JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
Volume **10**, Number **3**, July **2014**

doi:10.3934/jimo.2014.10.761

pp. **761–776**

A HYDROTHERMAL PROBLEM WITH NON-SMOOTH LAGRANGIAN

Luis Bayón, Jose Maria Grau
Maria del Mar Ruiz and Pedro Maria Suárez

University of Oviedo, Department of Mathematics, E.P.I
Campus of Viesques, Gijón, 33203, Spain

(Communicated by Helmut Maurer)

Abstract. This paper deals with the optimization of a hydrothermal problem that considers a non-smooth Lagrangian $L(t, z, z')$. We consider a general case where the functions $L_{z'}(t,\cdot,\cdot)$ and $L_z(t,\cdot,\cdot)$ are discontinuous in $\{(t, z, z')/z' = \phi(t, z)\}$, which is the borderline point between two power generation zones. This situation arises in problems of optimization of hydrothermal systems where the thermal plant input-output curve considers the shape of the cost curve in the neighborhood of the valve points. The problem shall be formulated in the framework of nonsmooth analysis, using the generalized (or Clarke's) gradient. We shall obtain a necessary minimum condition and we shall generalize the known result (smooth transition) that the derivative of the minimum presents a constancy interval. Finally, we shall present an example.

1. **Introduction.** The economic dispatch (ED) problem ([18], [5]) is one of the important optimization problems in a hydrothermal power system. In two previous papers ([2], [3]) a problem of hydrothermal optimization with pumped-storage plants was considered. The problem consisted in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0, T]$. The mathematical problem was stated in the following terms:

$$\min_{z\in\Theta} F(z) = \min_{z\in\Theta} \int_0^T \Psi\left[P_d(t) - H(t, z(t), z'(t))\right] dt = \min_{z\in\Theta} \int_0^T L(t, z(t), z'(t))dt, \tag{1.1}$$

$$\Theta = \{z \in \widehat{C}^1[0, T] \mid z(0) = 0, z(T) = b\}.$$

By ($\widehat{C}^1$), we denote the set of piecewise C^1 functions from $[0, T]$ to $\mathbb{R}$, b is the volume of water that must be discharged during the entire optimization interval, $P_d : [0, T] \to \mathbb{R}$ is a continuous function that represents the power demand at each instant of the optimization interval, $\Psi : \mathbb{R} \to \mathbb{R}$ is the cost function of the thermal plant, $H : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the function of effective hydraulic generation, $z(t)$ the volume that is discharged up to the instant t by the hydroplant, and $z'(t)$ the rate of water discharge at the instant t by the hydroplant. We shall assume that

2010 *Mathematics Subject Classification.* Primary: 65K10, 49J52.

Key words and phrases. Optimization, nonsmooth analysis, variational methods, hydrothermal problem, valve-point.

This work was supported by the Oviedo University, project: UNOV-12-MA-01.

Ψ is strictly increasing and strictly convex and that H is strictly increasing and concave with respect to z'. In this kind of problem, the derivative of H with respect to z' ($H_{z'}$) presents discontinuity at $z' = 0$, which is the border between the power generation zone (positive values of z') and the pumping zone (negative values of z'). Thus, the Lagrangian $L(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $L_z(\cdot,\cdot,\cdot)$ belong to class C^0 and the function $L_{z'}(t,z,\cdot)$ is piecewise continuous ($L_{z'}(t,z,\cdot)$ is discontinuous in $z' = 0$).

Denoting by $¥_z(t), z \in \Theta$ the function:

$$¥_z(t) := -L_{z'}(t,z(t),z'(t)) + \int_0^t L_z(s,z(s),z'(s))ds \tag{1.2}$$

and by $¥_z^+(t)$ and $¥_z^-(t)$ the expressions obtained when considering the lateral derivatives with respect to z'. The problem was formulated within the framework of nonsmooth analysis [7], using the generalized (or Clarke's) gradient, the following result being proven:

Theorem 1. *If q is a solution of (1.1), then $\exists K \in \mathbb{R}^+$ such that:*

$$\begin{cases} ¥_q^+(t) = ¥_q^-(t) = K & \textit{if} \quad q'(t) \neq 0, \\ ¥_q^+(t) \leq K \leq ¥_q^-(t) & \textit{if} \quad q'(t) = 0. \end{cases} \tag{1.3}$$

In another previous paper [4], we presented a qualitative aspect of the solution: the *smooth transition*. The following result was proven: under certain convexity conditions, the discontinuity of the derivative of the Lagrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verified that the derivative of the extremal where the minimum is reached presents an interval of constancy, the constant being the value for which $L_{z'}(t,z,\cdot)$ presents discontinuity. The character C^1 of the solution is thus guaranteed.

This paper generalizes the two previous studies, considering a more general and nonsmooth Lagrangian: $L(\cdot,\cdot,\cdot)$ belongs to class C^0, but $L_{z'}(t,\cdot,\cdot)$ and $L_z(t,\cdot,\cdot)$ are continuous, except in

$$\Xi = \{(t,z,z')/z' = \phi(t,z)\}, \tag{1.4}$$

where ϕ belongs to class C^1.

This situation arises in problems of optimization of hydrothermal systems in which the thermal plant input-output curve considers the shape of the cost curve in the neighborhood of the valve-points. Traditionally, the cost function for each thermal plant in the ED problem has been approximately represented by a single quadratic function

$$\Psi(P) = \quad \alpha + \beta P + \gamma P^2 \qquad \text{if} \qquad P_{\min} \leq P < P_{\max}, \tag{1.5}$$

where $\Psi(Euro/h)$ is the cost, $P(MW)$ is the power generated, $P_{\min}$ and $P_{\max}$ are the minimum and maximum power outputs of the unit, valve-point effects being ignored.

The ED problem with valve-point effects is represented as a nonsmooth optimization problem. Some studies of the ED problem, such as the improved genetic algorithm (IGA) [6], [1], a particle swarm optimization (PSO) technique [14], hybrid solution methodology integrating the particle swarm optimization (PSO) algorithm with the sequential quadratic programming (SQP) method [19], the differential evolution (DE) algorithm combined with the sequential quadratic programming (SQP) technique [8], the Taguchi method [10], a novel Stochastic Search (SS) method [16], and a group search optimizer (GSO) methodology [20] consider valve-point effects.

