) = H_{i\max}$ and therefore

$$\mathbb{Y}^i_{\mathbf{q}_n}(t) \geq C_{i_n} \Longrightarrow \mathbb{Y}^i_{\mathbf{q}}(t) \geq C_i$$

(ii) If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$, then in virtue of Proposition 2, we have that $\exists\{\mathbf{q}_{n_k}\}_{k\in\mathbb{N}} \subset \{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *and* $\{\mathbf{q}_{n_s}\}_{s\in\mathbb{N}} \subset \{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *such that* $\left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_k}}\right)^+\right\}_{k\in\mathbb{N}}$ c.p. to $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^+$, and/or $\left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_s}}\right)^-\right\}_{s\in\mathbb{N}}$ c.p. to $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^-$. Moreover, we know that it is verified that

$$\left(\mathbb{Y}^i_{\mathbf{q}_{n_k}}\right)^+(t) \leq C_{n_k,i} \leq \left(\mathbb{Y}^i_{\mathbf{q}_{n_k}}\right)^-(t) \text{ and } \left(\mathbb{Y}^i_{\mathbf{q}_{n_s}}\right)^+(t) \leq C_{n_s,i} \leq \left(\mathbb{Y}^i_{\mathbf{q}_{n_s}}\right)^-(t)$$

from which it may be deduced that

$$\left(\mathbb{Y}^i_{\mathbf{q}}\right)^+(t) \leq C_i \leq \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-(t)$$

□

We shall now show the conservation of convergence by means of the i-th minimizing map Φ_i.

Proposition 3 *If* $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *and* $\{\Phi_i(\mathbf{q}_n)\}_{n\in\mathbb{N}}$ *converge in* $(\Theta, ||\ \ ||^*)$ *then*

$$\{\Phi_i(\mathbf{q}_n)\}_{n\in\mathbb{N}} \textit{ converges to } \Phi_i(\lim_{n\to\infty}(\mathbf{q}_n))$$

Proof Let $\mathbf{s}_n := \Phi_i(\mathbf{q}_n)$ which converges uniformly to $\mathbf{s}$ and $\dot{\mathbf{s}}_n$ to $\dot{\mathbf{s}}$. Proposition 1, together with Corollary 1, guarantees that

(i) If $(t, \mathbf{s}(t), \dot{\mathbf{s}}(t)) \notin S_i$ then

$$\mathbb{Y}^i_{\mathbf{s}}(t) \text{ is } \begin{cases} \leq C_i \text{ if } H_i(t, \mathbf{s}(t), \dot{\mathbf{s}}(t)) = H_{i\,\min} \\ = C_i \text{ if } H_{i\,\min} < H_i(t, \mathbf{s}(t), \dot{\mathbf{s}}(t)) < H_{i\,\max} \\ \geq C_i \text{ if } H_i(t, \mathbf{s}(t), \dot{\mathbf{s}}(t)) = H_{i\,\max} \end{cases}$$

(ii) If $(t, \mathbf{s}(t), \dot{\mathbf{s}}(t)) \in S_i$ then

$$\left(\mathbb{Y}^i_{\mathbf{s}}\right)^+(t) \leq C_i \leq \left(\mathbb{Y}^i_{\mathbf{s}}\right)^-(t)$$

and, hence, $\Phi_i(\mathbf{s}) = \mathbf{s}$.

Now note that $\mathbf{q}_n$ and $\mathbf{s}_n = \Phi_i(\mathbf{q}_n)$ differ only in their i-th component, as do their limits. Thus,

$$\Phi_i\left(\lim_{n\to\infty}(\mathbf{q}_n)\right) = \Phi_i(\mathbf{s}) = \mathbf{s} = \lim_{n\to\infty}(\Phi_i(\mathbf{q}_n))$$

□

In the following corollary, we present a result, which proof is in [12], it establishes the sequential continuity of Φ.

Corollary 2 *If* $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *converges to* $\mathbf{q}$ *in* $(\Theta, ||\ \ ||^*)$ *and* $\{\Phi(\mathbf{q}_n)\}_{n\in\mathbb{N}}$ *and* $\{\dot{\Phi}(\mathbf{q}_n)\}_{n\in\mathbb{N}}$ *are equicontinuous and uniformly bounded then:*

$$\{\Phi(\mathbf{q}_n)\}_{n\in\mathbb{N}} \textit{ converges to } \Phi(\mathbf{q}) \textit{ in } (\Theta, ||\ \ ||^*)$$

In the following proposition, we establish the application framework of the extension of Zangwill's Theorem.

