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Generalization of the Firm's Profit Maximization Problem: An Algorithm for the Analytical and Nonsmooth Solution

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Abstract In this paper we present a generalization of the classic Firm's profit maximization problem, using the linear model for the production function, considering a non constant price and maximum constraints for the inputs. We formulate the problem by previously calculating the analytical minimum cost function. This minimum cost function will be calculated for each production level via the infimal convolution of quadratic functions and the result will be a piecewise quadratic function. To solve this family of optimization problems, we present an algorithm of quasi-linear complexity. Moreover, the resulting cost function in certain cases is not C^1 and the profit maximization problem will be solved within the framework of nonsmooth analysis. Finally, we present a numerical example.

Keywords Firm's profit maximization · Infimal convolution · Quadratic functions · Nonsmooth analysis · Computational complexity

1 Introduction

When the *Firm's profit maximization* (FPM) *problem* is considered (Varian 2005; Nicholson 2002), several generalizations can be presented. The classic approach to

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the FPM problem of Microeconomics is the following:

$$\pi(p, \mathbf{w}) = \max_{\mathbf{x}, y}(py - \mathbf{wx})$$
$$\text{s.t. } y = f(\mathbf{x}) \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ are the inputs, $\mathbf{w} \in \mathbb{R}^n$ are the factor prices, p is the price, y is the output and $f(\mathbf{x})$ is the *production function.*

To solve the FPM problem, we use the short-cut-via-cost minimization; i.e. we formulate the problem by previously calculating the analytical minimum cost function and then maximizing over the output quantity. So, the FPM problem is solved in two stages:

(I) First, the *Firm's cost minimization* (FCM) *problem*:

$$c(\mathbf{w}, y) = \min_{\mathbf{x}} \mathbf{wx}$$
$$\text{s.t. } f(\mathbf{x}) = y \tag{2}$$

(II) Second, having calculated the cost function $C(y) := c(\mathbf{w}, y)$, the *FPM problem:*

$$\pi(p, \mathbf{w}) = \max_{y}(py - c(\mathbf{w}, y)) = \max_{y}(py - C(y)) \tag{3}$$

In the case of $C(y)$ belonging to class C^1, this means that it is necessary to determine the optimum level of output y for which the marginal cost $C'(y)$ coincides with the price p.

The production function $f(\mathbf{x})$ express how inputs are transformed into outputs. Popular production functions models (Jehle and Reny 2001; Luenberger 1995) include:

$$\text{Leontief production function:} f(\mathbf{x}) = \min(a_1 x_1, a_2 x_2, ..., a_n x_n)$$
$$\text{Cobb-Douglas model:} f(\mathbf{x}) = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n} = \prod_{i=1}^{n} x_i^{\alpha_i}$$
$$\text{Linear production function:} f(\mathbf{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{i=1}^{n} a_i x_i \tag{4}$$

The formulae for the corresponding cost function $c(\mathbf{w}, y)$ are well known (Hamermesh 1996) when the production function follows the Cobb-Douglas model:

$$c(\mathbf{w}, y) = \alpha y^{\frac{1}{\alpha}} \prod_{i=1}^{n} \left(\frac{w_i}{\alpha_i}\right)^{\frac{\alpha_i}{\alpha}}, \quad \text{with } \alpha = \sum_{i=1}^{n} \alpha_i \tag{5}$$

These formulae, which can be obtained simply via the Lagrange multipliers method, present the drawback that they are not applicable when upper limit constraints are considered for the different inputs.

In this paper we consider the Linear production function model, and shall generalize the classic FPM problem, adding box constraints for the inputs and non constant prices:

$$w_i(x_i) = b_i + c_i x_i \tag{6}$$

The affine function model for the prices (6) represents a supposed production model in which the prices of the inputs are not independent of the amount, but increase linearly. Likewise, the function (6) may be directly interpreted as the cost (quadratic cost) of employing an amount x_i of the i-th input. On the other hand, the Linear production function is presented in a natural way when the output is the result of the sum of the inputs ($a_i = 1$) or, in general, a specific fraction of each of these. We thus have:

$$\begin{aligned}
&\pi(p, \mathbf{w}) = \max_{\mathbf{x}, y} \left(py - \sum_{i=1}^{n} x_i w_i(x_i) \right) \\
&\text{s.t.} \sum_{i=1}^{n} a_i x_i = y; \; a_i \neq 0, \quad i = 1, \ldots, n \\
&0 \leq x_i \leq U_i; \quad i = 1, \ldots, n
\end{aligned} \tag{7}$$

Problems of this kind, with box constraints, become complicated in the presence of boundary solutions.

