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Cyclic coordinate descent in a class of bang–singular–bang problems

L. Bayón *, P. Fortuny Ayuso, J.A. Otero, P.M. Suárez, C. Tasis

University of Oviedo, Department of Mathematics, E.P.I. Campus of Viesques, Gijón, 33203, Spain

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ABSTRACT

In this paper, we study a class of optimal control problems inspired by the hydroelectric context. These problems model a continuous production process with several interrelated pre-set availability inputs in a finite time interval with production functions which are linear with respect to the consumption rate over time.

It constitutes a bang–singular–bang control problem, which we solve using a cyclic coordinate descent strategy combined with a suitable adaptation of the shooting method. Finally, the proposed algorithm is implemented using the Mathematica package and applied to a hydraulic optimization problem in which the potential of the algorithm is evidenced.

1. Introduction

If we consider a continuous process of production in a time interval, $[0, T]$, whose output is sold on the market at a variable price, $p(t)$ and the production function is linear with respect to the consumption rate over time of the inputs, which are of limited availability, we have posed a problem of maximization of the following functional:

$$\int_0^T p(t)(f_1(t)z_1'(t) + \cdots + f_n(t)z_n'(t))dt$$

with $z_i \in (\widehat{C}^1[0, T]) \mid z_i(0) = 0, z_i(T) = b_i$, where technical constraints are considered for the consumption rate over time: $m_i \leq \dot{z}_i(t) \leq M_i, \ \forall i = 1, \ldots, n$. Recall that $\hat{C}^1[0, T]$ is the set of piecewise C^1 functions defined on $[0, T]$.

The solution to this problem for the one-dimensional case is relatively simple (see, for example, [1]). It basically consists in determining an input level above which the input should be consumed at its maximum rate and below which it should be consumed at its minimum rate, such that the total available input is used in the interval $[0, T]$. There is a wide variety of different types of problems (chemical, economic, electrical, etc.) that respond to these approaches. [2,3] solve a control problem to calculate the optimal enzyme concentrations in a chemical process by considering the minimization of the transition time. [1] put forward an algorithm to solve an optimal control problem that arises when a hydraulic system with fixed-head hydro-plants is considered. [4] in turn present a numerical scheme for computing optimal bang–bang controls and the computational technique is illustrated via three example applications: the Rayleigh problem, a batch reactor and the control of two-link robots.

* Corresponding author.
E-mail address: bayon@uniovi.es (L. Bayón).

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It is natural to ask what happens if the inputs involved in the production process are interrelated in such a way that the profits obtained from the produced output not only come from its market price, but also from the efficiency of the production process, which depends on the stock of various inputs. This is what occurs, for instance, in the process of generating electricity at several hydro-plants, each of which produces at a level of efficiency that depends on the water consumed by the plant itself and by those situated upstream in the same basin due to the influencing level of water in the reservoir of said plant l.

In this paper, we abstract from this situation to pose a general problem which studies a hypothetical production process in which the efficiency of the production process with respect to each input is a function that depends on time, on the stock of the various inputs, and even on the consumption rate of other inputs. As explained in Section 2, our model is quadratic in consumption rates and stocks, with an added requirement: that it is linear on both parameters in each coordinate (i.e. there are no quadratic terms in a single variable); this condition will be expressed imposing that the diagonal of some matrices describing the model are zero. This will imply that it is a bang–singular–bang optimal control problem. We propose an efficient method for finding the bang–singular–bang solution using a cyclic coordinate descent strategy combined with a suitable adaptation of the shooting method.

Different methods for determining optimal controls with a possibly singular part have already been developed. A popular approach [5–7] involves solving the singular/bang–bang optimal control problem as the limit of a series of nonsingular problems. It is important to establish the limitations of these perturbation-based methods for practical problems. In fact, the convergence criterion described in [5] requires that the perturbation parameter, ε, be sufficiently small; however, numerical difficulties result when ε approaches a zero limit. In the context of the known induced optimization problem, [8] presents an improvement and formulates the problem as a new finite-dimensional optimization problem involving the initial states, the switching times and the final time, t_f, as optimization variables, but with the limitation of assuming that the optimal control structure is known. The same limitation exists in [9] when considering the optimal control problem with bound constraints. These authors assume that the structure of the concatenation of bang and singular arcs of the optimal solution and an approximation of its switching times are known. Hence an initial guess of the solution must be obtained.

