The Exact Solution of the Environmental/Economic Dispatch Problem

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Abstract—**In this paper, the exact analytical solution for the environmental/economic dispatch (EED) optimization problem is presented for the first time. The EED, which simultaneously satisfies multiple contradictory criteria, is stated as a multiobjective optimization problem (MOP). Our paper has improved several aspects of a previous analytical approach. First, we take into account the unit capacity constraints in the exact formulae. Second, we obtain the set of compromise solutions known as Pareto optimal solutions and, third, our treatment of transmission losses satisfies the power balance constraint. In contrast with the known heuristic methods used in the literature, which only provide a reasonable solution (suboptimal, nearly global optimal), our method provides the global solution. Moreover, our method can obtain the Pareto optimal set under different loading conditions. The performance of the proposed technique is validated using a standard test system.**

Index Terms—**Environmental/economic dispatch, multiobjective optimization problem, Pareto optimal set, transmission losses.**

I. Introduction

POWER generation plants have traditionally been dispatched following minimum fuel cost criteria without considering the pollution produced. The basic objective of economic cost dispatch (ECD) is defined as finding an optimal distribution of system load to the generators in order to minimize the total generation cost while meeting all the system constraints. However, due to the ever increasing requirements of environmental regulations and social awareness, the use of these types of alternative strategies is becoming fundamental. Emission dispatch (ED) is similar to ECD, with emission being the objective to be minimized, instead of cost. The two functions are conflicting in nature and both have to be considered simultaneously to find the overall optimal dispatch. Environmental/economic dispatch (EED) is thus a multi-objective optimization problem (MOP) with non-commensurable and contradictory objectives.

Different techniques have been reported in the literature to solve the EED problem. In the first references [1] and [2], the EED problem was reduced to a single objective problem and the emission function was treated as a constraint. In [3], the EED problem was solved using linear programming. An ε-constraint method was proposed in [4] and [5]. This approach consists in optimizing the preferred objectives and treats the others as constraints. Recently, multi-objective evolutionary algorithms (MOEAs) [6] have been widely used. In many of their applications to solve the EED, [7]–[9] and [10] applied several algorithms to locate the Pareto optimal solutions, such as the niched Pareto genetic algorithm (NPGA), the non-dominated sorting genetic algorithm (NSGA) and the strength Pareto evolutionary algorithm (SPEA). Other techniques recently used to solve the EED problem are a particle swarm optimization (PSO) algorithm [11] and a differential evolution (DE) algorithm [12].

The solution of the EED problem using the different methods proposed in literature consumes considerable computing time. It should be noted that, in all the above references, the EED problem was addressed considering only one loading condition for a given system. So, its real-time exploitation is impossible when dealing with a load curve. Moreover, these heuristic methods do not always guarantee discovery of the globally optimal solution; they only provide a reasonable solution (suboptimal, nearly global optimal).

To date, only one analytical approach [13] has been developed to find the global solution to EED. In this paper we shall develop various improvements with respect to [13]. First of all, in our paper capacity constraints will be incorporated into the analytical solution. Second, we shall propose combining the classic iterative method to incorporate losses [14] with the analytic solution of our paper to obtain a solution that actually verifies the power balance equation. Third, for the treatment of the multiobjective problem, we shall present the set of compromise solutions known as Pareto optimal solutions. We shall see that our method can easily obtain the Pareto optimal set under different loading conditions.

The paper is organized as follows. In Section II, we state the mathematical models of our thermal system and show the multiobjective problem formulation using the Pareto optimal set. In Section III, we present the simple case when transmission losses are neglected. We describe the mathematical environment of our work: the *thermal equivalent* plant, and use it to obtain the exact analytical optimal solution. In Section IV, we present an iterative method that takes into account transmission losses. The results of the application of the method to a numerical example are presented in Section V, where the solution of our method is compared with the best results of [13]. Finally, Section VI summarizes the main conclusions of our research and presents a discussion to highlight the contribution compared to existing techniques.

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II. Formulation of the Multiobjective Problem

Let us see the different components of our problem.

A. Mathematical Models

The cost function of each generator is traditionally approximated by a single quadratic function

$$C_i(P_i) = a_i + b_i P_i + c_i P_i^2 \tag{1}$$

where $C_i(P_i)$ is the fuel cost ($/h), P_i is the power generated (MW), and a_i, b_i, c_i are the fuel cost coefficients of the ith unit, and we suppose $c_i > 0$.

The form of the emission function model depends on the emission type. For SO_2, it is generally acknowledged that emissions will be of the same form as that of the fuel cost function. The NO_x emission function is less straightforward to represent as it is highly nonlinear in P and highly dependent on the type of boiler. We focus on a particular type of boiler in this paper: the circulating fluidized bed combustion (CFBC) boiler. As an alternative to pulverized coal combustion (PCC) power plants, fluidized bed combustion (FBC) has emerged as a viable alternative, presenting significant advantages over the conventional firing system and offering multiple benefits. One of the main advantages of CFBC is the possibility of reducing the sulphur dioxide (SO_2) formed during combustion from the sulphur content of the fuel by adding a cheap absorbent material to the bed such as limestone $(CaCO_3)$. Increasing the ratio Ca/S sufficiently may, in theory, completely reduce the sulphur in the bed. This is the reason for not considering SO_2 emissions in our study.