If we focus on the first stage, the FCM problem, there is a vast array of software packages for numerically solving nonlinear optimization problems (Griva et al. 2009), (Neos). These methods only obtain an approximate solution for specific values of the output y, but do not provide the analytical expression of the cost function $c(\mathbf{w}, y)$. It is thus not possible to know the marginal cost expression $\partial c(\mathbf{w}, y)/\partial y$ needed to solve the FPM problem. For this reason, we shall address this problem in an exact way in this paper, which we state as a constrained infimal convolution problem (Hiriart-Urruty and Lemaréchal 1996). Several optimal algorithms have been presented for this non-linear separable programming problem with box constraints. The present paper generalizes prior studies (Bayón et al. 2010a), presenting an algorithm of quasi-linear complexity for the family of infimal convolution problems.

As regards the second stage, i.e. the determination of the optimum level of output y for which the marginal cost $\partial c(\mathbf{w}, y)/\partial y$ coincides with the price p, we shall see that for the model (8) the character of the resulting cost function cannot be C^1. This second stage may thus become complex, in which case the study falls within the scope of nonsmooth analysis and the *generalized* (or Clarke's) *gradient* (Clarke 1983), (Loewen 1993) must be considered.

The paper is organized as follows. In the next section we state the generalized problem (8). An algorithm for calculating the infimal convolution of the resulting quadratic functions is presented in Sect. 3, while Sect. 4 analyzes the nonsmooth FPM problem. In Sect. 5, we discuss the results of a numerical example. Finally, Sect. 6 summarizes the main conclusions of our research.

2 Statement of the Generalized Problem

We shall consider the first stage of the generalized problem to solve to be:

$$\begin{aligned} C(y) &= \min_{\mathbf{x}} \sum_{i=1}^{n} x_i w_i(x_i) \\ \text{s.t.} & \sum_{i=1}^{n} a_i x_i = y;\, a_i \neq 0, \quad i = 1, \ldots, n \\ & 0 \leq x_i \leq U_i; \quad i = 1, \ldots, n \end{aligned} \tag{8}$$

Using (6), and making

$$\begin{aligned} a_i x_i &= z_i; \quad a_i U_i = M_i \\ \frac{b_i}{a_i} &= \beta_i; \quad \frac{c_i}{a_i^2} = \gamma_i \end{aligned} \tag{9}$$

this problem may be re-written as follows:

$$\begin{aligned} C(y) &= \min_{\mathbf{z}} \sum_{i=1}^{n} \beta_i z_i + \gamma_i z_i^2 \\ \text{s.t.} & \sum_{i=1}^{n} z_i = y \\ & 0 \leq z_i \leq M_i; \quad i = 1, \ldots, n \end{aligned} \tag{10}$$

which makes $C(y)$ the infimal convolution of the quadratic functions:

$$F_i(z_i) := \beta_i z_i + \gamma_i z_i^2 \tag{11}$$

respectively constrained to the domains $[0, M_i]$; i.e.

$$C = F_1 \bigodot F_2 \bigodot \ldots \bigodot \tag{12}$$

In this case, $C(y)$ is guaranteed to be a convex function in view of the fact that each of the F_i functions is convex. However, there is no guarantee that it belongs to class C^1. Thus, the determination of the optimum level of output y for which the marginal cost coincides with the price p, which for the classic problem may be represented by the simple equation

$$p = C'(y) \tag{13}$$

should be substituted by

$$p \in \partial C(y) \tag{14}$$

where $\partial C(y)$ is the generalized (or Clarke's) gradient.

3 An Algorithm for the Infimal Convolution of the Quadratic Functions

The infimal convolution operator (Stromberg 1996) is well known within the context of convex analysis.