In this paper, an adaptation of the classic shooting method is used to compute the solution of the stated optimal control problem [10,9] for the unidimensional case. We shall thus solve the corresponding boundary value problem derived from Pontryagin's Maximum Principle without any initial guess regarding the structure of the solution.

Furthermore, we use the coordinate descent method to address the multidimensional case. The convergence of the iterates generated by this classic method has not been widely studied. Convergence typically requires restrictive assumptions such assuming that the cost function has bounded level sets and is in some sense strictly convex [11,12]. In a previous paper [13], we presented an application of the algorithm of the cyclic coordinate descent in multidimensional variational problems with constrained speed and proved its convergence under weak assumptions. In fact [13] will be used as a reference to ensure the convergence of the algorithm put forward in the present paper.

The combination of all the stated techniques (Pontryagin's maximum principle, the theory of singular control, the shooting method and cyclic coordinate descent) provides the theoretical basis that has enabled us to construct an algorithm for solving the problem approximately and, in some cases, even analytically. As we shall see, the mathematical framework of application of the theory presented here is very broad, including a very general class of functional.

The paper is organized as follows. In Section 2 we present the statement of the multidimensional variational problem. In Section 3 the unidimensional variational problem is analyzed in detail: a maximum necessary and sufficient condition is proved; the possible presence of singular arcs and their calculation (should they exist) is analyzed and the construction of the solution (via adaptation of the shooting method) is presented. After completing the one-dimensional case, the cyclic coordinate descent algorithm for solving the multidimensional case is presented in Section 4. In Section 5, the proposed algorithm is applied to a hydraulic optimization problem in which the potential of the algorithm is evidenced. Furthermore, we shall provide a theoretical example to illustrate the behavior of the algorithm in the presence of singular arcs. Finally, Section 6 summarizes the main conclusions and outlines future prospects.

2. Statement of the multidimensional variational problem

We consider the problem of maximizing the multidimensional functional

$$J(\mathbf{z}) = \int_0^T L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))dt$$

on $\mathbb{D} := \prod_{i=1}^m \mathbb{D}_i$, where $\mathbf{z}(t) = (z_1(t), \ldots, z_m(t))$ and

$$\mathbb{D}_i := \{z_i \in (\widehat{C}^1[0,T]) \mid z_i(0) = 0, z_i(T) = b_i \text{ and } m_i \le \dot{z}_i(t) \le M_i\} \neq \varnothing,$$

assuming that L depends as follows on $\mathbf{z}$ and $\dot{\mathbf{z}}$:

$$\frac{\partial^2 L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{\partial z_i \partial z_j} = a_{ij}(t), \qquad \frac{\partial^2 L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{\partial z_j \partial \dot{z}_i} = b_{ij}(t), \qquad \frac{\partial^2 L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))}{\partial \dot{z}_j \partial \dot{z}_i} = c_{ij}(t)$$

and

$$\frac{\partial^2 L(t,\mathbf{z}(t),\dot{\mathbf{z}}(t))}{\partial z_i \partial \dot{z}_i} = p_i, \qquad \frac{\partial^2 L(t,\mathbf{z}(t),\dot{\mathbf{z}}(t))}{\partial \dot{z}_i \partial \dot{z}_i} = 0.$$