Proposition 4 *Let* $\mathbb{U} := \Theta \cap \widehat{C}^2$. *Then* $\exists M \in \mathbb{R}$ *such that, being* $\mathbb{U}_M := \{\mathbf{z} \in \mathbb{U} /$ $||\ddot{\mathbf{z}}||_\infty < M\}$, *it is verified that:*

(i) $\Phi(\mathbb{U}_M) \subseteq \mathbb{U}_M$.
(ii) $\mathbb{U}_M$ *is relatively sequentially compact in* $(\Theta, ||\ ||^*)$.
(iii) $\Phi : (\Theta, ||\ ||^*) \longrightarrow (\Theta, ||\ ||^*)$ *is sequentially continuous.*
(iv) $F : (\Theta, ||\ \ ||^*) \longrightarrow (\mathbb{R}, ||)$ *is sequentially continuous satisfying* $\Phi(\mathbf{x}) \neq \mathbf{x} \Longrightarrow F(\Phi(\mathbf{x})) < F(\mathbf{x})$.

Proof $\forall \mathbf{p} = (p_1, \ldots, p_m) \in \Theta$ we have that by hypothesis, for every i-th component there exists $A_i, B_i \in \mathbb{R}$ such as $A_i < \dot{p}_i < B_i\ \forall t \in [0, T]$

$$||\dot{p}_i||_\infty \leq M_i := \max\{|A_i|, |B_i|\} \Longrightarrow ||p_i||_\infty < N_i := \alpha + M_i \cdot T$$

Let

$$\mathbb{O} := [0, T] \times \prod_{i=1}^{n} [-N_i, N_i] \times \prod_{i=1}^{n} [-M_i, M_i]$$

and let $f_i : [0, T] \times \mathbb{R}^{2m} \longrightarrow \mathbb{R}$ be the continuous functions associated with the i-th Euler equation of the functional $F^i_{\mathbf{p}}$

$$\ddot{z}_i(t) = f_i(t, p_1(t), \cdots, z_i(t), \cdots, p_m(t), \dot{p}_1(t), \cdots, \dot{z}_i(t), \cdots, \dot{p}_m(t))$$

Let

$$C_i := \max_{\mathbb{O}} f_i(t, x_1, \cdots, x_m, y_1, \cdots, y_m),$$

$$\underset{\mathbb{O}}{Max} \left| \frac{\partial H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{\partial z_i} \dot{z}_i \right| = c_i, \;\; \underset{\mathbb{O}}{Max} \left| \frac{\partial H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{\partial t} \right| = d_i$$

$$\underset{\mathbb{O}}{Min} \left| \frac{\partial H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{\partial \dot{z}_i} \right| = e_i$$

and let $\phi_i(\mathbf{p})$=$(p_1, \ldots, p^*_i, \ldots, p_m)$, where p^*_i minimizes the functional $F^i_{\mathbf{p}}$ and, hence, $\phi_i(\mathbf{p})$ verifies Proposition 1. Thus, we have that:

- If $\dot{p}^*_i$ is a point of discontinuity of $\left(L^i_{\mathbf{p}}\right)_{\dot{z}_i}$ then, by hypothesis, $\dot{p}^*_i(t) = 0$, from which $\ddot{p}^*_i(t) = 0$.
- If $H_{i\,\min} < H_i(t, p^*_i(t), \dot{p}^*_i(t)) < H_{i\,\max}$, p^*_i satisfies the Euler equation of the functional $F^i_{\mathbf{p}}$, then

  $$\ddot{p}^*_i(t) = f_i(t, p_1(t), \cdots, p^*_i(t), \cdots, p_m(t), \dot{p}_1(t), \cdots, \dot{p}^*_i(t), \cdots, \dot{p}_m(t)) \leq C_i$$

- If $H_i(t, p^*_i(t), \dot{p}^*_i(t)) = H_{i\,\min}$ or $H_i(t, p^*_i(t), \dot{p}^*_i(t)) = H_{i\,\max}$, calculating the total derivative with respect to t,

  $$\frac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial t} + \frac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial z_i} \dot{p}^*_i(t) + \frac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial \dot{z}_i} \ddot{p}^*_i(t) = 0$$

  from which

  $$\ddot{p}^*_i(t) = \frac{-\dfrac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial z_i} \dot{p}^*_i(t) - \dfrac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial t}}{\dfrac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial \dot{z}_i}} \tag{2}$$