Definition 1 Let $F, G : \mathbb{R} \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ be two functions. We denote as the Infimal Convolution of F and G the operation defined as follows:

$$(F \bigodot G)(x) := \inf_{y \in \mathbb{R}} \{F(x) + G(y - x)\} \tag{15}$$

When the functions are considered to be constrained to a certain domain, $Dom(F_i) = [0, M_i]$, the equivalent definition may be expressed as follows:

$$C(y) := \left(\bigodot_{i=1}^{n} F_i \right)(y) = \min_{\substack{\sum_{i=1}^{n} z_i = y \\ 0 \le z_i \le M_i}} \sum_{i=1}^{n} F_i(z_i) \tag{16}$$

We shall now expound the adaptation to the problem (10) of the results presented in (Bayón et al. 2010a), which were further developed in (Bayón et al. 2010b).

We first present the definitions needed to build our algorithm. Let $A = \{1, \ldots, n\}$.

Definition 2 If $C(y) := \left(\bigodot_{i=1}^{n} F_i \right)(y)$, we denote by the i-th distribution functions, the functions

$$\Psi_i : \left[0, \sum_{i=1}^{n} M_i \right] \longrightarrow [0, M_i] \tag{17}$$

that satisfy

$$\sum_{i=1}^{n} \Psi_i(y) = y \quad \text{and} \quad (\Psi_1(y), \ldots, \Psi_n(y)) \in \prod_{i=1}^{n} [0, M_i] \tag{18}$$

and are the solution of (10).

Definition 3 Let us consider, in the set $A \times \{m, M\}$, the binary relation $\preccurlyeq$ defined as follows:

$$\begin{aligned}
(i, m) \preccurlyeq (j, m) &\iff F_i'(0) \le F_j'(0) \quad \text{or} \quad (F_i'(0) = F_j'(0) \text{ and } i \le j) \\
(i, m) \preccurlyeq (j, M) &\iff F_i'(0) \le F_j'(M_j) \quad \text{or} \quad (F_i'(0) = F_j'(M_j) \text{ and } i \le j) \\
(i, M) \preccurlyeq (j, m) &\iff F_i'(M_i) \le F_j'(0) \quad \text{or} \quad (F_i'(M_i) = F_j'(0) \text{ and } i \le j) \\
(i, M) \preccurlyeq (j, M) &\iff F_i'(M_i) \le F_j'(M_j) \quad \text{or} \quad (F_i'(M_i) = F_j'(M_j) \text{ and } i \le j)
\end{aligned} \tag{19}$$

Obviously, $\preccurlyeq$ is a total order relation and $(A \times \{m, M\}, \preccurlyeq)$ is isomorphic with respect to $(\{1, 2, \ldots, 2n\}, \le)$.

Definition 4 We denote by g the isomorphism

$$g(i) := (g_1(i), g_2(i)), \quad g : (\{1, 2, \ldots, 2n\}, \leq) \longrightarrow (A \times \{m, M\}, \preccurlyeq) \tag{20}$$

which at each natural number $i \in \{1, 2, \cdots, 2n\}$ corresponds to the i-th element of $A \times \{m, M\}$ following the order established by $\preccurlyeq$.

We now present the optimization algorithm that leads to the determination of the optimal solution. The algorithm generates all the feasible states of activity/inactivity of the constraints on the solution to the problem. We build a sequence $(\Omega_i, \Theta_i, \Xi_i)$ starting with the triad $(A, \varnothing, \varnothing)$, which represents the fact that all the constraints on minimum are active, and ending with the triad $(\varnothing, \varnothing, A)$, which represents the fact that all the constraints on maximum are active. Let us consider the following recurrent sequence $X_i := (\Omega_i, \Theta_i, \Xi_i)$, $i = 0, \ldots, 2n$:

$$\begin{array}{lll} \Omega_0 = A & \Theta_0 = \varnothing & \Xi_0 = \varnothing \\ \text{If } g_2(i) = M : \Omega_i = \Omega_{i-1} & \Theta_i = \Theta_{i-1} - \{g_1(i)\} & \Xi_i = \Xi_{i-1} \cup \{g_1(i)\} \\ \text{If } g_2(i) = m : \Omega_i = \Omega_{i-1} - \{g_(i)\} & \Theta_i = \Theta_{i-1} \cup \{g_1(i)\} & \Xi_i = \Xi_{i-1} \end{array} \tag{21}$$

The following propositions will allow us to construct the infimal convolution.