This is equivalent to saying that $L(t,\mathbf{z}(t),\dot{\mathbf{z}}(t))$ can be written as

$$L(t,\mathbf{z}(t),\dot{\mathbf{z}}(t)) = \mathbf{z}^t A(t)\mathbf{z} + \mathbf{z}^t B(t)\dot{\mathbf{z}} + \dot{\mathbf{z}}^t C(t)\dot{\mathbf{z}} + \mathbf{s}(t)^t \cdot \dot{\mathbf{z}} + \mathbf{z}^t P\dot{\mathbf{z}} + \mathbf{r}(t)^t \cdot \mathbf{z}$$

where $A(t)$, $B(t)$ and $C(t)$ symmetric are matrices of order m with 0 on the main diagonal, P is a matrix of order m with constant p_i on the main diagonal and 0 at other places, and $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are vectors of dimension m. We shall assume throughout the paper that $a_{ij}(t)$, $b_{ij}(t)$, $c_{ij}(t)$, $s_i(t)$ and $r_i(t)$ are continuous functions.

Hence, we can write

$$\begin{aligned} L(t,\mathbf{z}(t),\dot{\mathbf{z}}(t)) \;=\; & \sum_{\substack{i,j=1\\ i\neq j}}^{m} a_{ij}(t)z_i(t)z_j(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{m} b_{ij}(t)z_i(t)\dot{z}_j(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{m} c_{ij}(t)\dot{z}_i(t)\dot{z}_j(t) \\ & + \sum_{i=1}^{m} s_i(t)\dot{z}_i(t) + \sum_{i=1}^{m} p_i z_i(t)\dot{z}_i(t) + \sum_{i=1}^{m} r_i(t)z_i(t). \end{aligned}$$

3. Statement of the unidimensional variational problem

We shall present an algorithm for solving the general problem by means of tackling the one-dimensional version and performing a cyclic iteration. Thus, what we need first is to solve the one-dimensional problem. To this end, assume that all the components of $\mathbf{z}$ and $\dot{\mathbf{z}}$ are fixed but the ith one. Let $\mathbf{q} = (q_1,\ldots,q_m) \in \mathbb{D}$ and write

$$L^i_{\mathbf{q}}(t,z_i,\dot{z}_i) := L(t,q_1(t),\ldots,q_{i-1}(t),z_i,q_{i+1}(t),\ldots,q_m(t),\dot{q}_1(t),\ldots,\dot{z}_i,\ldots,\dot{q}_m(t)).$$

We wish to solve the problem of maximizing the functional $J^i_{\mathbf{q}} : \mathbb{D}_i \longrightarrow \mathbb{R}$

$$J^i_{\mathbf{q}}(z_i) := J(q_1,\ldots,q_{i-1},z_i,q_{i+1},\ldots,q_m) = \int_0^T L^i_{\mathbf{q}}(t,z_i(t),\dot{z}_i(t))dt \tag{1}$$

on

$$\mathbb{D}_i := \{z_i \in (\widehat{C}^1[0,T]) \mid z_i(0)=0, z_i(T)=b_i \text{ and } m_i \le \dot{z}_i(t) \le M_i\}$$

where one can write:

$$L^i_{\mathbf{q}}(t,z_i,\dot{z}_i) = F^i_{\mathbf{q}}(t) + G^i_{\mathbf{q}}(t)z_i(t) + \left(H^i_{\mathbf{q}}(t) + p_i z_i(t)\right)\dot{z}_i(t)$$

with

$$\begin{aligned} F^i_{\mathbf{q}}(t) \;=\; & \sum_{\substack{k,j=1\\ k,j\neq i}}^{m} a_{kj}(t)q_k(t)q_j(t) + \sum_{\substack{k,j=1\\ k,j\neq i}}^{m} b_{kj}(t)q_k(t)\dot{q}_j(t) + \sum_{\substack{k,j=1\\ k,j\neq i}}^{m} c_{kj}(t)\dot{q}_k(t)\dot{q}_j(t) \\ & + \sum_{\substack{j=1\\ j\neq i}}^{m} s_j(t)\dot{q}_j(t) + \sum_{\substack{j=1\\ j\neq i}}^{m} p_j q_j(t)\dot{q}_j(t) + \sum_{\substack{j=1\\ j\neq i}}^{m} r_j(t)q_j(t) \\ G^i_{\mathbf{q}}(t) =\; & \sum_{j=1}^{m} 2a_{ij}(t)q_j(t) + \sum_{j=1}^{m} b_{ij}(t)\dot{q}_j(t) + r_i(t) \\ H^i_{\mathbf{q}}(t) =\; & \sum_{j=1}^{m} b_{ji}(t)q_j(t) + \sum_{\substack{j=1\\ i\neq j}}^{m} 2c_{ij}(t)\dot{q}_j(t) + s_i(t). \end{aligned}$$