The cost function is obtained from a data point taken during tests when input and output data are measured, as the thermal plant slowly varies through its operating region. The shape of the cost curve in the neighborhood of the valve-points is difficult to determine by testing. Wire drawing effects, which occur as each steam admission valve in a turbine starts to open, produce a rippling effect on the cost curve. This curve contains higher order nonlinearity and discontinuity due to the valve-point effect and should be refined. Therefore, (1.5) can be modified by a sine function

$$\Psi(P) = \alpha + \beta P + \gamma P^2 + e \sin\left[f(P_{\min} - P)\right],$$

where e and f are constants of the valve-point effect of generators, or by several piecewise quadratic functions

$$\Psi(P) = \begin{cases} \alpha_1 + \beta_1 P + \gamma_1 P^2 & \text{if} \quad P_{\min} \leq P < P_1 \\ \alpha_2 + \beta_2 P + \gamma_2 P^2 & \text{if} \quad P_1 \leq P < P_2 \\ \dots & \dots \quad \dots \\ \alpha_k + \beta_k P + \gamma_k P^2 & \text{if} \quad P_k \leq P < P_{\max}. \end{cases} \tag{1.6}$$

We accept this more approximate model (1.6). So, let us consider a thermal plant defined by several quadratic cost function such that Ψ is continuous but Ψ' is discontinuous at the valve points (a piecewise C^1 quadratic function). In Fig. 1 we see that Ψ' is discontinuous at P_1 and P_2. At P_1, for example, we have that

$$P_1 = P_d(t) - H(t, z(t), z'(t)) \Rightarrow z' = \phi(t, z),$$

so $L_{z'}(t,\cdot,\cdot)$ and $L_z(t,\cdot,\cdot)$ are discontinuous in $z' = \phi(t, z)$.

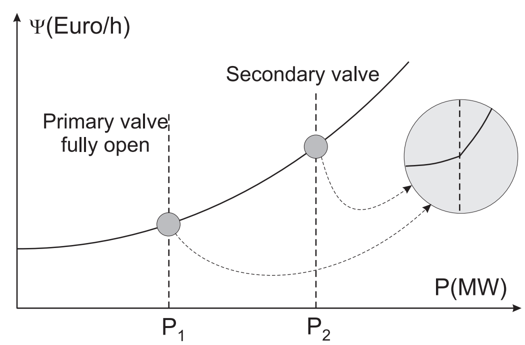

Fig. 1. Thermal plant cost curve.

The paper is organized as follows. In Section 2, we shall obtain a necessary minimum condition using the generalized (or Clarke's) gradient. The study of nonsmooth variational problems has been widely developed in recent years and the development of nonsmooth analysis has allowed researchers to deal with nonsmooth problems. In particular, it has permitted the achievement of nonsmooth versions of the classical Euler–Lagrange condition [11], [9], [17], [12], [13]. In Section 3, we shall generalize the smooth transition and shall prove that the derivative of the minimum presents a interval where $z' = \phi(t, z)$ is verified. In Section 4, we shall present a solution algorithm and shall apply it to an example. Finally, Section 5 summarizes the main conclusions of our research.

2. **A necessary condition.** Let us once more consider the mathematical problem (1.1):

$$\min_{z\in\Theta} F(z) = \min_{z\in\Theta} \int_0^T \Psi\left[P_d(t) - H(t,z(t),z'(t))\right] dt = \min_{z\in\Theta} \int_0^T L(t,z(t),z'(t))dt, \tag{2.1}$$

$$\Theta = \{z \in \widehat{C}^1[0,T] \mid z(0)=0, z(T)=b\},$$

though now assuming new conditions: We shall assume a known $P_d(t)$, that Ψ is strictly increasing and strictly convex and that Ψ' is discontinuous at the valve points represented by the set $\Xi = \{(t,z,z')/z' = \phi(t,z)\}$, and that H verifies $H_{z'} > 0$ and $H_z < 0$. In this problem, the Lagrangian $L(\cdot,\cdot,\cdot)$ belongs to class C^0, but $L_{z'}(t,\cdot,\cdot)$ and $L_z(t,\cdot,\cdot)$ are discontinuous in Ξ.

We shall establish the necessary minimum condition for this problem with nonsmooth Lagrangian, employing nonsmooth analysis for this purpose.

Nonsmooth analysis [9] works with locally Lipschitz functions that are differentiable almost everywhere (the set of points at which f fails to be differentiable is denoted by Ω_f). Let $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschitz near x, and let us assume that S is any set of Lebesgue measure 0 in $\mathbb{R}^n$. The generalized (or Clarke's) gradient ∂f can be calculated as a convex hull of (almost) all converging sequences of the gradients

$$\partial f(x) = \text{co}\left\{\lim \nabla f(x_i) : x_i \longrightarrow x,\ x_i \notin S,\ x_i \notin \Omega_f\right\}. \tag{2.2}$$

We now extend this study to integral functionals, which will be taken over the σ-finite positive measure space $(\mathbb{T}, \Im, \mu) = [0,T]$ with Lebesgue measure. $L^\infty(\mathbb{T}, Y)$ denotes the space of measurable essentially bounded functions mapping $\mathbb{T}$ to Y, equipped with the usual supremum norm, with Y being the separable Banach space $Y = \mathbb{R} \times \mathbb{R}$. We are also given a closed subspace X of $L^\infty(\mathbb{T}, Y)$

$$X = \left\{(s,v) \in L^\infty(\mathbb{T}, Y) \ \text{ for some } \ c \in \mathbb{R},\ s(t) = c + \int_0^t v(\tau)d\tau\right\}$$

and a family of functions $f_t : Y \longrightarrow \mathbb{R}$ $(t \in \mathbb{T})$ with $f_t(s,v) = L(t,s,v)$. We define a function f

$$f(s,v) = \int_0^T L(t,s(t),v(t))dt.$$

Under the above hypotheses, f is Lipschitz in a neighborhood of $(\widehat{s},\widehat{v}) \in X$ and the following [7] holds:

$$\partial f(\widehat{s},\widehat{v}) \subset \int_0^T \partial L(t,\widehat{s}(t),\widehat{v}(t))dt. \tag{2.3}$$

Hence, if $\xi \in \partial f(\widehat{s},\widehat{v})$, we deduce the existence of a measurable function $\xi_t = (r(t),p(t))$ such that

$$(r(t),p(t)) \in \partial L(t,\widehat{s}(t),\widehat{v}(t)) \ a.e.$$

(∂L denotes the generalized gradient with respect to (s,v)) and where, for any $(s,v)\in X$

$$< \xi,(s,v) >= \int_0^T < \xi_t,(s,v) > dt = \int_0^T [r(t)s(t)+p(t)v(t)]\,dt.$$

If $\xi = 0$ (as when F attains a local minimum at $\widehat{s}$), then $0\in \partial f(\widehat{s},\widehat{v})$, it hence follows easily ([15], Dubois-Reymond lemma) that $p(\cdot)$ is absolutely continuous and that $r = p'$ *a.e.* In this case, therefore, we have a nonsmooth version (generalized subgradient version) of the Euler-Lagrange equation

$$(p'(t),p(t))\in \partial L(t,\widehat{s}(t),\widehat{s}'(t))\ a.e. \tag{2.4}$$

For our problem, we assume the following notations throughout the paper: denoting by z the state variable, and by q the optimal value of said variable. We denote:

$$L(t,z,z') := \begin{cases} L^+(t,z,z') & if \quad z'\geq \phi(t,z), \\ L^-(t,z,z') & if \quad z'\leq \phi(t,z). \end{cases} \tag{2.5}$$

We shall use L_z^+ and $L_{z'}^+$ to represent the derivatives of L with respect to z and z', respectively, when $z'\geq\phi(t,z)$, and, similarly, L_z^- and $L_{z'}^-$ when $z'\leq\phi(t,z)$.