Thus,

$$|\ddot{p}^*_i(t)| \leq \frac{\left| \dfrac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial z_i} \dot{p}^*_i(t) \right| + \left| \dfrac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial t} \right|}{\left| \dfrac{\partial H_i(t, \mathbf{p}^*_i(t), \dot{\mathbf{p}}^*_i(t))}{\partial \dot{z}_i} \right|} \leq \frac{c_i + d_i}{e_i} = \aleph_i$$

and, in short,

$$|\ddot{p}^*_i(t)| \leq B_i := Max\{C_i, \aleph_i\}$$

Let $M := \max_{i=1,\ldots,m} \{B_i\}$. From all the above, it is clear that $||\{\Phi(\ddot{\mathbf{p}})\}||_\infty \leq M$.

(i) If $p = (p_1, \ldots, p_m) \in U_M \Longrightarrow \Phi_i(p) = (p_1, \cdots, p_i^*, \cdots, p_m)$, where p_i^* is $\widehat{C}^2$, since this either satisfies the Euler equation of the functional $F_{\mathbf{q}}^i$ or it satisfies (2) or $\ddot{p}_i^*(t) = 0$. Hence, $\phi_i(p) \in U_M$ $\forall i = 1, \cdots, m$ and, obviously, $\Phi(p) \in U_M$.

(ii) Any sequence $\{\mathbf{p}_n\}_{n\in\mathbb{N}} \subset \mathbb{U}_M$ is uniformly bounded ($||\mathbf{p}_n||_\infty < \max\{N_i\}$) and so are all the sequences $\{\dot{\mathbf{p}}_n\}_{n\in\mathbb{N}}$ ($||\dot{\mathbf{p}}_n||_\infty < \max\{M_i\}$) and $\ddot{\mathbf{p}}_n\}_{n\in\mathbb{N}}$ ($||]\ddot{\mathbf{p}}_n||_\infty < M$).
Thus, in virtue of Lemma 2, $\{\mathbf{p}_n\}_{n\in\mathbb{N}}$ and $\{\dot{\mathbf{p}}_n\}_{n\in\mathbb{N}}$ are equicontinuous. We are therefore in a situation to use Arzela–Ascoli's Theorem, which guarantees that there exists a subsequence $\{\mathbf{p}_{n_k}\}_{k\in\mathbb{N}}$ that converges uniformly to a certain $\mathbf{p} \in \Theta$. What's more, in virtue of Lemma 4, $\{\dot{\mathbf{p}}_{n_k}\}_{k\in\mathbb{N}}$ also converges uniformly to $\dot{\mathbf{p}}$ and, in short, $\{\mathbf{p}_{n_k}\}_{k\in\mathbb{N}}$ converges in the topological space $(\Theta, || \,||^*)$.

(iii) Let the sequence $\{\mathbf{p}_n\}_{n\in\mathbb{N}}$ be convergent in $(\Theta, || \,||^*)$, i.e. $\{\mathbf{p}_n\}_{n\in\mathbb{N}}$ converges uniformly to $\mathbf{p}$ and $\{\dot{\mathbf{p}}_n\}_{n\in\mathbb{N}}$ converges uniformly to $\dot{\mathbf{p}}$.
As for each $n \in \mathbb{N}$, $\mathbf{p}_n \in \left(C^1[0, T]\right)^m$, in virtue of Proposition 1, $\Phi_i\left(\mathbf{p}_n\right) \in \left(C^1[0, T]\right)^m$. Thus, it may be deduced that $\{\Phi_i\left(\mathbf{p}_n\right)\}_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous in Θ.
The reiterative application of this reasoning guarantees that $\{\Phi\left(\mathbf{p}_n\right)\}_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous in Θ.
Furthermore, as $\{\Phi_i(\dot{\mathbf{p}}_n)\}_{n\in\mathbb{N}} \in \left(C^0[0, T]\right)^m$, simply by taking the maximum of the bounds for each of these, $\{\Phi_i(\dot{\mathbf{p}}_n)\}_{n\in\mathbb{N}}$ is uniformly bounded. On the other hand, following the considerations set out at the beginning of this proof, each i-th component, $\ddot{p}_{n,i}^*$ of $\{\Phi_i(\ddot{\mathbf{p}}_n)\}_{n\in\mathbb{N}}$ is bounded. Hence $\{\Phi(\ddot{\mathbf{p}}_n)\}_{n\in\mathbb{N}}$ is uniformly bounded. Taking into account Lemma 2, we may state that $\{\Phi(\dot{\mathbf{p}}_n)\}_{n\in\mathbb{N}}$ is equicontinuous in Θ.
As $\{\Phi(\mathbf{q}_n)\}_{n\in\mathbb{N}}$ and $\{\Phi(\dot{\mathbf{q}}_n)\}_{n\in\mathbb{N}}$ are uniformly bounded and equicontinuous in Θ, Corollary 2 ensures that Φ is sequentially continuous.