As regards NO_x emissions, as the reactions involving thermal NO_x are only significant at high temperatures ($> 1200°$C), this extra NO_x is avoided in CFBC boilers. This allows us to use a second-order polynomial function (U-shaped) (see [15]) for the NO_x emission function

$$E_i(P_i) = d_i + e_i P_i + f_i P_i^2 \tag{2}$$

where $E_i(P_i)$ is the emission (kg/h), P_i is the power generated (MW), and d_i, e_i, f_i are the emission coefficients of the ith unit, and we suppose $f_i > 0$.

B. Environmental/Economic Dispatch

1) Problem Objectives: The EED problem, with N plants, is to minimize two competing objective functions: fuel cost and emission.

Minimization of Fuel Cost: The total ($/h) fuel cost $C(P)$ can be expressed as

$$C(P) = \sum_{i=1}^{N} C_i(P_i). \tag{3}$$

Minimization of NO_x Emission: The total (kg/h) emission can be expressed as

$$E(P) = \sum_{i=1}^{N} E_i(P_i). \tag{4}$$

2) Problem Constraints: The EED problem is subject to two constraints:

Power balance constraint: The total power generated must supply total load demand and transmission losses:

$$\sum_{i=1}^{N} P_i = P_D + P_L \tag{5}$$

where P_D is the total load demand (MW) and P_L is the total transmission losses (MW).

Unit capacity constraint: The power generated by each generator, P_i, is constrained between its minimum and maximum limits, i.e.,

$$P_i^{\min} \leq P_i \leq P_i^{\max} \tag{6}$$

where $P_i^{\min}$ and $P_i^{\max}$ are the minimum and maximum power outputs of the ith unit.

Aggregating the two conflicting objectives (3), (4) and the two constraints (5), (6), the EED problem can be mathematically formulated as follows:

$$\begin{aligned} \text{minimize}: \quad & [C(P), E(P)] \\ \text{subject to}: \quad & g(P) = 0 \\ & h(P) \leq 0 \end{aligned} \tag{7}$$

where g is the equality constraint representing the power balance and h is the inequality constraint representing the unit generation capacity. In general, the EED can be formulated either as an emissions constrained economic dispatch (ECED) or as a multiobjective optimization problem (MOP). In [13], the EED is modeled in a simple way using a price penalty factor ($/kg) to combine the two objectives. But actually there is no single optimal solution because the exact preference or "weight" of the two objectives is unknown. In the present paper we use a more accurate approach: the Pareto optimal solution.

C. Pareto Optimal Solution

A general MOP consists of a number of objectives to be optimized simultaneously and is associated with a number J of equality constraints and a number K of inequality constraints:

$$\begin{aligned} \text{minimize}: \quad & f_i(x), \quad i = 1, \ldots, N_{obj} \\ \text{subject to}: \quad & g_j(x) = 0, \quad j = 1, \ldots, J \\ & h_k(x) \leq 0, \quad k = 1, \ldots, K \end{aligned}$$

where f_i is the ith objective function, x represents a solution, and N_{obj} is the number of objectives. For a MOP, any two solutions x_1 and x_2 can have one of two possibilities: one dominates the other or neither dominates the other. In a minimization problem, a solution x_1 dominates x_2 if the following two conditions are satisfied:

$$\begin{aligned} &\forall i \in \{1, \ldots, N_{obj}\} : f_i(x_1) \leq f_i(x_2) \\ &\exists j \in \{1, \ldots, N_{obj}\} : f_j(x_1) < f_j(x_2). \end{aligned}$$

If any of the above conditions is violated, the solution x_1 does not dominate the solution x_2. If x_1 dominates the solution x_2, x_1

is called the nondominated solution within the set $\{x_1, x_2\}$. The solutions that are nondominated within the entire search space are denoted as Pareto optimal and constitute the Pareto optimal set.

Our MOP (7) thus gives rise to a set of optimal solutions instead of one optimal solution. The reason for the optimality of many solutions is that none can be considered to be better than any other with respect to all the objective functions. This set of compromise solutions is known as the Pareto optimal solutions set. When optimizing all objectives simultaneously, Pareto optimal solutions show the tradeoffs between conflicting objective functions. Methods of generating Pareto sets continue to be a topic of research: random sampling, weighting method, distance method, constrained trade-off method, normal boundary intersection method, goal programming, Pareto genetic algorithm, etc. The weighted sum method of generating Pareto sets was shown to work well with convex problems decades ago and while it is still a very popular method, some deficiencies have been noted. As we shall see, some of these deficiencies can be avoided with the combined use of our analytical technique. Thus, using the weighting method, our approach converts it into a single function optimization problem using the weighted sum of C_i and E_i

$$\begin{aligned} \text{minimize}: \quad & \delta \sum_{i=1}^{N} C_i(P_i) + (1-\delta) \sum_{i=1}^{N} E_i(P_i) \\ \text{subject to}: \quad & \sum_{i=1}^{N} P_i = P_D + P_L \\ & P_i^{\min} \le P_i \le P_i^{\max}, \quad \forall i = 1, \ldots, N \end{aligned} \tag{8}$$

where δ is a constant in the range of [0,1]. To obtain the Pareto optimal solutions set, multiple runs with different weights are required to create the tradeoff curve. The important aspect of this weighted sum method is that a set of noninferior solutions can be obtained by varying the weights. The traditional disadvantage of the weighted sum method, when traditional methods are used, is that it requires multiple runs. With our analytical method, multiple runs can be performed without any difficulty. As we have exact formulae for the optimal solution, each point on the Pareto front is obtained immediately. As we have chosen the weighting method in our case to populate the Pareto set, we need only assign values to the weighting functions. There is no need for iterative processes, the estimation of initial values or problems of convergence of any algorithm. Simply substitute in the given formulae.