We define by $\Upsilon_q^1(t)$ the set of instants prior or equal to t, where $\phi(\cdot,q(\cdot))-q'(\cdot)$ is no longer null and the Lagrangian becomes smooth,

$$\Upsilon_q^1(t) := \left\{ s\ /\ 0\leq s\leq t\ \wedge\ \exists\delta>0\ /\ \left\{ \begin{array}{l} \forall\varepsilon\in[s-\delta,s), q'(\varepsilon)=\phi(\varepsilon,q(\varepsilon)) \\ \forall\varepsilon\in[s,s+\delta], q'(\varepsilon)\neq\phi(\varepsilon,q(\varepsilon)) \end{array} \right\} \right\}$$

and we denote by $N(t)$ its cardinal:

$$N(t) = card\left(\Upsilon_q^1(t)\right)$$

and its elements by

$$\Upsilon_q^1(t) = \{t_{2k}/1\leq k\leq N(t)\}.$$

We define by $\Gamma_q^1(t)$ the set of instants prior or equal to t, where $\phi(\cdot,q(\cdot))-q'(\cdot)$ becomes null; i.e. the Lagrangian becomes nonsmooth,

$$\Gamma_q^1(t) := \left\{ s\ /\ 0\leq s\leq t\ \wedge\ \exists\delta>0\ /\ \left\{ \begin{array}{l} \forall\varepsilon\in[s-\delta,s), q'(\varepsilon)\neq\phi(\varepsilon,q(\varepsilon)) \\ \forall\varepsilon\in[s,s+\delta], q'(\varepsilon)=\phi(\varepsilon,q(\varepsilon)) \end{array} \right\} \right\}$$

and we denote by $m(t)$ its cardinal:

$$m(t) = card\left(\Gamma_q^1(t)\right)$$

and its elements by

$$\Gamma_q^1(t) = \{t_{2k-1}/1\leq k\leq m(t)\}.$$

We shall assume that the set Ξ in (1.4) is not "active" at $t=0$ and $t=T$, the reasoning would be analogous in any other case. Moreover, there are no "transition points" where formally $t_{2k-1}=t_{2k}$ holds (a fact which will be demonstrated in the following section and which we shall call "smooth transition"). We hence have:

$$0<t_1<t_2<\ldots<t_{2k-1}<t_{2k}<\ldots<t_{2N(T)-1}<t_{2N(T)}<T.$$

We shall call times t_{2k-1} "entry-times" and times t_{2k} "exit-times". We define by $\Upsilon_q^2(t)$ the set formed by those intervals of continuity of $L_{z'}(s,q(s),q'(s))$ and $L_z(s,q(s),q'(s))$ lower than or equal to t, denoted by $t_0=0$,

$$\Upsilon_q^2(t) := \left\{ \bigcup_{i=0}^{N(t)} [t_{2i}, t_{2i+1}] \,/ t_{2i+1} \le t \right\}.$$

We denote by $\delta_q(t)$ the value of the integral of L_z in the areas of discontinuity it presents prior to t :

$$\delta_q(t) := \begin{cases} \sum_{i=1}^{N(t)} \int_{t_{2i-1}}^{t_{2i}} L_z^+(\tau,q(\tau),q'(\tau))d\tau & \text{if} \quad \begin{array}{l} \exists\mu>0 \ / \\ \forall\varepsilon\in(t_{2i},t_{2i}+\mu), q'(\varepsilon)>\phi(\varepsilon,q(\varepsilon)) \end{array} \\ \sum_{i=1}^{N(t)} \int_{t_{2i-1}}^{t_{2i}} L_z^-(\tau,q(\tau),q'(\tau))d\tau & \text{if} \quad \begin{array}{l} \exists\mu>0 \ / \\ \forall\varepsilon\in(t_{2i},t_{2i}+\mu), q'(\varepsilon)<\phi(\varepsilon,q(\varepsilon)) \end{array} \end{cases}$$

and by $¥_q^+(t)$ and $¥_q^-(t)$ the functions:

$$\begin{aligned} ¥_q^+(t) &:= -L_{z'}^+(t,q(t),q'(t)) + \int_{\Upsilon_q^2(t)} L_z(\tau,q(\tau),q'(\tau))d\tau + \\ &\quad + \delta_q(t) + \int_{t_{2N(t)}}^{t} L_z^+(\tau,q(\tau),q'(\tau))d\tau, \\ ¥_q^-(t) &:= -L_{z'}^-(t,q(t),q'(t)) + \int_{\Upsilon_q^2(t)} L_z(\tau,q(\tau),q'(\tau))d\tau + \\ &\quad + \delta_q(t) + \int_{t_{2N(t)}}^{t} L_z^-(\tau,q(\tau),q'(\tau))d\tau. \end{aligned} \tag{2.6}$$

With the hypotheses imposed on problem (2.1) at Section 2, which are verified in our hydraulic model, is easy to prove that $L_{z'}^+ > L_{z'}^-$ and $L_z^+ < L_z^-$ in Ξ. With the above definitions, we can prove the following result (necessary condition for minimum).

Theorem 2. *If q is a solution of* (2.1)*, then $\exists K\in\mathbb{R}^+$ such that:*

$$\begin{cases} ¥_q^+(t) = ¥_q^-(t) = K & \text{if} \quad q'(t)\neq\phi(t,q(t)), \\ ¥_q^+(t) \le K \le ¥_q^-(t) & \text{if} \quad q'(t)=\phi(t,q(t)). \end{cases} \tag{2.7}$$

Proof. It is easy to see that the functional (2.1) satisfies the necessary conditions to verify (2.3). Bearing in mind that the functions $L_z(t,\cdot,\cdot), L_{z'}(t,\cdot,\cdot)$ are discontinuous in Ξ, using (2.2), we have that the generalized gradient is

$$\partial L(t,q(t),q'(t)) = \begin{cases} (L_z, L_{z'}) & \text{if} \quad q'(t)\neq\phi(t,q(t)) \\ \left(L_z^-, L_{z'}^-\right) + u(t)(L_z^+ - L_z^-, L_{z'}^+ - L_{z'}^-) & \text{if} \quad q'(t)=\phi(t,q(t)) \end{cases}$$

for $0\le u(t)\le 1$, where $(t,q(t),q'(t))$ is the argument of the functions L_z^+, L_z^-, $L_{z'}^+$ and $L_{z'}^-$.