Proposition 1 *There exist*

$$\{\phi_i\}_{i=1}^{2n} \subset \mathbb{R}, \quad 0 = \phi_1 \leq \cdots \leq \phi_{2n} = \sum_{i=1}^{n} M_i \tag{22}$$

such that $\forall y \mid \phi_i \leq y < \phi_{i+1}$, *the solution to the problem* $(\Psi_1(y), \ldots, \Psi_n(y))$ *satisfies:*

$$\Psi_k(y) = \begin{cases} \dfrac{2\widehat{\gamma}_i(y - \mu_i) + \widehat{\beta}_i - \beta_k}{2\gamma_k} & \text{if } k \in \Theta_i \\ 0 & \text{if } k \in \Omega_i \\ M_k & \text{if } k \in \Xi_i \end{cases} \tag{23}$$

being

$$\phi_1 = 0; \quad \phi_i = \phi_{i-1} + \frac{1}{2}\left[s_i - s_{i-1}\right]\frac{1}{\widehat{\gamma}_{i-1}}$$

$$s_1 = 0; \quad s_i = \begin{cases} s_{i-1} & \text{if } \Theta_{i-1} = \varnothing \\ F'_{g_1(i)}(0) & \text{if } g_2(i) = m \wedge \Theta_{i-1} \neq \varnothing \\ F'_{g_1(i)}(M_{g_1(i)}) & \text{if } g_2(i) = M \wedge \Theta_{i-1} \neq \varnothing \end{cases} \tag{24}$$

$$\widehat{\gamma}_i := \frac{1}{\displaystyle\sum_{j \in \Theta_i} \frac{1}{\gamma_j}}; \quad \widehat{\beta}_i := \widehat{\gamma}_i \sum_{j \in \Theta_i} \frac{\beta_j}{\gamma_j}; \quad \mu_i := \sum_{j \in \Xi_i} M_j \tag{25}$$

Proposition 2 *The function $C(y)$ (infimal convolution) is continuous function and piecewise quadratic C^1 function. Specifically, if $\phi_i \leq y < \phi_{i+1}$, (being ϕ_i the coefficients defined in proposition (1) we have that*

$$C(y) = \widehat{\alpha}_i + \widehat{\beta}_i (y - \mu_i) + \widehat{\gamma}_i (y - \mu_i)^2 \tag{26}$$

where

(i) If $\Theta_i = \varnothing$:

$$\widehat{\alpha}_i = \widehat{\alpha}_{i-1} + F_{g_1(i)}\left(M_{g_1(i)}\right) = \sum_{j \in \Xi_i} F_j(M_j); \quad \widehat{\beta}_i := 0; \quad \widehat{\gamma}_i := 0$$

(ii) If $\Theta_i \neq \varnothing \wedge \Theta_{i-1} \neq \varnothing$:

$$\mu_i = \begin{cases} \mu_{i-1} & \text{if } g_2(i) = m \\ \mu_{i-1} + M_{g_1(i)} & \text{if } g_2(i) = M \end{cases} \tag{27}$$

$$\widehat{\alpha}_i = \begin{cases} \widehat{\alpha}_{i-1} - \frac{\left(\widehat{\beta}_{i-1} - \beta_{g_1(i)}\right)^2}{4\left(\widehat{\gamma}_{i-1} + \gamma_{g_1(i)}\right)} & \text{if } g_2(i) = m \\ \widehat{\alpha}_{i-1} - \frac{\left(\widehat{\beta}_{i-1} - \beta_{g_1(i)}\right)^2}{4\left(\widehat{\gamma}_{i-1} - \gamma_{g_1(i)}\right)} + F_{g_1(i)}\left(M_{g_1(i)}\right) & \text{if } g_2(i) = M \end{cases} \tag{28}$$

$$\widehat{\beta}_i = \begin{cases} \frac{1}{\widehat{\gamma}_{i-1} + \gamma_{g_1(i)}}\left[\widehat{\beta}_{i-1} \cdot \gamma_{g_1(i)} + \beta_{g_1(i)} \cdot \widehat{\gamma}_{i-1}\right] & \text{if } g_2(i) = m \\ \frac{1}{\widehat{\gamma}_{i-1} - \gamma_{g_1(i)}}\left[-\widehat{\beta}_{i-1} \cdot \gamma_{g_1(i)} + \beta_{g_1(i)} \cdot \widehat{\gamma}_{i-1}\right] & \text{if } g_2(i) = M \end{cases} \tag{29}$$

$$\widehat{\gamma}_i = \begin{cases} \frac{\widehat{\gamma}_{i-1} \cdot \gamma_{g_1(i)}}{\widehat{\gamma}_{i-1} + \gamma_{g_1(i)}} & \text{if } g_2(i) = m \\ -\frac{\widehat{\gamma}_{i-1} \cdot \gamma_{g_1(i)}}{\widehat{\gamma}_{i-1} - \gamma_{g_1(i)}} & \text{if } g_2(i) = M \end{cases} \tag{30}$$