It should also be noted that the weighting method is not suitable for achieving the whole Pareto set in some problems, since it can map only the solutions which belong to the boundary of the convex hull of the efficient solution set. Therefore, using the weighting method, the whole Pareto set can be achieved only in convex problems, such as ours (see the hypothesis introduced in models (1) and (2)). The whole Pareto set can be mapped in non-convex problems using an enhanced scalarization method, such as the ϵ-constrained approach [4] and [5]. In this method, one of the problem objectives is considered as the objective function and the other objectives are treated as constraints. The solutions of the Pareto set are obtained through variations in the constraint. One limitation of this approach is that this technique changes the structure of problem, adding new constraints to it, which can make the problem nonviable in terms of computational cost. A way of mapping the whole Pareto set in non-convex problems is the hybrid method [16], which uses a combination of the weighting and ϵ-constraint methods.

III. Problem Without Transmission Losses: Analytical Solution

In this section we first present the simple case when transmission losses are neglected. Our method is based on constructing the *equivalent thermal plant*: a single one that behaves equivalently to the entire set. This study constitutes the generalization of prior papers. We summarize the main results obtained by Bayon *et al.* in previous papers, which we consider necessary for a better understanding of the present paper.

In [17], we considered the case where the objective functions are second-order polynomials and we imposed the natural constraint of positivity of the thermal power:

$$\begin{aligned} \text{minimize}: \quad & \sum_{i=1}^{N} F_i(P_i) = \sum_{i=1}^{N} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \\ \text{subject to}: \quad & \sum_{i=1}^{N} P_i = P_D \\ & P_i \ge 0, \quad \forall i = 1, \ldots, N. \end{aligned} \tag{9}$$

Without loss of generality, we assumed that $\beta_1 \le \beta_2 \le \ldots \le \beta_N$. We denote by $\Psi(P_D)$ the minimum value of $\sum_{i=1}^{N} F_i(P_i)$ and by $(\Psi_1(P_D), \ldots, \Psi_N(P_D))$ the vector where said minimum value is reached. Following the nomenclature employed in [17], we shall call Ψ the *equivalent minimizer* of $\{F_i\}_{i=1}^{N}$ and each Ψ_i the *ith distribution function*. We proved that if $\delta_k \le P_D < \delta_{k+1}$, then the equivalent minimizer is a second-order polynomial with piece-wise constant coefficients:

$$\Psi(P_D) = \sum_{i=1}^{N} F_i\left(\Psi_i(P_D)\right) = \widetilde{\alpha}_k + \widetilde{\beta}_k P_D + \widetilde{\gamma}_k P_D^2$$

with the coefficients

$$\begin{aligned} \widetilde{\gamma}_k &= \frac{1}{\sum\limits_{i=1}^{k} \frac{1}{\gamma_i}}; \quad \widetilde{\beta}_k = \widetilde{\gamma}_k \sum_{i=1}^{k} \frac{\beta_i}{\gamma_i} \\ \widetilde{\alpha}_k &= \sum_{i=1}^{N} \alpha_i + \frac{\widetilde{\beta}_k^2}{4\widetilde{\gamma}_k} - \sum_{i=1}^{k} \frac{\beta_i^2}{4\gamma_i} \end{aligned}$$

and δ_k being

$$\delta_k = \frac{1}{2}\left[\beta_k \sum_{i=1}^{k} \frac{1}{\gamma_i} - \sum_{i=1}^{k} \frac{\beta_i}{\gamma_i}\right]$$

the value of the demanded power, P_D, below which the kth plant is kept at its minimum value: $P_k = 0$.

Moreover, we proved that for every $k=1,\ldots,N$, the kth distribution function $\Psi_k(P_D)$ (i.e., the optimal power) is

$$\begin{cases} (1)\ \text{If } P_D \le \delta_k : \Psi_k(P_D) = 0 \\ (2)\ \text{If } \delta_k \le \delta_j \le P_D < \delta_{j+1} : \Psi_k(P_D) \\ \quad = \dfrac{\sum_{i=1}^{j} \frac{\beta_i}{2\gamma_i} + P_D}{\gamma_k \sum_{i=1}^{j} \frac{1}{\gamma_i}} - \frac{\beta_k}{2\gamma_k}. \end{cases}$$