Hence, Equation (2.4) is

$$(p'(t),p(t)) \in \begin{cases} (L_z, L_{z'}) & \text{if} \quad q'(t)\neq\phi(t,q(t)) \\ \left((L_z^-, L_{z'}^-) + u(t)(L_z^+ - L_z^-, L_{z'}^+ - L_{z'}^-)\right) & \text{if} \quad q'(t)=\phi(t,q(t)) \end{cases} \quad a.e. \tag{2.8}$$

We shall first prove that the theorem is verified in an interval $[t_0, t_2]$, $t_0 = 0$, $t_2 \in \Upsilon_q^1(t)$ where the solution q of (2.1) reaches points in Ξ. We shall then generalize for $t \geq t_2$, where the solution shall be formed by a concatenation of arcs $q'(t) \neq \phi(t, q(t))$ and $q'(t) = \phi(t, q(t))$, like those already studied in $[t_0, t_2]$.

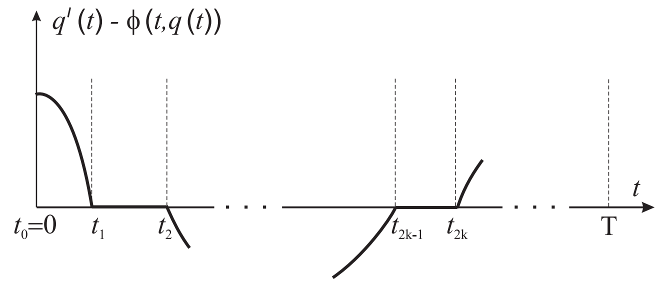

Fig. 2. Concatenating the extremal arcs and the boundary arcs.

Let us assume that $q'(t) > \phi(t, q(t))$ in $[0, t_1)$ and $q'(t) = \phi(t, q(t))$ for $t \in [t_1, t_2]$. We shall reason analogically in the case that $q'(t) < \phi(t, q(t))$ in $[0, t_1)$ and $q'(t) = \phi(t, q(t))$ for $t \in [t_1, t_2]$.

Hence, from (2.8), for $q'(t) = \phi(t, q(t))$, we have that:

$$\left\{ \begin{array}{l} p'(t) = u(t)L_z^+(t, q(t), q'(t)) + (1 - u(t))L_z^-(t, q(t), q'(t)) \\ p(t) = u(t)L_{z'}^+(t, q(t), q'(t)) + (1 - u(t))L_{z'}^-(t, q(t), q'(t)) \end{array} \right. ; 0 \leq u(t) \leq 1. \quad (2.9)$$

An expression that is also verified for $q'(t) \neq \phi(t, q(t))$ by simply bearing in mind (2.5) and (2.6), since:

$$\begin{array}{c} L_{z'}(t, z, z') = L_{z'}^+(t, z, z') = L_{z'}^-(t, z, z'), \\ L_z(t, z, z') = L_z^+(t, z, z') = L_z^-(t, z, z'). \end{array}$$

From (2.9), integrating $p'(t)$ in $[0, t], t \leq t_2$, we have that

$$\left\{ \begin{array}{l} p(t) - p(0) = \displaystyle\int_0^t (u(t)L_z^+(\tau, q(\tau), q'(\tau)) + (1 - u(t))L_z^-(\tau, q(\tau), q'(\tau)))\, d\tau, \\ -p(t) = -u(t)L_{z'}^+(t, q(t), q'(t)) - (1 - u(t))L_{z'}^-(t, q(t), q'(t)), \end{array} \right.$$

with $0 \leq u(t) \leq 1$. Summing, and denoting $-p(0) = K$, we have that

$$\begin{aligned} K = &-u(t)L_{z'}^+(t, q(t), q'(t)) - (1 - u(t))L_{z'}^-(t, q(t), q'(t)) + \\ &+ \int_0^t \left(u(t)L_z^+(\tau, q(\tau), q'(\tau)) + (1 - u(t))L_z^-(\tau, q(\tau), q'(\tau))\right) d\tau, \end{aligned}$$

with $0 \leq u(t) \leq 1$ and as $L_{z'}^+ > L_{z'}^-$ and $L_z^+ < L_z^-$, we can conclude that

$$\begin{aligned} &- L_{z'}^+(t, q(t), q'(t)) + \int_0^t L_z^+(\tau, q(\tau), q'(\tau))d\tau \leq K \leq \\ &\qquad\qquad \leq -L_{z'}^-(t, q(t), q'(t)) + \int_0^t L_z^-(\tau, q(\tau), q'(\tau))d\tau. \quad (2.10) \end{aligned}$$

i) If $0 \leq t \leq t_1, N(t) = 0$ and $q'(t) \neq \phi(t, q(t))$, then $L^+_{z'} \equiv L^-_{z'}, L^+_z \equiv L^-_z$ and from (2.10)

$$-L^+_{z'}(t, q(t), q'(t)) + \int_0^t L^+_z(\tau, q(\tau), q'(\tau))d\tau = K$$

$$K = -L^-_{z'}(t, q(t), q'(t)) + \int_0^t L^-_z(\tau, q(\tau), q'(\tau))d\tau,$$

from which

$$\yen^+_q(t) = \yen^-_q(t) = K.$$

ii) If $t < t_2, N(t) = 0$ and in such a case from (2.10), we have that

$$\yen^+_q(t) \leq K \leq \yen^-_q(t).$$

iii) If $t = t_2, t_2 \in \Upsilon^1_q(t_2), N(t_2) = 1$ and, in (2.10) an equality will be verified, for example

$$K = -L^-_{z'}(t_2, q(t_2), q'(t_2)) + \int_0^{t_2} L^-_z(\tau, q(\tau), q'(\tau))d\tau.$$

Moreover,

$$\begin{aligned}\int_0^{t_2} L^-_z(\tau, q(\tau), q'(\tau))d\tau &= \int_0^{t_1} L_z(\tau, q(\tau), q'(\tau))d\tau + \int_{t_1}^{t_2} L^-_z(\tau, q(\tau), q'(\tau))d\tau \\ &= \int_{\Upsilon^2_q(t_2)} L_z(\tau, q(\tau), q'(\tau))d\tau + \delta_q(t),\end{aligned}$$

from which

$$K = -L^-_{z'}(t_2, q(t_2), q'(t_2)) + \int_{\Upsilon^2_q(t_2)} L_z(\tau, q(\tau), q'(\tau))d\tau + \delta_q(t) = \yen^-_q(t_2). \quad (2.11)$$

iv) If $t \geq t_2, \forall s \in [t_2, t], q'(s) \neq \phi(s, q(s))$, we shall reiterate the process once more, taking t_2 as the initial point. For the concatenation of arcs, we must bear in mind that:

On the one hand, we are in a zone in which $q'(t) \neq \phi(t, q(t))$ $(L^+_{z'} \equiv L^-_{z'}, L^+_z \equiv L^-_z)$ and, hence, $\exists K^*$ such that

$$K^* = -L_{z'}(t, q(t), q'(t)) + \int_{t_2}^t L_z(\tau, q(\tau), q'(\tau))d\tau,$$

with

$$K^* = -L_{z'}(t_2, q(t_2), q'(t_2)).$$

On the other hand, from (2.11) it is verified that

$$K = -L^-_{z'}(t_2, q(t_2), q'(t_2)) + \int_{\Upsilon^2_q(t)} L_z(\tau, q(\tau), q'(\tau))d\tau + \delta_q(t),$$

from which

$$-L^-_{z'}(t_2, q(t_2), q'(t_2)) = K - \int_{\Upsilon^2_q(t)} L_z(\tau, q(\tau), q'(\tau))d\tau - \delta_q(t).$$

Therefore,

$$K^* = K - \int_{\Upsilon^2_q(t)} L_z(\tau, q(\tau), q'(\tau))d\tau - \delta_q(t).$$

Thus,

$$K-\int_{\Upsilon_q^2(t)} L_z(\tau,q(\tau),q'(\tau))d\tau-\delta_q(t)=-L_{z'}(t,q(t),q'(t))+\int_{t_2}^{t} L_z(\tau,q(\tau),q'(\tau))d\tau,$$

from which

$$K=-L_{z'}(t,q(t),q'(t))+\int_{\Upsilon_q^2(t)} L_z(\tau,q(\tau),q'(\tau))d\tau+\delta_q(t)+\int_{t_2}^{t} L_z(\tau,q(\tau),q'(\tau))d\tau,$$

and in such a case, once more

$$K=\yen_q^+(t)=\yen_q^-(t).$$

Once the arcs of the extremal have been concatenated in t_2, the theorem is demonstrated, simply be bearing in mind that only situations like those already analyzed above may arise. □

3. **Smooth transition.** In this section, we present a qualitative aspect of the solution of (2.1). We prove that, under certain conditions, the discontinuity of the derivative of the Lagrangian does not translate as discontinuity in the derivative of the solution. In fact, it is verified that the derivative of the extremal where the minimum is reached presents an interval where $z'=\phi(t,z)$ is verified. The character C^1 of the solution is thus guaranteed.

Theorem 3. *Let $L(\cdot,\cdot,\cdot)$ be the Lagrangian of the functional F in the conditions stated above, and let us assume that the function $L_{z'}(t_0,z(t_0),\cdot)$ is strictly increasing and discontinuous in $\phi(t_0,z(t_0))$. If q is minimum for F, and $q'(t_0)=\phi(t_0,q(t_0))$, then $q'(t)=\phi(t,q(t))$ in some interval that contains t_0 and q' is continuous in t_0.*

Proof. Let us see, first, that $q'(t)=\phi(t,q(t))$ in some interval that contains t_0. We shall proceed by contradiction.

Let $q\in\Theta$ be a minimum of F, and let us first assume that for $t_0\in(0,T)$ there exist $\varepsilon>0$ such that:

$$q'(t)<\phi(t,q(t)),\ \forall t\in(t_0-\varepsilon,t_0),$$
$$q'(t)>\phi(t,q(t)),\ \forall t\in(t_0,t_0+\varepsilon).$$

The strict growth of $L_{z'}$, implies that:

$$L_{z'}(t,q(t),q'(t))<L_{z'}^-(t_0,q(t_0),\phi(t_0,q(t_0))),\ \forall t\in(t_0-\varepsilon,t_0)$$
$$L_{z'}^+(t_0,q(t_0),\phi(t_0,q(t_0)))<L_{z'}(\widetilde{t},q(\widetilde{t}),q'(\widetilde{t})),\ \forall\widetilde{t}\in(t_0,t_0+\varepsilon)$$

and, together with the discontinuity of $L_{z'}$, we have:

$$\begin{aligned}&L_{z'}(t,q(t),q'(t))<L_{z'}^-(t_0,q(t_0),\phi(t_0,q(t_0)))<\\&<L_{z'}^+(t_0,q(t_0),\phi(t_0,q(t_0)))<L_{z'}(\widetilde{t},q(\widetilde{t}),q'(\widetilde{t})),\forall t\in(t_0-\varepsilon,t_0),\forall\widetilde{t}\in(t_0,t_0+\varepsilon).\end{aligned}\tag{3.1}$$

We consider the auxiliary function $h_\varepsilon^{t_0}$ defined on $[0,T]$:

$$h_\varepsilon^{t_0}(t):=\begin{cases}0 & \text{if}\quad t\in[0,t_0-\varepsilon]\cup[t_0+\varepsilon,T],\\(t-t_0+\varepsilon) & \text{if}\quad t\in[t_0-\varepsilon,t_0],\\-(t-t_0-\varepsilon) & \text{if}\quad t\in[t_0,t_0+\varepsilon].\end{cases}$$

Notice that $h_\varepsilon^{t_0} \in \widehat{C}^1[0,T]$, $0 \le h_\varepsilon^{t_0}(t) \le \varepsilon$, $\forall t \in [0,T]$, and

$$(h_\varepsilon^{t_0})'(t) = \begin{cases} 0 & \text{if} \quad t \in [0, t_0-\varepsilon) \cup (t_0+\varepsilon, T], \\ 1 & \text{if} \quad t \in (t_0-\varepsilon, t_0), \\ -1 & \text{if} \quad t \in (t_0, t_0+\varepsilon). \end{cases}$$

It is evident of (3.1) that we may choose the previous ε sufficiently small for the following inequality to be verified:

$$\sup_{t\in(t_0-\varepsilon,t_0)} \left[L_{z'}(t,q(t),q'(t)) + h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))\right] <$$
$$< \inf_{t\in(t_0,t_0+\varepsilon)} \left[L_{z'}(t,q(t),q'(t)) - h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))\right],$$

from which the following inequalities are deduced:

$$\begin{aligned} I_1 = \int_{t_0-\varepsilon}^{t_0} &[L_{z'}(t,q(t),q'(t)) + h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]dt \le \\ &\le \varepsilon \cdot \sup_{t\in(t_0-\varepsilon,t_0)} \left[L_{z'}(t,q(t),q'(t)) + h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))\right] < \\ &< \varepsilon \cdot \inf_{t\in(t_0,t_0+\varepsilon)} \left[L_{z'}(t,q(t),q'(t)) - h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))\right] \le \\ &\qquad \le \int_{t_0}^{t_0+\varepsilon} [L_{z'}(t,q(t),q'(t)) - h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]dt = I_2. \end{aligned}$$

Let us now take into account that

$$h_\varepsilon^{t_0}(t) = 0,\ \forall t \in [0,t_0-\varepsilon] \cup [t_0+\varepsilon, T];\ (h_\varepsilon^{t_0})'(t) = 0,\ \forall t \in [0,t_0-\varepsilon) \cup (t_0+\varepsilon, T],$$

then

$$\begin{aligned} \delta^+ F(q, h_\varepsilon^{t_0}) &:= \lim_{x\to 0^+} \frac{F(q + x h_\varepsilon^{t_0}) - F(q)}{x} \\ &= \int_{t_0-\varepsilon}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + (h_\varepsilon^{t_0})'(t)\cdot L_{z'}(t,q(t),q'(t))]dt, \end{aligned}$$

and hence

$$\begin{aligned} \delta^+ F(q, h_\varepsilon^{t_0}) = \int_{t_0-\varepsilon}^{t_0} &[h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + 1\cdot L_{z'}(t,q(t),q'(t))]dt + \\ &+ \int_{t_0}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t)) + (-1)\cdot L_{z'}(t,q(t),q'(t))]dt, \end{aligned}$$

and we have that

$$\begin{aligned} \delta^+ F(q, h_\varepsilon^{t_0}) = \int_{t_0-\varepsilon}^{t_0} &[L_{z'}(t,q(t),q'(t)) + h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]dt \\ &- \int_{t_0}^{t_0+\varepsilon} [L_{z'}(t,q(t),q'(t)) - h_\varepsilon^{t_0}(t)\cdot L_z(t,q(t),q'(t))]dt = I_1 - I_2 < 0, \end{aligned}$$

which contradicts the assumption that q is a minimum of F.