3.1. Maximum necessary and sufficient condition

Before giving a necessary and sufficient conditions for an element of $\mathbb{D}_i$ to be a maximum of $J^i_{\mathbf{q}}$ we need the following definition, which makes use of the notation described above:

Definition 1. The ith *efficiency function* associated to $\mathbf{q} \in \mathbb{D}$ is

$$\mathbb{Y}^i_{\mathbf{q}}(t) := \int_0^t G^i_{\mathbf{q}}(s)ds - H^i_{\mathbf{q}}(t).$$

The characterization of the solutions of the optimization problem is given by:

Theorem 1. *Given* $\mathbf{q} \in \mathbb{D}$, *its ith component* $q_i(t)$ *solves the optimization problem* (1) *for* $J^i_{\mathbf{q}}$ *if and only if there exists* $k_i \in \mathbb{R}$ *satisfying:*

$$\mathbb{Y}^i_{\mathbf{q}}(t) \text{ is } \begin{cases} \leq k_i & \text{if } \dot{q}_i(t) = m_i \\ = k_i & \text{if } m_i < \dot{q}_i(t) < M_i \\ \geq k_i & \text{if } \dot{q}_i(t) = M_i. \end{cases}$$

Proof. ($\Rightarrow$) Let $z_i(t)$ be the state variable and $u_i(t)$ the control variable. The state equation is $\dot{z}_i(t) = u_i(t)$. The optimal control problem is, in this notation:

$$\max_{u_i} \int_0^T \left(F^i_{\mathbf{q}}(t) + G^i_{\mathbf{q}}(t) z_i(t) + \left(H^i_{\mathbf{q}}(t) + p_i z_i(t)\right) \dot{z}_i(t)\right) dt \quad \text{with} \begin{cases} \dot{z}_i(t) = u_i(t) \\ z_i(0) = 0, \qquad z_i(T) = b_i \\ m_i \leq u_i(t) \leq M_i. \end{cases}$$

Let $q_i(t)$ be the solution of the problem and let the optimal control be $u^*_i(t) = \dot{q}_i(t)$. Let $\hbar$ be the Hamiltonian associated with the problem

$$\hbar(t, z_i, u, \lambda) = F^i_{\mathbf{q}}(t) + G^i_{\mathbf{q}}(t) z_i(t) + \left(H^i_{\mathbf{q}}(t) + p_i z_i(t)\right) u_i(t) + \lambda \cdot u_i.$$

Because the functional $L(z, z_i(t), \dot{z}_i(t))$ is differentiable and the control variables are piecewise continuous with $u_i(t) \subset [m_i, M_i]$, we can apply Pontryagin's Principle: there exists a piecewise C^1 function (the co-state variable), $\lambda^*_i(t)$ satisfying the two following conditions:

$$\dot{\lambda}^*_i(t) = -\frac{\partial \hbar(t, q_i(t), u^*_i(t), \lambda^*_i(t))}{\partial z_i} = -\left(G^i_{\mathbf{q}}(t) + p_i u^*_i(t)\right) \tag{2}$$
$$\hbar(t, q_i(t), u^*_i(t), \lambda^*_i(t)) \geq \hbar(t, q_i(t), u_i, \lambda^*_i(t)), \quad \forall u_i \in [m_i, M_i].$$

By definition, we also have

$$\frac{\partial \hbar(t, q_i(t), u^*_i(t), \lambda^*_i(t))}{\partial u_i} = \lambda^*_i(t) + H^i_{\mathbf{q}}(t) + p_i q_i(t). \tag{3}$$