In [18], we generalized the previous paper and calculated the equivalent minimizer in the case where the cost functions are a general (non-quadratic) model F_i and the inequality constraint is $P_i \ge P_i^{\min}$. We assumed $\{F_i\}_{i=1}^N \subset C^1[0,\infty)$, F_i' strictly increasing, and with $F_i'(P_i^{\min}) \le F_{i+1}'(P_{i+1}^{\min})$, $(i=1,\ldots,N-1)$. The problem considered was

$$\begin{aligned} \text{minimize}: &\quad \sum_{i=1}^{N} F_i(P_i) \\ \text{subject to}: &\quad \sum_{i=1}^{N} P_i = P_D \\ &\quad P_i \ge P_i^{\min}, \quad \forall i = 1,\ldots,N. \end{aligned} \tag{10}$$

We proved that for every $k=1,\ldots,N$, the kth distribution function $\Psi_k(P_D)$ is

$$\begin{cases} (1)\ \text{If } P_D \le \delta_k : \Psi_k(P_D) = P_k^{\min} \\ (2)\ \text{If } \delta_k \le \delta_j \le P_D < \delta_{j+1} : \Psi_k(P_D) \\ \quad = \left(\sum_{i=1}^{j} F_i'^{-1} \circ F_k'\right)^{-1} \left(P_D - \sum_{i=j+1}^{N} P_i^{\min}\right) \end{cases}$$

with

$$\delta_k = \sum_{i=1}^{k} \left(F_i'^{-1} \circ F_k'\right)\left(P_k^{\min}\right) + \sum_{i=k+1}^{N} P_i^{\min}$$

the value of the demanded power, P_D, below which the kth plant is kept at its minimum value, $P_k^{\min}$.

We shall now generalize the particular cases (9) and (10) to our more general MOP (8), where

$$F_i(P_i) = \delta C_i + (1-\delta)E_i = \alpha_i + \beta_i P_i + \gamma_i P_i^2 \tag{11}$$

and where we simultaneously have minimum and maximum constraints

$$P_i^{\min} \le P_i \le P_i^{\max}.$$

We assume $F_i'(P_i^{\min}) < F_j'(P_j^{\max}), \forall i,j$. Following the same technique as in [18], we obtain the values δ_k

$$\delta_k = \sum_{i=1}^{k} \left(F_i'^{-1} \circ F_k'\right)\left(P_k^{\min}\right) + \sum_{i=k+1}^{N} P_i^{\min} \tag{12}$$

and new values, which we shall call θ_k

$$\theta_k = \sum_{i=k}^{N} \left(F_{\sigma(i)}'^{-1} \circ F_{\sigma(k)}'\right)\left(P_{\sigma(k)}^{\max}\right) + \sum_{i=1}^{k-1} P_{\sigma(i)}^{\max} \tag{13}$$

which represent the value of the demanded power, P_D, above which the kth plant is kept at its maximum value, $P_k^{\max}$. The order of the plants is given by the permutation $\sigma \in \Sigma_N$ such that

$$F_{\sigma(i)}'\left(P_{\sigma(i)}^{\max}\right) \le F_{\sigma(i+1)}'\left(P_{\sigma(i+1)}^{\max}\right), \quad \forall i = 1,\ldots,N-1.$$

It is easy to prove that for every $k=1,\ldots,N$, the kth distribution function $\Psi_k(P_D)$ (i.e., the optimal power) is

$$\begin{cases} (1)\ \text{If } P_D \le \delta_k : \Psi_k(P_D) = P_k^{\min} \\ (2)\ \text{If } \delta_k \le \delta_j \le P_D < \delta_{j+1} : \Psi_k(P_D) \\ \quad = \left(\sum_{i=1}^{j} F_i'^{-1} \circ F_k'\right)^{-1} \left(P_D - \sum_{i=j+1}^{N} P_i^{\min}\right) \\ (3)\ \text{If } \delta_N \le P_D < \theta_1 : \Psi_k(P_D) \\ \quad = \left(\sum_{i=1}^{N} F_i'^{-1} \circ F_k'\right)^{-1} (P_D) \\ (4)\ \text{If } \theta_j \le P_D < \theta_{j+1} : \Psi_k(P_D) \\ \quad = \left(\sum_{i=j+1}^{N} F_{\sigma(i)}'^{-1} \circ F_k'\right)^{-1} \left(P_D - \sum_{i=1}^{j} P_{\sigma(i)}^{\max}\right) \\ (5)\ \text{If } \theta_{\sigma^{-1}(k)} \le P_D : \Psi_k(P_D) = P_k^{\max}. \end{cases} \tag{14}$$

Applying (12), (13) and (14) to the functional (11) and expressing it in terms of the coefficients of F_i, we have that the kth distribution function $\Psi_k(P_D)$ (i.e., the exact optimal power) is