If $q'(t) > \phi(t,q(t))$ in some interval to the left of t_0 and $q'(t) < \phi(t,q(t))$ in some interval to the right of t_0, the proof would be analogous, taking $\delta^+ F(q, -h_\varepsilon^{t_0})$.

We therefore conclude that $q'(t) = \phi(t, q(t))$ in some interval that contains t_0.

Let us also see that q' is continuous at t_0. We shall proceed by contradiction.

Let us assume that $q'(t_0^-) < q'(t_0^+)$ (if we assume that $q'(t_0^-) > q'(t_0^+)$, the proof will be analogous). Bearing in mind what was demonstrated above, q' is discontinuous at t_0 only in the following cases:

$$\begin{array}{l}(a)\ q'(t_0^-) < \phi(t_0, q(t_0)); \quad q'(t_0^+) = \phi(t_0, q(t_0)),\\ (b)\ q'(t_0^-) = \phi(t_0, q(t_0)); \quad q'(t_0^+) > \phi(t_0, q(t_0)).\end{array}$$

For (a), there will exist an $\varepsilon > 0$ such that $q'(x) = \phi(x, q(x))$ at $[t_0, t_0 + \varepsilon]$. We may choose ε such that $q'(x) < \phi(x, q(x))$ at $[t_0 - \varepsilon, t_0)$. In this case

$$\begin{aligned}\delta^+ F(q, h_\varepsilon^{t_0}) = &\int_{t_0-\varepsilon}^{t_0} [h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) + (h_\varepsilon^{t_0})'(t) \cdot L_{z'}(t, q(t), q'(t))]dt+\\ &+ \int_{t_0}^{t_0+\varepsilon} [h_\varepsilon^{t_0}(t) \cdot L_z^-(t, q(t), q'(t)) + (h_\varepsilon^{t_0})'(t) \cdot L_{z'}^-(t, q(t), q'(t))]dt\end{aligned}$$

and, by identical reasoning to that used above, we shall have that $\delta^+ F(q, h_\varepsilon^{t_0}) < 0$, which once more means a contradiction of the fact that q is a minimum of F.

Finally, for (b), there will exist an $\varepsilon > 0$ such that $q'(x) = \phi(x, q(x))$ at $[t_0 - \varepsilon, t_0]$. We may choose ε such that $q'(x) > \phi(x, q(x))$ at $(t_0, t_0 + \varepsilon]$. In this case,

$$\begin{aligned}\delta^+ F(q, -h_\varepsilon^{t_0}) = &\int_{t_0-\varepsilon}^{t_0} [-h_\varepsilon^{t_0}(t) \cdot L_z^+(t, q(t), q'(t)) - (h_\varepsilon^{t_0})'(t) \cdot L_{z'}^+(t, q(t), q'(t))]dt+\\ &+ \int_{t_0}^{t_0+\varepsilon} [-h_\varepsilon^{t_0}(t) \cdot L_z(t, q(t), q'(t)) - (h_\varepsilon^{t_0})'(t) \cdot L_{z'}(t, q(t), q'(t))]dt,\end{aligned}$$

where, by identical reasoning to that used in (a), we shall once more have the contradiction

$$\delta^+ F(q, -h_\varepsilon^{t_0}) < 0.$$

□

This result has a very clear interpretation: under optimum operating conditions, thermal plants never switch brusquely from one generating power zone to another, but rather carry out a smooth transition, remaining on the boundary $q'(t) \equiv \phi(t, q(t))$ for a certain interval.

4. **Application to a hydrothermal problem.** A program that solves the optimization problem was elaborated using the Mathematica package. The Optimization Algorithm, briefly described below, is very similar to the algorithm that we present in [4]. The solution to the problem consists in finding for each K the function q_K that satisfies the conditions of Theorem 2 and, from among these functions, an admissible function $q_K \in \Theta$. Theorem 2 allows the extremals q_K to be constructed in a simple way:

Stage 1) For each K we construct q_K, where q_K satisfies the conditions of Theorem 2 and the initial condition $q_K(0) = 0$. In general, the construction of q'_K cannot be carried out all at once over all the interval $[0, T]$. The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs ($q'(t) \neq \phi(t, q(t))$) and boundary arcs ($q'(t) = \phi(t, q(t))$) until completing the interval $[0, T]$. This is relatively simple to implement using a discretized version of Equations (2.7).

To calculate the exit instant of each smooth transition zone, we shall verify the conditions in Theorem 2. That is, the exit time will be the first value of t in which the following condition is no longer fulfilled for K:

$$¥_q^+(t) \leq K \leq ¥_q^-(t).$$

Stage 2) K is calculated such that $q_K \in \Theta$. The procedure is similar to the shooting method used to resolve second-order differential equations with boundary conditions. Effectively, we may consider the function $\varphi(K) := q_K(T)$ and calculate the root of $\varphi(K) - b = 0$, which may be realized approximately using elemental procedures like the secant method.

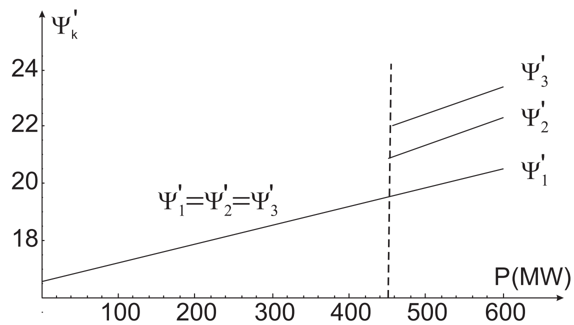

Fig. 3. Slopes of the cost curves of the thermal plant.

The program was applied to an example of a hydrothermal system made up of one thermal plant and one hydroplant. We shall analyze the behavior of the thermal plant assuming the existence of a valve point ($P_1 = 450$). This study may be easily extended to the remaining valve points.