(iv) If $\{\mathbf{p}_n\}_{n\in\mathbb{N}}$ converges to $\mathbf{p} \in \Theta$ with the topology $|| \,||^*$, making $\Gamma_{\mathbf{z}}(t) := L(t, \mathbf{z}(t), \mathbf{z}(t))$, we are in a situation to use Lemma 3, which guarantees that

$$\{\Gamma_{\mathbf{p}_n}\}_{n\in\mathbb{N}} \text{ converges uniformly to } \Gamma_{\mathbf{p}}$$

and, thus

$$\left\{F(\mathbf{p}_n) = \int_a^b \Gamma_{\mathbf{p}_n}(t)dt\right\}_{n\in\mathbb{N}} \text{ converges to } \int_a^b \Gamma_{\mathbf{p}}(t)dt = F(\mathbf{p})$$

□

Theorem 3 *For every $\mathbf{q}_0 \in \mathbb{U}_M$, the sequence generated by the algorithm $\{\mathbf{q}_n = \Phi(\mathbf{q}_{n-1})\}_{n\in\mathbb{N}}$ possesses a subsequence that converges in $(\Theta, || \,||^*)$ and the limit* *is a fixed point of* Φ. *Moreover, any convergent subsequence of* $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *will converge at a fixed point on* Φ.

Proof It suffices to prove that, in fact, the sequence $\{\mathbf{q}_n\}_{n\in\mathbb{N}} \subset \mathbb{U}_M$ possesses a subsequence that converges uniformly and, as we have guaranteed the verification of the hypothesis of Theorem 2, in virtue of Proposition 4, we may conclude that $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ possesses a subsequence that converges uniformly to a fixed point on Φ. By virtue of Theorem 2 itself, we have guaranteed that any convergent subsequence of $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ will converge to a fixed point on Φ.

We see that the sequence $\{\mathbf{q}_n\}_{n\in\mathbb{N}} \subset \mathbb{U}_M$ possesses a subsequence that converges uniformly.

For any $\mathbf{q}_0 \in \mathbb{U}_M$, from (i) in Proposition 4, we know that the sequence $\mathbf{q}_n = \Phi(\mathbf{q}_{n-1})$ is contained in $\mathbb{U}_M$ and following analogous reasoning to point (ii) in this same proposition, we may conclude that there exists a subsequence $\{\mathbf{q}_{n_k}\}_{k\in\mathbb{N}}$ that converges in Θ. □

6 Example

A program that solves the optimization problem was developed using the Mathematica package and was then applied to one example of a hydrothermal system made up of 8 thermal plants and 5 hydro-plants of variable head, two of which have pumping capacity. For the thermal plants, the cost function Ψ_i used is a quadratic model:

$$\Psi_i(x) = \alpha_i + \beta_i x + \gamma_i x^2$$

and we consider Kirchmayer's model for the transmission losses: $l_i(x) = b_{ii} \cdot x^2$, where b_{ii} is termed the loss coefficient. The data of the plants are summarized in Table 1. The units for the coefficients are: α_i in $(\$/h)$, β_i in $(\$/h.Mw)$, γ_i in $(\$/h.Mw^2)$, and the loss coefficients b_{ii} in $(1/Mw)$. We construct the equivalent thermal plant as we saw in [7], obtaining: $\alpha_{\text{eq}} = 10696.1$; $\beta_{\text{eq}} = 16.5477$; $\gamma_{\text{eq}} = 0.00329982$.