(iii) If $\Theta_i \neq \varnothing \wedge \Theta_{i-1} = \varnothing$:

$$\mu_i = \mu_{i-1}; \quad \widehat{\alpha}_i = \widehat{\alpha}_{i-1}; \quad \widehat{\beta}_i = \beta_{g_1(i)}; \quad \widehat{\gamma}_i = \gamma_{g_1(i)} \tag{31}$$

being

$$\widehat{\gamma}_1 = \gamma_{g_1(1)}; \quad \widehat{\beta}_1 = \beta_{g_1(1)}; \quad \widehat{\alpha}_1 = 0; \quad \mu_1 = 0 \tag{32}$$

Corollary 1 *Under the same conditions as in the above proposition, if $\Theta_i \neq \varnothing, \forall i, 0 < i < 2n$, then the function $C(y)$(infimal convolution) also belongs to class C^1.*

The demonstrations of these two propositions and the corollary are adaptations of those presented in (Bayón et al. 2010b).

Remark When $\Theta_i = \varnothing$ and hence $\phi_i = \phi_{i+1}$, it may be that the infimal convolution is not derivable in ϕ_i. This occurs, for example, in the case of two functions F_1 and

F_2 with domains $[0, M_1]$ and $[0, M_2]$, where $F_1'[M_1] < F_2'[0]$. The convolution of F_1 and F_2 is given by

$$(F_1 \odot F_2)(x) = \begin{cases} F_1(x) & \textit{if } x \in [0, M_1] \\ F_2(x - M_1) + F_1(M_1) & \textit{if } x \in [M_1, M_1 + M_2] \end{cases} \tag{33}$$

and is obviously not derivable in M_1. This is what occurs in the example we shall see in Sect. 5.

4 Nonsmooth FPM Problem

Nonsmooth analysis works with locally Lipschitz functions, f, which are differentiable almost everywhere (the set of points at which f fails to be differentiable is denoted by Ω_f).

Definition 5 Let $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschitz near x, and let us assume S is any set of Lebesgue measure 0 in $\mathbb{R}^n$. The generalized (or Clarke's) gradient, ∂f, is

$$\partial f(x) = \mathrm{co}\left\{\lim \nabla f(x_i) : x_i \longrightarrow x, x_i \notin S, x_i \notin \Omega_f\right\} \tag{34}$$

The meaning of Eq. (34) is the following: consider any sequence x_i converging to x while avoiding both S and points at which f is not differentiable, and such that the sequence $\nabla f(x_i)$ converges; then the convex hull of all such limit points is $\partial f(x)$.

It follows simply from this reasoning that if f is a piecewise C^1 function, then $\partial f = [f_-'(x), f_+'(x)]$, where $f_-'(x)$ and $f_+'(x)$ represent the lateral derivatives of f.

In the case of the functions for which the generalized (or Clarke's) gradient exists, we have (Clarke 1983) the following result:

Proposition 3 *If f is convex (resp. concave), the function reaches its absolute minimum (resp. maximum) at x_0*

$$0 \in \partial f(x_0) \tag{35}$$

In the case we are dealing with here, the cost function $C(y)$ (see Eq. 26) is of quadratic piecewise character C^1. Consequently, the profit-maximization problem:

$$\pi(p) = \max_y (py - C(y)) \tag{36}$$

translates into the determination of the optimum level of output y^* for which

$$0 \in \partial(py^* - C(y^*)) \tag{37}$$

in which case, $\pi(p) = py^* - C(y^*)$.

Applying the above definition to $C(y)$, we immediately have that the generalized (or Clarke's) gradient is

$$\partial C(y) = [C_-'(y), C_+'(y)] \tag{38}$$

where $C_-'(y)$ and $C_+'(y)$ represent the lateral derivatives of C.