$$\begin{cases} (1)\ \text{If } P_D \le \delta_k : \Psi_k(P_D) = P_k^{\min} \\ (2)\ \text{If } \delta_k \le \delta_j \le P_D < \delta_{j+1} : \Psi_k(P_D) \\ \quad = \dfrac{\left(P_D - \sum_{i=j+1}^{N} P_i^{\min} + \sum_{i=1}^{j} \frac{\beta_i}{2\gamma_i}\right)}{\gamma_k \sum_{i=1}^{j} \frac{1}{\gamma_i}} - \frac{\beta_k}{2\gamma_k} \\ (3)\ \text{If } \delta_N \le P_D < \theta_1 : \Psi_k(P_D) \\ \quad = \dfrac{\left(P_D + \sum_{i=1}^{N} \frac{\beta_i}{2\gamma_i}\right)}{\gamma_k \sum_{i=1}^{N} \frac{1}{\gamma_i}} - \frac{\beta_k}{2\gamma_k} \\ (4)\ \text{If } \theta_j \le P_D < \theta_{j+1} : \Psi_k(P_D) \\ \quad = \dfrac{\left(P_D - \sum_{i=1}^{j} P_{\sigma(i)}^{\max} + \sum_{i=j+1}^{N} \frac{\beta_{\sigma(i)}}{2\gamma_{\sigma(i)}}\right)}{\gamma_k \sum_{i=j+1}^{N} \frac{1}{\gamma_{\sigma(i)}}} - \frac{\beta_k}{2\gamma_k} \\ (5)\ \text{If } \theta_{\sigma^{-1}(k)} \le P_D : \Psi_k(P_D) = P_k^{\max} \end{cases} \tag{15}$$

with

$$\delta_k = \frac{1}{2} \sum_{i=1}^{k} \frac{\beta_k + 2\gamma_k P_k^{\min} - \beta_i}{\gamma_i} + \sum_{i=k+1}^{N} P_i^{\min}$$

$$\theta_k = \frac{1}{2} \sum_{i=k}^{N} \frac{\beta_{\sigma(k)} + 2\gamma_{\sigma(k)} P_{\sigma(k)}^{\max} - \beta_{\sigma(i)}}{\gamma_{\sigma(i)}} + \sum_{i=1}^{k-1} P_{\sigma(i)}^{\max}.$$

Moreover, the function $\Psi(P_D)$ (equivalent minimizer) is

$$\begin{cases} \text{(i) If } \delta_k \le \delta_j \le P_D < \delta_{j+1} : \Psi(P_D) \\ \quad = \widetilde{\alpha}_k + \widetilde{\beta}_k \left(P_D - \sum_{i=k+1}^{N} P_i^{\min}\right) \\ \qquad + \widetilde{\gamma}_k \left(P_D - \sum_{i=k+1}^{N} P_i^{\min}\right)^2 \\ \text{(ii) If } \delta_N \le P_D < \theta_1 : \Psi(P_D) \\ \quad = \widetilde{\alpha}_N + \widetilde{\beta}_N P_D + \widetilde{\gamma}_N P_D^2 \\ \text{(iii) If } \theta_k \le P_D < \theta_{k+1} : \Psi(P_D) \\ \quad = \widehat{\alpha}_k + \widehat{\beta}_k \left(P_D - \sum_{i=1}^{k} P_{\sigma(i)}^{\max}\right) \\ \qquad + \widehat{\gamma}_k \left(P_D - \sum_{i=1}^{k} P_{\sigma(i)}^{\max}\right)^2 \end{cases}$$

with the coefficients

$$\widetilde{\gamma}_k = \frac{1}{\sum_{i=1}^{k} \frac{1}{\gamma_i}}; \quad \widetilde{\beta}_k = \widetilde{\gamma}_k \sum_{i=1}^{k} \frac{\beta_i}{\gamma_i}$$
$$\widetilde{\alpha}_k = \sum_{i=1}^{N} \alpha_i + \frac{\widetilde{\beta}_k^2}{4\widetilde{\gamma}_k} - \sum_{i=1}^{k} \frac{\beta_i^2}{4\gamma_i} + \sum_{i=k+1}^{N} \left(\beta_i P_i^{\min} + \gamma_i P_i^{\min\,2}\right)$$

and

$$\widehat{\gamma}_k = \frac{1}{\sum_{i=k+1}^{N} \frac{1}{\gamma_{\sigma(i)}}}; \quad \widehat{\beta}_k = \widehat{\gamma}_k \sum_{i=k+1}^{N} \frac{\beta_{\sigma(i)}}{\gamma_{\sigma(i)}}$$
$$\widehat{\alpha}_k = \sum_{i=1}^{N} \alpha_i + \frac{\widehat{\beta}_k^2}{4\widehat{\gamma}_k} - \sum_{i=k+1}^{N} \frac{\beta_{\sigma(i)}^2}{4\gamma_{\sigma(i)}} + \sum_{i=1}^{k} \left(\beta_{\sigma(i)} P_{\sigma(i)}^{\max} + \gamma_{\sigma(i)} P_{\sigma(i)}^{\max\,2}\right).$$

It is evident that our formulae provide a very notable improvement with respect to those obtained in [13], as they take into account the inequality constraints. So, in this paper we have obtained the exact formulae corresponding to the MOP without losses.

The construction of all the parameters involved in the aforementioned formulas, which defines the piecewise equivalent minimizer, can be carried out in linear $O(n)$ or quasi-linear $O(n \log n)$ time if we take into account the time spent in calculating the sigma permutation. Once all the aforementioned parameters have been constructed, the calculation of the optimal cost to produce a given level of power is of complexity $O(\log_2 n)$, since the only calculation that requires a significant period of time, depending on the size of the problem, is the determination of the interval in which P_D is found (of the $2n-1$ possible polynomials). This is achieved with less than $\log_2(2n)$ operations using a very simple bisection strategy.