The hydro-network has the configuration shown in Fig. 1. Hydro-plants 1 and 2 are isolated and have pumping capacity. Hydro-plants 3, 4 and 5 are conventional hydro-plants and they are in the same river basin, so the rate

Table 1 Coefficients of the thermal plants

Plant i	α_i	β_i	γ_i	b_{ii}	$P_{i\,\min}$	$P_{i\,\max}$
1	2,248.16	−7.984	0.17026	0.000353	5	600
2	1,625.43	6.347	0.09803	0.000220	5	150
3	1,615.35	16.676	0.01659	0.000100	10	340
4	1,227.83	17.621	0.01325	0.000103	10	300
5	2,155.62	17.745	0.01982	0.000097	10	320
6	743.78	20.842	0.00211	0.000072	15	550
7	77.72	21.277	0.00286	0.000172	10	310
8	1,459.44	21.569	0.01489	0.000121	5	240

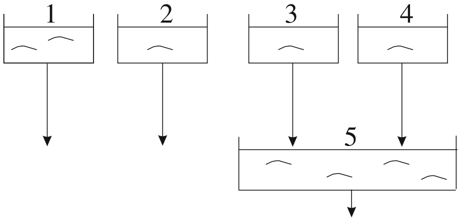

Fig. 1 The hydro-network

of discharge at the upstream plants affects the behaviour at the downstream plants: the hydraulic system has *hydraulic coupling*. We do not consider any time delay between plants.

We use a *variable head* model and the i-th hydro-plant's active power generation P_{hi} (for a conventional hydro-plant) is given by

$$P_{hi}(t, z_i(t), \dot{z}_i(t)) = A_i(t)\dot{z}_i(t) - B_i\dot{z}_i(t)[z_i(t) - Coup_i(t)]; \quad \dot{z}_i(t) \geq 0,$$

where $A_i(t)$ and B_i are the coefficients: $A_i(t) = \frac{1}{G_i} B_{y_i}(S_{0i} + t \cdot i_i)$; $B_i = \frac{B_{y_i}}{G_i}$, and $Coup_i(t)$ represents the hydraulic coupling between plants. In variable-head models, the term $-B_i\dot{z}_i(t)[z_i(t) - Coup_i(t)]$ represents the negative influence of the consumed volume and reflects the fact that consuming water lowers the effective height and hence the performance of the plant. The plants in our system verify:

$$Coup_i(t) = 0, \ i = 1, 2, 3, 4; \ Coup_5(t) = z_3(t) + z_4(t)$$

For the pumped-storage plants, P_{hi} is defined piecewise, taking in the pumping zone:

$$P_{hi}(t, z_i(t), \dot{z}_i(t)) = M_i \cdot \left[A_i(t)\dot{z}_i(t) - B_i\dot{z}_i(t)z_i(t)\right]; \quad \dot{z}_i(t) < 0,$$

where M is the efficiency of the hydroplant in the pumping zone. The parameters that appear in the formula are: the efficiency G in $(m^4/h.Mw)$, the constraint on the volume b in $(10^4 m^3)$, the loss coefficient b_{ll} in $(1/Mw)$, the natural inflow i in $(10^6 m^3/h)$, the initial volume S_0 in $(10^9 m^3)$, and the coefficient B_y (a parameter that depends on the geometry of the reservoir) in $(10^{-12} m^{-2})$. The data of the hydro-plants is summarized in Table 2. We consider that the transmission losses for the hydro-plants are also expressed

Table 2 Hydro-plant coefficients

Plant i	G_i	b_i	b_{lli}	i_i	S_{0i}	B_{yi}	M_i	$H_{i\,\min}$	$H_{i\,\max}$
1	534,660	141.6	0.000180	108.176	193.885	150.1	1.033	−100	115
2	536,315	135.2	0.000195	98.176	203.904	149.5	1.030	−100	100
3	520,834	2,130.5	0.000160	30.952	197.808	138.7	–	0	105
4	544,800	3,510.0	0.000200	30.118	224.234	140.9	–	0	120
5	547,770	4,950.8	0.000210	298.204	283.904	156.1	–	0	205