Table 1 Example data

i	1	2	3	4	5
a_i	0.95	0.97	0.98	0.95	0.97
M_i	360	543	253	350	250
b_i	51	31	67	25	2
c_i	0.191	0.016	0.14	0.024	0.011

Summing up, the optimum level of production should satisfy the following relation:

$$C'_{-}(y) \leq p \leq C'_{+}(y) \tag{39}$$

However, also bearing in mind that the cost function is piecewise quadratic, the correct calculation of the output level requires prior investigation of the interval $[\phi_i, \phi_{i+1}]$ *for which:*

$$C'_{-}(\phi_i) \leq p \leq C'_{+}(\phi_{i+1}) \tag{40}$$

This question is trivial, as we already have the analytical expression of $C(y)$.

5 Example

The problem (8) has been developed in the previous sections and may be tackled using the aforementioned algorithm. We now present an example. A program that solves the Firm's Profit-optimization Problem was written using the Mathematica package and was then applied to one example using the linear model for the production function: $f(\mathbf{x}) = \sum_{i=1}^{n} a_i x_i$, the affine function model for the prices: $w_i(x_i) = b_i + c_i x_i$, and maximum constraints M_i for the $n = 5$ inputs. The data on the inputs is summarized in Table 1.

We shall now apply the theory developed previously. Taking into consideration the values of $F_i'(0)$ and $F_i'(M_i)$ in accordance with the order $\preceq$:
we have that the elements of $(A \times \{m, M\})$, in accordance with the order $\preceq$ and the sequence $X_i := (\Omega_i, \Theta_i, \Xi_i), i = 0, \ldots, 10$, are:

i	$g(i)$	Ω_i	Θ_i	Ξ_i
0		$\{1, 2, 3, 4, 5\}$	$\{\}$	$\{\}$
1	$\{5, m\}$	$\{1, 2, 3, 4\}$	$\{5\}$	$\{\}$
2	$\{5, M\}$	$\{1, 2, 3, 4\}$	$\{\}$	$\{5\}$
3	$\{4, m\}$	$\{1, 2, 3\}$	$\{4\}$	$\{5\}$
4	$\{2, m\}$	$\{1, 3\}$	$\{4, 2\}$	$\{5\}$
5	$\{4, M\}$	$\{1, 3\}$	$\{2\}$	$\{5, 4\}$
6	$\{2, M\}$	$\{1, 3\}$	$\{\}$	$\{5, 4, 2\}$
7	$\{1, m\}$	$\{3\}$	$\{1\}$	$\{5, 4, 2\}$
8	$\{3, m\}$	$\{\}$	$\{1, 3\}$	$\{5, 4, 2\}$
9	$\{3, M\}$	$\{\}$	$\{1\}$	$\{5, 4, 2, 3\}$
10	$\{1, M\}$	$\{\}$	$\{\}$	$\{1, 2, 3, 4, 5\}$

Table 2 Derivatives at box constraints

$F_5'(0)$	$F_5'(M_5)$	$F_4'(0)$	$F_2'(0)$	$F_4'(M_4)$
2.06186	7.90732	26.3158	31.9588	44.9307
$F_2'(M_2)$	$F_1'(0)$	$F_3'(0)$	$F_3'(M_3)$	$F_1'(M_1)$
50.4262	53.6842	68.3673	142.128	206.061

Table 3 Values of ϕ_i

$\phi_1 = 0$	$\phi_2 = 250$	$\phi_3 = 250$	$\phi_4 = 356.1$	$\phi_5 = 981.417$
$\phi_6 = 1143$	$\phi_7 = 1143$	$\phi_8 = 1177.69$	$\phi_9 = 1604.95$	$\phi_{10} = 1756$

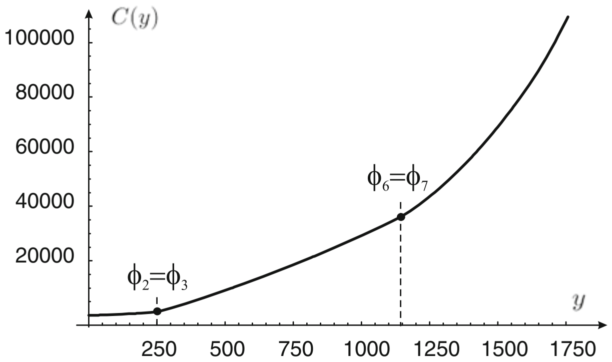

Fig. 1 Cost function

The family $\{\phi_i\}_{i=1}^{10} \subset \mathbb{R}$, where

$$0 = \phi_1 \leq \cdots \leq \phi_{2n} = \sum_{i=1}^{5} M_i = 1756 \tag{41}$$

is:

The coincidences $\phi_2 = \phi_3$ and $\phi_6 = \phi_7$ are due to the fact that $\Theta_2 = \varnothing = \Theta_6$ (Tables 2, 3).

The fact that the solution of (10) has all its constraints active ($\Theta_2 = \varnothing$) for $y = \phi$ means that it is impossible for this situation to be produced in any interval of the form $[\phi_2, \phi_2 + \varepsilon)$ with $\varepsilon > 0$. Hence, ϕ_3 must necessarily coincide with ϕ_2, in which case the cost function presents angular points at $\phi_2 = \phi_3 = 250$ and at $\phi_6 = \phi_7 = 1143$ (see Fig. 1).

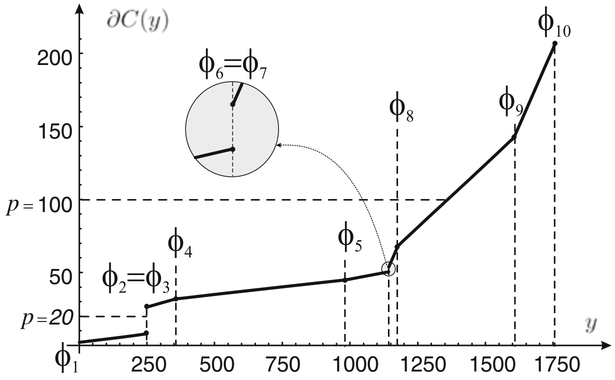

Fig. 2 Marginal cost function

Next, we show the analytical expression for the cost function, considering constraints on the inputs. We can see that it is a piece-wise quadratic function:

$$C(y) = \begin{cases} 2.06186y + 0.0116909y^2 & \text{if} \quad \phi_1 \le y \le \phi_2 = \phi_3 \\ -3670.75 + 13.0194y + 0.0265928y^2 & \text{if} \quad \phi_3 \le y \le \phi_4 \\ -5727.62 + 24.5716y + 0.0103723y^2 & \text{if} \quad \phi_4 \le y \le \phi_5 \\ 660.832 + 11.5528y + 0.017005y^2 & \text{if} \quad \phi_5 \le y \le \phi_6 = \phi_7 \\ 251210 - 430.112y + 0.211634y^2 & \text{if} \quad \phi_7 \le y \le \phi_8 \\ 77401.7 - 134.943y + 0.0863175y^2 & \text{if} \quad \phi_8 \le y \le \phi_9 \\ 400203 - 537.199y + 0.211634y^2 & \text{if} \quad \phi_9 \le y \le \phi_{10} \end{cases}$$

Remark Coincidences in the ϕ_i may also arise without any Θ_i being empty. In fact, this occurs whenever we have situations of the type: $F_i'(0) = F_j'(0)$ or $F_i'(0) = F_j'(M_j)$ or $F_i'(M_i) = F_j'(M_j)$. In these cases, however, the infimal convolution does not cease to belong to class C^1.

As an example to illustrate this, let us consider two functions F_1 and F_2 with domains $[0, M_1]$ and $[0, M_2]$, where $F_1'[M_1] = F_2'[0]$.

The convolution of F_1 and F_2 is given by

$$(F_1 \odot F_2)(x) = \begin{cases} F_1(x) & \textit{if } x \in [0, M_1] \\ F_2(x - M_1) + F_1(M_1) & \textit{if } x \in [M_1, M_1 + M_2] \end{cases} \tag{42}$$

and is obviously derivable throughout its domain even though there is a coincidence in the values ϕ_2 and ϕ_3 :

$$0 = \phi_1 < \phi_2 = \phi_3 = M_1 < \phi_4 = M_1 + M_2$$

Figure 2 shows the graph of the marginal cost function, which, as has already been established, is piecewise continuous (i.e. the cost function is piecewise C^1). It can be seen that there are two points for which the marginal cost "jumps", at $\phi_2 = \phi_3 = 250$ and at $\phi_6 = \phi_7 = 1143$.