From (2), follows that

$$\lambda^*_i(t) = -\int_0^t \left(G^i_{\mathbf{q}}(s) + p_i u_i(s)\right) ds + k_i.$$

Taking into account (3), there are three possibilities:

(1) $m_i < u^*_i(t) < M_i \Longrightarrow 0 = \lambda^*_i(t) + H^i_{\mathbf{q}}(t) + p_i q_i(t) = -\mathbb{Y}^i_{\mathbf{q}}(t) + k_i \Longrightarrow \mathbb{Y}^i_{\mathbf{q}}(t) = k_i$

(2) $m_i = u^*_i(t) \Longrightarrow 0 \leq \lambda^*_i(t) + H^i_{\mathbf{q}}(t) + p_i q_i(t) = -\mathbb{Y}^i_{\mathbf{q}}(t) + k_i \Longrightarrow \mathbb{Y}^i_{\mathbf{q}}(t) \leq k_i$

(3) $M_i = u^*_i(t) \Longrightarrow 0 \geq \lambda^*_i(t) + H^i_{\mathbf{q}}(t) + p_i q_i(t) = -\mathbb{Y}^i_{\mathbf{q}}(t) + k_i \Longrightarrow \mathbb{Y}^i_{\mathbf{q}}(t) \geq k_i$.

($\Leftarrow$) Because $p_i \dot{z}_i z_i = \left(\frac{p_i}{2} z_i^2\right)'$, there is a continuous function $W(x)$ such that

$$\int_0^T \left(F^i_{\mathbf{q}}(t) + G^i_{\mathbf{q}}(t) z_i(t) + \left(H^i_{\mathbf{q}}(t) + p_i z_i(t)\right) \dot{z}_i(t)\right) dt = \int_0^T \left(W(T) + F^i_{\mathbf{q}}(t) + G^i_{\mathbf{q}}(t) z_i(t) + H^i_{\mathbf{q}}(t) \dot{z}_i(t)\right) dt$$

and the concavity hypotheses of Mangasarian's Theorem [14] hold, which gives the result.

Definition 2. Given $\mathbf{q} \in \mathbb{D}$, the constant k_i of Theorem 1 shall be called *the i-th critical efficiency level.*

One can interpret the Theorem as follows: given fixed inputs q_j, for $j \neq i$, there is a critical efficiency level k_i such that the optimal use of the ith input is as follows: spend it at its maximal rate M_i if its efficiency function is above the critical efficiency level and at rate m_i otherwise. When the efficiency function is equal to the critical efficiency level, the rate may be anyone (as long as it is admissible). From this description, it is clear that whenever the efficiency function is constantly equal to the critical efficiency level on an interval, *singular arcs* exist. We tackle this issue in the following section.

3.2. On the existence of singular arcs

In singular optimal control problems, the singular solution is *usually* determined by solving the algebraic equation which results from successively differentiating the switching function until the control appears explicitly. When systems are affine in all the control variables, there is a classical technique to find the control on a singular arc, which uses the fact that H_u remains zero along the whole arc. Hence, all time derivatives are zero along it as well. After successive differentiation, one of these time derivatives may contain the control u. An important result (see [15]) is the necessary condition for a singular arc to be optimal, which is called the Generalized Legendre–Clebsch (GLC) condition and can be stated as: *If $x^*(t)$, $u^*(t)$ are optimal on a singular arc, then, for scalar u,*

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{d^{2q}(H_u)}{dt^{2q}}\right] \leq 0.$$

If the control never appears, then the problem is called an *infinite-order* singular problem. We shall show in this section that singular arcs may exist in our problem and that they are in fact of infinite order. However, we shall show that if there is no open interval on which the efficiency function is constant, then there are none. In the next section we describe how to compute solutions to our problem both when there are and where there are no singular arcs. We also show how the appearing of singular arcs is equivalent to the non-uniqueness of the solution to the optimization problem.