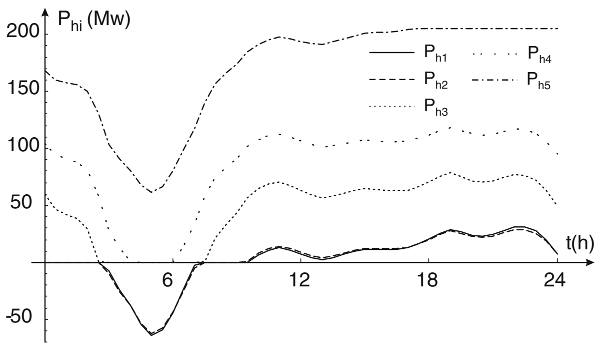

Fig. 2 Optimal hydro-power

by Kirchmayer's model (where b_{ll} is the loss coefficient). Hence, the function of effective hydraulic generation is

$$H_i(t, z_i(t), \dot{z}_i(t)) := P_{hi}(t, z_i(t), \dot{z}_i(t)) - b_{lli} P_{hi}^2(t, z_i(t), \dot{z}_i(t)), \ \forall \dot{z}_i(t)$$

This model verifies: $\partial H_i/\partial \dot{z}_i > 0$, $\left[\partial H_i/\partial z_i\right]_{\dot{z}_i=0} = 0$ and $\partial^2 H_i/\partial \dot{z}_i^2 < 0$.

We consider short-term hydrothermal scheduling (24 h) with an optimization interval $[0, 24]$ and we consider a discretization of 48 subintervals. The optimal power for the hydro-plants is shown in Fig. 2, and the optimal power for the thermal plants in Fig. 3.

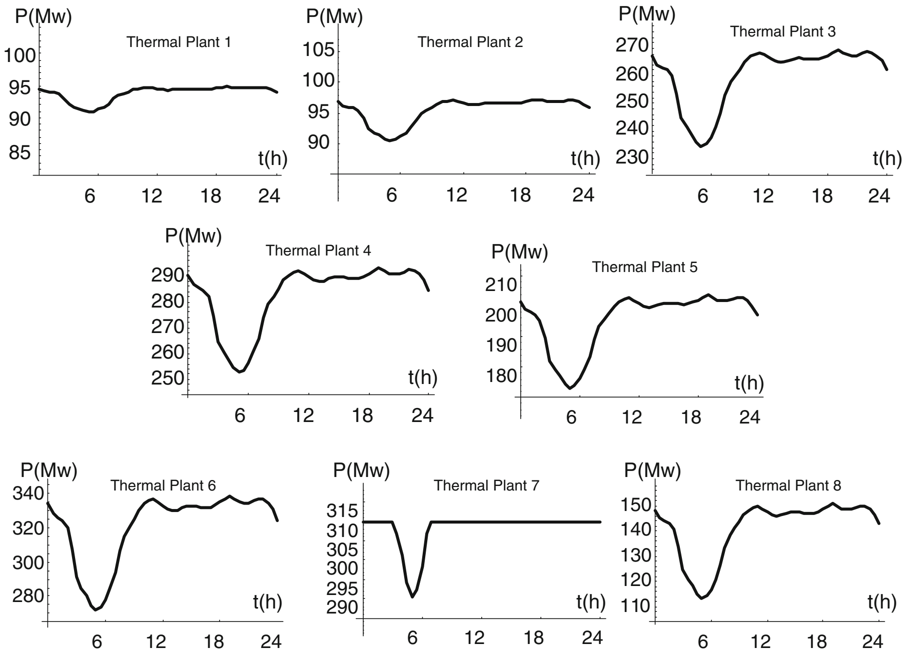

Fig. 3 Optimal thermal power

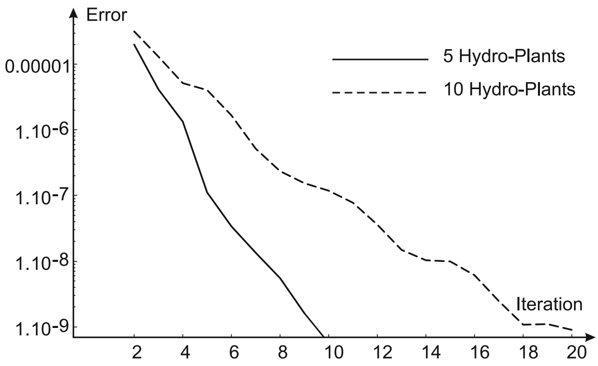

Fig. 4 Convergence with 5 and 10 hydro-plants, with losses in the pumping zone

The vector $\mathbf{C}^n = (C_1, \ldots, C_m)$ was considered as the stopping criterion for the algorithm in each iteration, the components of which are the coordination constants associated with the different hydro-plants, the tolerance being defined as

$$Tol(n) = \left\| \mathbf{C}^n - \mathbf{C}^{n-1} \right\|$$

For our example, for the case of the 5 hydro-plants, the tolerance was less than 10^{-9} in 10 iterations, the time required by the program being 194 s on a personal computer (Pentium IV/2GHz). Figure 4 presents the obtained results. We can see how the method exhibits rapid convergence.