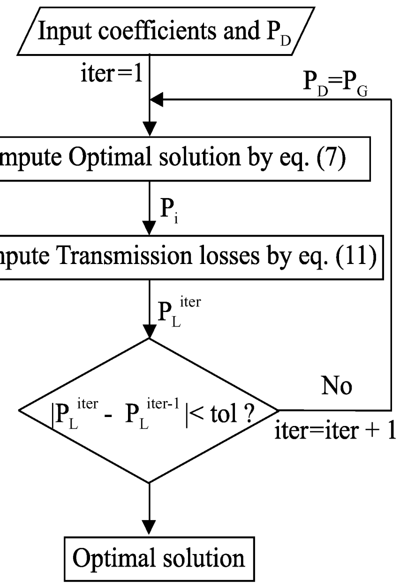

Fig. 1. Computational flow of the proposed iterative method.

IV. Problem With Transmission Losses: Iterative Method

To incorporate the transmission losses in the problem, an approximate technique is used in [13] that consists in representing the transmission loss in terms of demand. To do so, the aforementioned authors calculate an initial distribution of optimal power values (neglecting transmission losses) and use said power values to calculate the losses, subsequently incorporating the losses in the demand. A new solution is obtained with this new demand; it being evident that the solution thus obtained produces different losses to those initially estimated. The solution is analytical, but not exact. One of the advantages of our method is its versatility. So, in this section we shall combine the well known iterative method to incorporate transmission losses [14] with the exact formulae obtained in the previous section to achieve an iterative method that obtains a solution that actually verifies the power balance equation (5). The process (see Fig. 1) is similar to the method presented in [14], replacing the phase of solving the linear system of optimal conditions and the subsequent phase of meeting the constraints for our analytic solution (15). We represent the transmission losses by means of the classic B-coefficients

$$P_L = \sum_{i=1}^{N} \sum_{j=1}^{N} P_i B_{ij} P_j \tag{16}$$

where B_{ij} represents the transmission loss coefficients, P_i the generation of unit i (MW), and P_j the generation of unit j (MW). Since P_i and P_j are given by (15), substituting P_i and P_j in (16) and simplifying, we obtain an expression for transmission losses, P_L, in terms of demand, P_D

$$P_L = A + BP_D + CP_D^2$$

TABLE I
FUEL COST AND EMISSION COEFFICIENTS AND GENERATING CAPACITY CONSTRAINTS

Plant	Unit	a_i	b_i	c_i
1	G1	756.79886	38.53973	0.15247
	G2	451.32513	46.15916	0.10587
	G3	1049.32513	40.39655	0.02803
2	G4	1243.5311	38.30553	0.03546
	G5	1658.5696	36.32782	0.02111
3	G6	1356.65920	38.27041	0.01799

Plant	Unit	d_i	e_i	f_i
1	G1	13.85932	0.32767	0.00419
	G2	13.85932	0.32767	0.00419
	G3	40.2669	-0.54551	0.00683
2	G4	40.2669	-0.54551	0.00683
	G5	42.89553	-0.51116	0.00461
3	G6	42.89553	-0.51116	0.00461

Plant	Unit	$P_i^{\min}$	$P_i^{\max}$
1	G1	10	125
	G2	10	150
	G3	40	250
2	G4	35	210
	G5	130	325
3	G6	125	315

TABLE II
TRANSMISSION LOSS COEFFICIENTS

B_{ij}	1	2	3
1	0.000091	0.000031	0.000029
2	0.000031	0.000062	0.000028
3	0.000029	0.000028	0.000072

the coefficients A, B and C being known. The total power generated, P_G, must be equal to total load demand and transmission losses, so

$$P_G = \sum_{i=1}^{N} P_i = P_D + P_L$$
$$= A + (B+1)P_D + CP_D^2.$$

Introducing P_G as a new demand in (15), we obtain a new solution. Now, however, instead of accepting this solution, we iterate the process until the variation in losses from one iteration to the next is lower than the chosen tolerance.

In the following section, we shall show the rapid convergence of this algorithm on the chosen test example.

V. EXAMPLE: COMPARISON OF RESULTS

In this section the proposed method is applied to a test system. To compare with [13], the same realistic Indian system is presented here. The system (which is around 25 years old) has three plants and six generating units. The fuel cost and emission coefficients and the capacity constraints of the generating units are given in Table I, while Table II shows the transmission loss coefficients. The optimal solution was calculated on a personal computer (Pentium IV, 3.4-GHz PC) using the commercial program Mathematica 5.0.

TABLE III
ECONOMIC DISPATCH (WITHOUT TRANSMISSION LOSSES) $P_D = 900$ MW

$P_D = 900$ MW

Item		Our Method	[13] Method
Unit generation (MW)	1	32.497	32.50
	2	10.816	10.82
	3	143.646	143.64
	4	143.032	143.03
	5	287.104	287.10
	6	282.905	282.90
Total fuel cost ($/h)		45463.492	45463.42
Total emission (kg/h)		795.019	795.00
CPU time (ms)		31	168

TABLE IV
ECONOMIC DISPATCH (WITHOUT TRANSMISSION LOSSES) $P_D = 1170$ MW

$P_D = 1170$ MW

Item		Our Method
Unit generation (MW)	1	49.381
	2	35.132
	3	235.487
	4	210.000
	5	325.000
	6	315.000
Total fuel cost ($/h)		59095.180
Total emission (kg/h)		1291.278
CPU time (ms)		47

A. *Economic Dispatch Without Transmission Losses*

We shall commence by comparing the exact solution of our method: that is the formulae (15), with the best results of the method in [13]. The results are summarized in Table III. As was to be expected, the two solutions are identical (except for the discrepancies produced by rounding off), since none of the units reaches its technical minimum or maximum with the demanded power (free solution). Where our method shows its true value is in problems with a load curve when unit capacity constraints come into effect. No additional method is needed to impose the constraints, while the exact formulae provide us with the solution directly without the need to perform any iterative procedure. Table IV shows the solution for $P_D = 1170$ MW. As can be seen, units 3, 4 and 5 reach their maximum constraints, while units 1, 2 and 3 are in the free zone.