Finally, we present the solution for two price cases: Case (i) $p = 100$; Case (ii) $p = 20$.

- In case (i), the interval for which $C'_{-}(\phi_i) \leq 100 \leq C'_{+}(\phi_{i+1})$ is $[\phi_8, \phi_9]$. In this case, the solution is derivable; hence:

$$0 \in \partial(py^* - C(y^*)) \Longrightarrow 0 = p - C'(y^*) \Longrightarrow 100 = -134.943 + 0.172635y^* \tag{43}$$

  Thus, $y^* = 1360.92$ and the maximum profit is

$$\pi(100) = 100y^* - C(y^*) = 82469.2 \tag{44}$$

- In case (ii), the interval for which $C'_{-}(\phi_i) \leq 20 \leq C'_{+}(\phi_{i+1})$ is $[\phi_2, \phi_3]$. In this case, the solution is not derivable

$$C'_{-}(\phi_2) = 7.907; \quad C'_{+}(\phi_3) = 26.316 \tag{45}$$

  hence:

$$0 \in \partial(py^* - C(y^*)) \Longrightarrow 0 \in p - \partial C(y^*) \tag{46}$$

$$\Longrightarrow 20 \in [7.907, 26.316] \Longrightarrow y^* = 250 \tag{47}$$

  For $y^* = 250$, the maximum profit is

$$\pi(20) = 20y^* - C(y^*) = 3753.91 \tag{48}$$

  It is very important to highlight the fact that situations such as those considered in case (ii), in which there is no derivability, are very difficult to analyze if the analytic solution is not available.

The algorithm runs quickly despite our analytic solution method. The optimal solution was calculated on a personal computer (Pentium IV, 3.4 GHz PC) using the commercial program Mathematica 5.0 ®. The above solution presents an execution time of only 16 ms.

The total time represents the time that is consumed in each of the different phases of algorithm:

Phase I: Construction of the sequence $X_i := (\Omega_i, \Theta_i, \Xi_i)$.
Phase II: Ordering of the elements of $(A \times \{m, M\})$ and construction of g, in accordance with the order $\preceq$.
Phase III: Calculation of the exact solution $C(y)$ using the recurrent formulae of Proposition 1 and Proposition 2.
Phase IV: Calculation of the i such that $C'_{-}(\phi_i) \leq p \leq C'_{+}(\phi_{i+1})$.
Phase V: Calculation of the y^* for which $0 \in \partial(py^* - C(y^*))$.
Phase VI: Calculation of the solution $\pi(p) = py^* - C(y^*)$.

The numerous trials carried out using the algorithm show that, of all the phases, the one that consumes almost all the time is Phase II. In the above example, Phase II employed 15 ms.

Finally, it should be noted that, as we have proven in (Bayón et al. 2010b), the complexity of the aforementioned algorithm is quasi-linear:

$$O(n \log(n)) \tag{49}$$

The reason for this is that the algorithm has two fundamental parts:

(I) Ordering of the set $A \times \{m, M\}$ following the ordering relation $\preccurlyeq$
(II) Iterative construction of the sequences:

$$\begin{aligned} & s_i, i = 0, \ldots, 2n \\ & \widehat{\alpha}_i, \widehat{\beta}_i, \widehat{\gamma}_i, \mu_i, i = 1, \ldots, 2n-1 \\ & \phi_i, i = 1, \ldots, 2n \end{aligned}$$

The first, using merge sort, is of quasi-linear complexity: $O(n * \log(n))$. The second, which involves iteratively calculating the different coefficients is of linear complexity: $O(n)$. In conclusion, the complexity is quasi-linear. This fact means that our algorithm is a tool that able to tackle large-scale problems.

6 Conclusions

In this paper we have established the analytic solution for the classic firm's profit maximization problem in the general case with n inputs. We have considered, for the first time, maximum constraints for the inputs. For the considered models, the solution may not be derivable and our method calculates the exact solution without any kind of simplification.

Our study has a number of advantages over other methods: the exact boundary solution is obtained and the method is not affected by the size or the derivability of the problem. Moreover, the complexity of the algorithm is quasi-linear and it is able to tackle large-scale problems.

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