To improve the quality of the study, we shall analyze three cases (Fig. 3): In the first case, we shall assume that the valve point does not suppose a discontinuity in the derivative Ψ'. In the second, we shall assume that it implies a moderate jump in the slope, and in the third, a more pronounced jump. The cost function in each case is:

$$\text{Case 1: } \Psi_1(P) = \ 10696.0977 + 16.5477P + 0.003299P^2, \quad P_{\min} \leq P < P_{\max}$$

$$\text{Case 2: } \Psi_2(P) = \begin{cases} 10696.0977 + 16.5477P + 0.003299P^2, & P_{\min} \leq P \leq 450 \\ 10365.3252 + 16.6496P + 0.004706P^2, & 450 < P < P_{\max} \end{cases}$$

$$\text{Case 3: } \Psi_3(P) = \begin{cases} 10696.0977 + 16.5477P + 0.003299P^2, & P_{\min} \leq P \leq 450 \\ 9897.21269 + 17.6678P + 0.004755P^2, & 450 < P < P_{\max} \end{cases}$$

with $P_{\min} = 0$; $P_{\max} = 600$.

For the power production of the hydroplant (variable-head), we consider a function of $z(t)$ and $z'(t)$ defined as

$$P_h(t, z(t), z'(t)) := A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t),$$

with

$$A(t) = \frac{B_y}{G}(S_0 + t \cdot i), B = \frac{B_y}{G}.$$

So, the function of effective hydraulic generation is

$$H(t, z(t), z'(t)) := P_h(t, z(t), z'(t)) - b_{ll} P_h^2(t, z(t), z'(t)),$$

where $b_{ll} = 0.000220$ $(1/MW)$ is the loss coefficient. The values for the coefficients of the hydroplant are: the efficiency $G = 526315$ $(m^4/h.MW)$, the restriction on the volume $b = 941.6 \cdot 10^5$ (m^3), the natural inflow $i = 101.952 \cdot 10^5$ (m^3/h), the initial volume $S_0 = 203.904 \cdot 10^9$ (m^3) and the coefficient $B_y = 149.510^{-12}$ (m^{-2}) (a parameter that depends on the geometry of the tanks).

Next, we present the optimal solution. Table 1 shows the power demand $P_d(t)$ (MW), and the optimal solution, i.e. the state variable: $q(t)(m^3 \cdot 10^6)$, and the optimal thermal power: $P(t)(MW)$ for the three cases. Although the discretization used comprised 48 instants for the optimization interval $[0, 24]$, for the sake of brevity we present only the integer values of $t(h)$. The objective value is 448699 $(Euros)$ in Case1, 448970 $(Euros)$ in Case 2, and 449339 $(Euros)$ in Case 3.

Table I. Optimal Solution and Power Demand.

		Case	1	Case	2	Case	3
t	P_d	q	P	q	P	q	P
0	1980	0	455.776	0	450.	0	450.
1	1916	3.6972	416.407	3.1952	450.	3.1952	450.
2	1871	7.0746	389.072	5.6752	440.572	5.5889	450.
3	1739	9.6070	310.749	7.2546	363.322	6.5251	399.559
4	1688	11.5648	281.154	8.2633	334.122	6.8933	370.616
5	1681	13.3826	277.071	9.1330	330.09	7.1233	366.619
6	1765	15.6948	325.782	10.4938	378.142	7.8419	414.24
7	1795	18.3556	343.400	12.2010	395.516	8.9054	431.454
8	1847	21.3486	374.301	14.2384	425.988	10.4027	450.
9	1930	24.9080	424.499	17.0774	450.	13.2328	450.
10	2024	29.1617	482.595	21.6465	455.497	17.8562	450.
11	2060	33.8033	505.155	26.8617	475.791	23.4603	456.422
12	2022	38.2901	481.232	31.8924	454.277	28.7071	450.
13	1989	42.4978	460.62	36.3819	450.	33.1966	450.
14	2015	46.7903	476.746	41.0932	450.256	37.9103	450.
15	2039	51.2823	491.716	46.1291	463.702	43.1624	450.
16	2034	55.7887	488.523	51.1818	460.833	48.4509	450.
17	2040	60.3068	492.232	56.2480	464.17	53.7687	450.
18	2074	65.0220	513.613	61.5484	483.431	59.4563	464.257
19	2116	70.0490	540.272	67.2193	507.533	65.5073	488.904
20	2084	74.9751	519.840	72.7700	489.055	71.4404	470.011
21	2082	79.8099	518.516	78.2116	487.861	77.2666	468.791
22	2113	84.8279	538.191	83.8709	505.654	83.3063	486.985
23	2090	89.8056	523.484	89.4819	492.352	89.2987	473.386
24	1980	94.1600	454.422	94.1600	450.	94.1600	450.

In Case 1, there is no smooth transition; hence K has the same value in all the instants: $-p(0) = K = 0.00101472651608$. In Cases 2 and 3, a smooth transition does occur. In the free instants, K takes the values: $K = 0.00106026202804$ and $K = 0.00109182791977$ in Case 2 and Case 3, respectively. In the smooth transition instants, the value of K remains within the interval $\left[\yen_q^+(t), \yen_q^-(t)\right]$. To see this in more detail, in Table II we present the entry and exit times and the value of K and the values of $\yen_q^+(t)$, and $\yen_q^-(t)$,which, as can be easily proven, verify Theorem 2.

Table II. Entry and Exit times. The smooth transition.

t	Case 2 $\left[\yen_q^+(t), \yen_q^-(t)\right]$	Case 3 $\left[\yen_q^+(t), \yen_q^-(t)\right]$
0	[0.0010095373118, 0.0010803096183]	[0.0010095373118, 0.00113525876399]
1	[0.0010446241595, 0.0011178550438]	[0.0010446241595, 0.00117471308241]
2	0.00106026202804	[0.0010686245231, 0.00120170075024]
3	0.00106026202804	0.00109182791977
4	0.00106026202804	0.00109182791977
5	0.00106026202804	0.00109182791977
6	0.00106026202804	0.00109182791977
7	0.00106026202804	0.00109182791977
8	0.00106026202804	[0.001081504409, 0.00121618141338]
9	[0.00103747734733, 0.001110202169]	[0.0010374776647, 0.0011666700436]
10	0.00106026202804	[0.00098522584484, 0.001107908297]
11	0.00106026202804	0.00109182791977
12	0.00106026202804	[0.0009864688517, 0.0011092992284]
13	[0.0010051729880, 0.0010756262490]	[0.0010051707305, 0.0011303273776]
14	0.00106026202804	[0.0009905539685, 0.0011138874728]
15	0.00106026202804	[0.0009768688274, 0.0010984949299]
16	0.00106026202804	[0.0009797949547, 0.0011017823000]
17	0.00106026202804	[0.0009763900265, 0.0010979501699]
18	0.00106026202804	0.00109182791977
19	0.00106026202804	0.00109182791977
20	0.00106026202804	0.00109182791977
21	0.00106026202804	0.00109182791977
22	0.00106026202804	0.00109182791977
23	0.00106026202804	0.00109182791977
24	[0.0010107682640, 0.0010815928155]	[0.0010107639988, 0.0011365782603]

The graphs of the optimal solution can be seen in Figs. 4 and 5. Note that,

as Theorem 3 guarantees, when the valve point supposes a discontinuity in Ψ', the optimum thermal power presents a constant interval. Furthermore, it can be appreciated (Fig. 4) that the more pronounced the jump in the derivative, the more prolonged this interval will be, being inexistent in Case 1 (continuous Ψ').