To verify this statement, another test was conducted considering the same 8 thermal plants from the above example and 10 hydro-plants, 5 from the above example and another 5 identical to these, in terms of both the model and configuration of the river basins. It is evident (assuming conditions that guarantee the uniqueness of the solution) that, in this second example, we should obtain the same solution for identical hydro-plants, such as, for instance, 1 and 6, or 2 and 7, etc. In this case the method requires twice the number of iterations (from 10 to 20) to achieve the established tolerance. This makes the

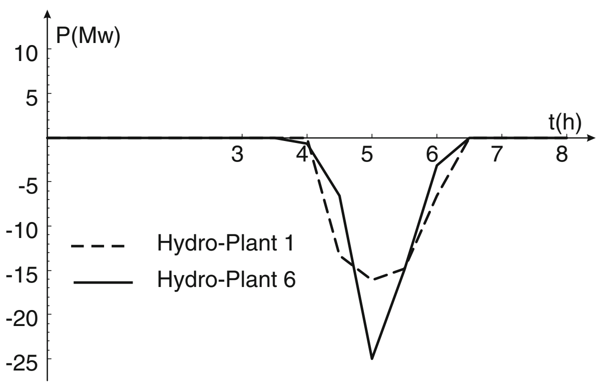

Fig. 5 Absence of the uniqueness of solution

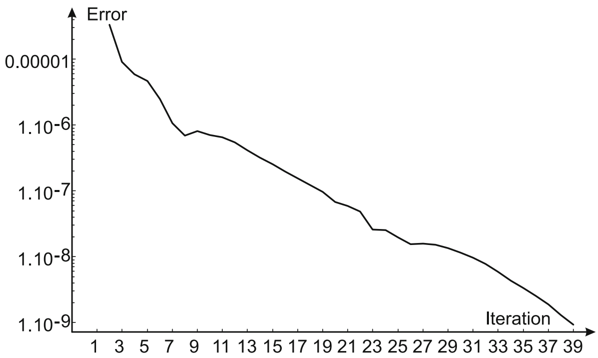

Fig. 6 Convergence with 10 hydro-plants, without losses in the pumping zone

method an ideal tool for working with large-scale systems. Figure 4 presents the obtained results.

In the above hydraulic models, fulfilment of the condition

$$\partial^2 H_i / \partial \dot{z}_i^2 < 0$$

is fundamental to solve the problem. We conducted a test with the same example of 10 hydro-plants as above, but with a Kirchmayer model that considers that there are no losses for the pumped-storage plants in the zone in which $\dot{z}_i(t) < 0$. The condition $\partial^2 H_i / \partial \dot{z}_i^2 < 0$ is no longer verified in this zone, and the uniqueness of the solution is lost for the pumped-storage plants, although the cost of both solutions is identical. Figure 5 shows the different solutions that are obtained for plants 1 and 6 in the pumping zone. This means that the convergence is much slower. The method requires 39 iterations to achieve the established tolerance (Fig. 6).

7 Conclusions

In this paper we describe a numerical relaxation method for computing the solution for a hydrothermal optimization problem that simultaneaously considers non-regular Lagrangian and non-holonomic inequality constraints. The problem can be naturally formulated in the framework of nonsmooth analysis. The proof of the convergence of the succession generated by the algorithm was based on the use of an appropriate adaptation of Zangwill's global theorem of convergence. Finally we ilustrate the performance of our work with a numerical example. The algorithm, based on coordinate descent, developed with the Mathematica package shows a rapid convergence to the optimal solution.

It would be most interesting that in future researc we can to apply this method to models in which the discontinuity of the Lagrangain is not produced at $z' = 0$, but rather to solutions of a differential equation of the form

$$z' = f(t, z)$$

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