B. *Economic Dispatch With Transmission Losses*

We shall now consider transmission losses. To obtain a solution that verifies the power balance constraint

$$\sum_{i=1}^{N} P_i = P_D + P_L$$

TABLE V
ECONOMIC DISPATCH (WITH TRANSMISSION LOSSES) $P_D = 900$ MW

$P_D = 900$ MW

Item		Our Method	[13] Method
Unit generation (MW)	1	33.872	33.77
	2	12.797	12.65
	3	151.128	150.56
	4	148.946	148.50
	5	297.038	296.29
	6	294.563	293.68
Total fuel cost ($/h)		47329.308	47188.29
Total emission (kg/h)		862.997	857.74
Transmission loss (MW)		38.3448	35.45
CPU time (ms)		157	189

TABLE VI
TRANSMISSION LOSSES

Iteration	P_L (MW)
1	35.2825
2	38.0955
3	38.3245
4	38.3431
5	38.3447
6	38.3448

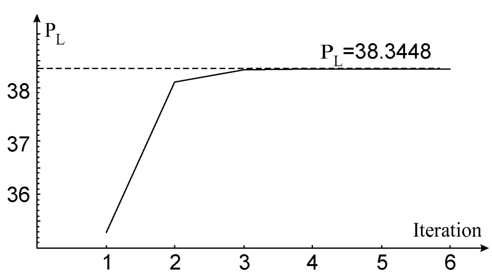

Fig. 2. Convergence of the iterative method.

we apply the iterative method presented in the previous section. In Table V we compare our solution with the one obtained in [13]. As can be seen, the discrepancies are now significant, sometimes even exceeding 1 MW, as in the case of unit 6. The reader will be able to appreciate that our solution verifies the power balance equation, whereas the solution in [13] does not. Even so, our CPU time is lower. The algorithm converges rapidly (see Fig. 2). Only 6 iterations are needed to reach the desired tolerance. Losses in two successive iteration should be less than tolerance which in this case was selected to be 10^{-3}:

$$\left|P_L^{iter} - P_L^{iter-1}\right| < tol = 10^{-3}.$$

Despite carrying out 6 iterations, it can be seen that the CPU time is very low. Table VI shows the variation in transmission losses with the iterations.

Finally, we compare the test presented in Table VI for $P_D = 1170$ MW, an example with technical minimum and maximum constraints, but without losses, with the solution obtained when losses are incorporated. Table VII shows the optimal solution. As can be seen, when incorporating transmission losses, unit 3

TABLE VII
ECONOMIC DISPATCH (WITH TRANSMISSION LOSSES) $P_D = 1170$ MW

$P_D = 1170$ MW

Item		Our Method
Unit generation (MW)	1	71.294
	2	66.689
	3	250.000
	4	210.000
	5	325.000
	6	315.000
Total fuel cost ($/h)		62923.514
Total emission (kg/h)		1373.550
CPU time (ms)		162

TABLE VIII
TRANSMISSION LOSSES

Iteration	P_L (MW)
1	60.1067
2	67.0280
3	67.8672
4	67.9696
5	67.9821
6	67.9836
7	67.9838

reaches its maximum constraint, besides the logical increase in units 1 and 2. Once again, the algorithm converges rapidly (162 ms), employing 7 iterations to reach the same tolerance as the previous example. Table VIII shows the variation in transmission losses with the iterations.

C. EED Dispatch

In this section the formulae (15) are used to tackle the EED multi-objective problem by adopting the weighting method. Note that each given weight provides a single solution in the Pareto optimal set. Fig. 3 shows the emission-cost tradeoff curve for a load of $P_D = 900$ MW and the system is considered lossless. Table IX presents the results of the EED. The solution presented in [13] is only one point of our Pareto optimal set. With our analytical method, the multiple runs that are the basis of the weighted sum method can be performed without any difficulty. The above solution presents the variation of δ from 0 to 1, in intervals of 0.05 (i.e., 20 solutions) with an execution time of only 140 ms. As can be seen in Fig. 3, for an even spread of the weights, the optimal solutions in the criterion space are usually not evenly distributed. This is one of the well-known drawbacks of the weighted sum method to populate the Pareto set. Fortunately, this drawback can be mitigated in our problem using the well-known technique [12] that introduces a scale conversion factor. This new approach makes it a single function optimization problem using the weighted sum of C_i and E_i

$$\text{minimize}: \quad \delta \sum_{i=1}^{N} C_i(P_i) + (1-\delta)\kappa_i \sum_{i=1}^{N} E_i(P_i) \qquad (17)$$

where δ is a weight selected between [0,1] and κ_i is a conversion factor that scales emissions into monetary units. Using the same