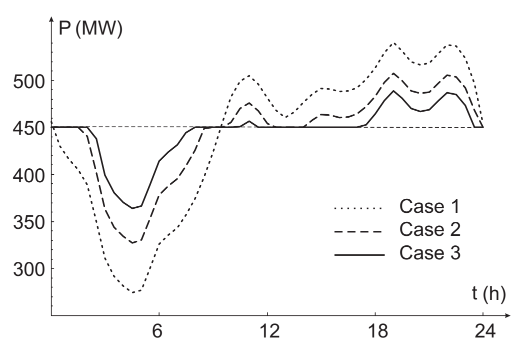

Fig. 4. Optimal Thermal Power.

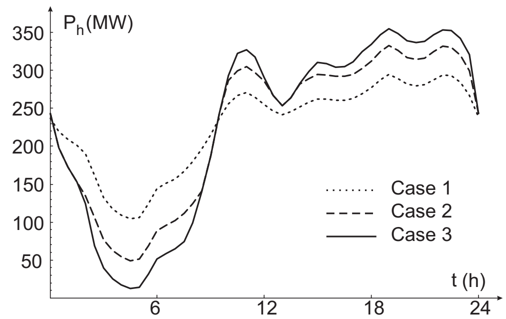

Fig. 5. Optimal Hydro-Power.

An optimization interval of $T = 24$ h. was considered, with a discretization of $24 \cdot 2$ subintervals. The secant method was used to calculate the approximate value of K for which $q_K(T) - b = 0$. In 4 iterations: $|q_K(T) - b| < 10^{-2}(m^3)$. The CPU time employed was 12.0 sec.

5. **Conclusions.** This paper proposes a new approach for solving economic dispatch problems with valve-point effects. The ED problem with valve-point effects is represented as a nonsmooth optimization problem in which the cost function for each thermal plant is approximately represented by several piecewise quadratic functions. We establish a necessary minimum condition for this problem with nonsmooth Lagrangian, employing nonsmooth analysis for this purpose. Furthermore, we present a qualitative aspect of the solution, the smooth transition: under optimum operating conditions, thermal plants never switch brusquely from one generating power zone to another, remaining above the boundary for a certain interval. The feasibility of our method was illustrated by a hydro-thermal system and we compared the results obtained using three different valve-point effects for the thermal plant.

REFERENCES

[1] N. Amjady and H. Nasiri-Rad, Solution of nonconvex and nonsmooth economic dispatch by a new Adaptive Real Coded Genetic Algorithm, *Expert Syst. Appl.,* **37** (2010), 5239–5245.

[2] L. Bayón, J. M. Grau, M. M. Ruiz and P. M. Suárez, Nonsmooth Optimization of Hydrothermal Problems, *J. Comput. Appl. Math.,* **192** (2006), 11–19.

[3] L. Bayón, J. M. Grau, M. M. Ruiz and P. M. Suárez, An optimization problem in deregulated electricity markets solved with the nonsmooth maximum principle, *Int. J. Comput. Math.,* **86** (2009), 237–249.

[4] L. Bayón, J. Grau, M. M. Ruiz and P.M. Suárez, A Constrained and Nonsmooth Hydrothermal Problem, *Appl. Math. Comput.,* **209** (2009), 10–18.

[5] L. Bayón, J. M. Grau, M. M. Ruiz and P. M. Suárez, Algorithm for calculating the analytic solution for economic dispatch with multiple fuel units, *Comput. Math. Appl.,* **62** (2011), 2225–2234.

[6] C. L. Chiang, Genetic algorithm for static power economic dispatch, *Computer Science and Information Engineering,* 2009 WRI World Congress (2009), 646–650.

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.

[8] L. S. Coelho and V. C. Mariani, Combining of chaotic differential evolution and quadratic programming for economic dispatch optimization with valve-point effect, *IEEE Trans. Power Syst.,* **21** (2006), 989–996.

[9] A. D. Ioffe and R. T. Rockafellar, The Euler and Weierstrass conditions for nonsmooth variational problems, *Calc. Var. Partial Dif.,* **4** (1996), 59–87.

[10] D. Liu and Y. Cai, Taguchi method for solving the economic dispatch problem with non-Smooth cost functions, *IEEE Trans. Power Syst.,* **20** (2005), 2006–2014.

[11] P. D. Loewen and R. T. Rockafellar, New necessary conditions for the generalized problem of Bolza, *SIAM J. Control Optim.,* **34** (1996), 1496–1511.

[12] C. Marcelli, Variational problems with nonconvex, noncoercive, highly discontinuous integrands: characterization and existence of minimizers, *SIAM J. Control Optim.,* **40** (2002), 1473–1490.

[13] C. Marcelli, E. Outkine and M. Sytchev, Remarks on necessary conditions for minimizers of one-dimensional variational problems, *Nonlinear Anal.,* **48** (2002), 979–993.

[14] J. B. Park, K. S. Lee, J. R. Shin and K. Y. Lee, A particle swarm optimization for economic dispatch with nonsmooth cost functions, *IEEE Trans. Power Syst.,* **20** (2005), 34–42.

[15] J. L. Troutman, *Variational Calculus with Elementary Convexity*, Springer, New York, 1983.

[16] M. T. Tsai, H. J. Gow and W. M. Lin, A novel stochastic search method for the solution of economic dispatch problems with non-convex fuel cost functions, *Int. J. Elec. Power,* **33** (2011), 1070–1076.

[17] R. Vinter and H. Zheng, The extended Euler-Lagrange condition for nonconvex variational problems, *SIAM J. Control Optim.,* **35** (1997), 56–77.

[18] A. J. Wood and B. F. Wollenberg, *Power Generation, Operation, and Control*, Wiley-Interscience, 1996.

[19] X. Yuan, L. Wang, Y. Zhang and Y. Yuan, A hybrid differential evolution method for dynamic economic dispatch with valve-point effects, *Expert Syst. Appl.,* **36** (2009), 4042–4048.

[20] K. Zare, M. T. Haque and E. Davoodi, Solving non-convex economic dispatch problem with valve point effects using modified group search optimizer method, *Electr. Pow. Syst. Res.,* **84** (2012), 83–89.

Received September 2012; 1st revision April 2013; 2nd revision June 2013.

E-mail address: bayon@uniovi.es
E-mail address: grau@uniovi.es
E-mail address: mruiz@uniovi.es
E-mail address: pedrosr@uniovi.es