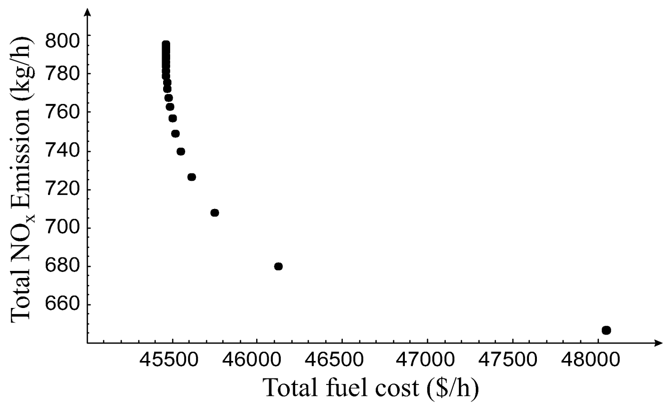

Fig. 3. Pareto-optimal front.

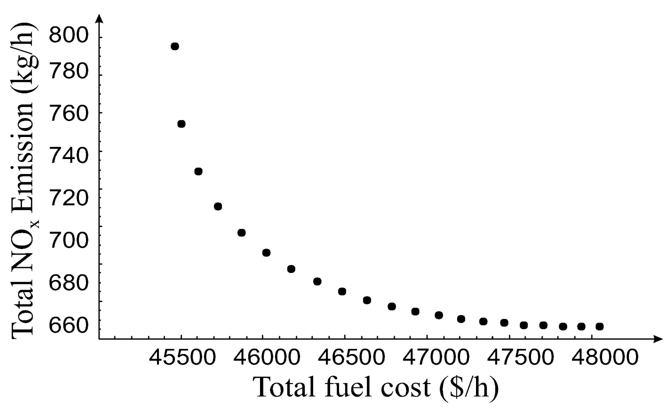

Fig. 4. Pareto-optimal front.

TABLE IX
EED (WITHOUT TRANSMISSION LOSSES) $P_D = 900$ MW

P_D = 900 MW

Item		Best Cost	Best Emissions
Unit generation (MW)	1	32.497	116.993
	2	10.816	116.993
	3	143.646	135.694
	4	143.032	135.694
	5	287.104	197.313
	6	282.905	197.313
Total fuel cost ($/h)		45463.492	48051.3
Total emission (kg/h)		795.019	646.128

value that [13], i.e., $\kappa_i = 47.8224(\$/\mathrm{kg})$, we obtain the well-distributed Pareto-optimal front presented in Fig. 4.

As the tests carried out show, despite being analytic and providing the exact solution, our method is very fast. It is not only useful as a comparison for the methods that seek approximate solutions, it is in itself a useful tool in practice for real systems. Let us recall that although we present the solution for a particular load, for reasons of comparison with other techniques, our method solves the problem for all loading conditions and, once this solution has been achieved, we assign a particular value to the power demand. The CPU time we present corresponds to the complete problem. It should also be noted that as it is neither an iterative nor approximate method, it does not require initial estimations of any parameter and, finally, it is not influenced by the size of the system. In fact, it has not limitations. It is a linear time method.

VI. DISCUSSION AND CONCLUSIONS

An exact analytical solution technique for combined economic and emissions dispatch has been presented in this paper. To date, only one analytical approach [13] has been developed to find the global solution to EED. In this paper we have developed three improvements with respect to [13].

1) First of all, the solution of [13] when transmission losses are neglected does not take into account unit capacity constraints. These constraints are imposed a posteriori in a barely effective way, above all when considering a high number of plants. In our paper, these constraints are incorporated into the analytical solution, thereby improving the previous analytical approach.
2) Second, to incorporate the transmission losses in the problem, [13] use an approximate technique that consists in representing the transmission loss in terms of demand. Once again, the solution is analytical, but not exact. In our paper, we have combined the classic iterative method to incorporate losses [14] with the analytical solution of our paper to obtain a solution that actually verifies the power balance equation.
3) Third, the treatment in [13] of the Multiobjective Problem is simply reduced to introducing a price penalty factor between both objectives. In this paper, we have presented a better approach: the Pareto set. When optimizing all objectives simultaneously, Pareto optimal solutions show the tradeoffs among conflicting objective functions.

With respect to heuristic methods, which only provide a nearly optimal solution, and for each loading condition, our approach presents the exact solution and does so without any additional time cost. To the contrary, as we have seen, our method is very fast and there is no need for iterative processes, the estimation of initial values or problems of convergence of any algorithm. We are able to obtain the solution for all the values of the load curve by simple substitution in the given formulae. In this environment, our paper presents a technique that we consider highly practical and useful for solving an EED problem in real-time for a forecast load curve.

On some occasions, exact techniques are usually complicated to understand, difficult to program and slow from the computational point of view. This means that they are usually substituted by approximate techniques. In this case, however, none of these conditions is fulfilled in our paper. Furthermore, our technique is highly versatile and we have shown in this paper that it is compatible with many other well-known techniques such as the iterative method to incorporate losses or the weighting method to populate the Pareto set.

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