Proposition 1. *If Problem (1) has singular solutions then, on some open subinterval of $[0, T]$, the efficiency function $\mathbb{Y}^i_{\mathbf{q}}(t)$ is constant, or what is the same, $G^i_{\mathbf{q}}(t) = \dot{H}^i_{\mathbf{q}}(t)$.*

Proof. The control problem has a singular solution when on some subinterval $[t', t'']$ of $[0, T]$, the Euler equation holds:

$$\frac{\partial L^i_{\mathbf{q}}(t, z_i, \dot{z}_i)}{\partial z_i} - \frac{d}{dt}\frac{\partial L^i_{\mathbf{q}}(t, z_i, \dot{z}_i)}{\partial \dot{z}_i} = 0.$$

If this is the case and $q_i(t)$ is such a singular solution, then

$$G_{\mathbf{q}}(t) + \dot{q}_i(t)\frac{\partial P_{\mathbf{q}}}{\partial z_i}(q_i(t)) - \frac{d}{dt}\left(H^i_{\mathbf{q}}(t) + P_{\mathbf{q}}(q_i(t))\right) = 0,$$

that is:

$$G^i_{\mathbf{q}}(t) + \dot{q}_i(t)\frac{\partial P^i_{\mathbf{q}}}{\partial z_i}(q_i(t)) - \dot{H}^i_{\mathbf{q}}(t) - \dot{q}_i(t)\frac{\partial P^i_{\mathbf{q}}}{\partial z_i}(q_i(t)) = 0,$$

so that

$$G^i_{\mathbf{q}}(t) = \dot{H}^i_{\mathbf{q}}(t)$$

on the specified subinterval.

The following lemma shall allow us to prove that when there are no singular arcs, then there is a continuous relation between the critical efficiency level of an input and the value of b_i (availability of the ith input).

Lemma 1. *Let $f : [0, T] \to \mathbb{R}$ be a continuous function and let $\underline{f} y \bar{f}$ be its minimum and its maximum, respectively. Given m, $M \in \mathbb{R}$ with $m < M$, define, for any $x \in [\underline{f}, \bar{f}]$, the function g_x as follows:*

$$g_x(t) = \begin{cases} M & \text{if } x < f(t) \\ m & \text{if } x \geq f(t) \end{cases}$$

and consider the function $I(x) = \int_0^T g_x(t)\, dt$, for $x \in [\underline{f}, \bar{f}]$.

Then: $I(x)$ is strictly decreasing on the interval on $[\underline{f}, \bar{f}]$, where it is defined and

$$I(x^-) - I(x^+) = (M - m)\mu(f^{-1}(x))$$

where μ denotes the Lebesgue measure on $[0, T]$.

Proof. Given $\epsilon > 0$, define

$$R_{x,\epsilon} = \{t \in [0, T] : x \leq f(t) < x + \epsilon\}$$

and let $R'_{x,\epsilon}$ be its complement in $[0, T]$. One can decompose $I(x + \epsilon)$ as

$$I(x + \epsilon) = \int_{R_{x,\epsilon}} g_{x+\epsilon}(t)\, dt + \int_{R'_{x,\epsilon}} g_{x+\epsilon}(t)\, dt.$$

From the definition of $g_{x+\epsilon}$ and g_x, it is clear that

$$g_{x+\epsilon}(t) = \begin{cases} g_x(t) - (M - m) & \text{if } t \in R_{x,\epsilon} \\ g_x(t) & \text{if } t \in R'_{x,\epsilon} \end{cases}$$

so that

$$I(x+\epsilon) = \int_{R_{x,\epsilon}} g_{x+\epsilon}(t)\,dt + \int_{R'_{x,\epsilon}} g_x(t)\,dt < \int_{R_{x,\epsilon}} g_x(t)\,dt + \int_{R'_{x,\epsilon}} g_x(t)\,dt = I(x)$$

where the inequality is strict because, being f continuous, the set $R_{x,\epsilon}$ has positive measure and $M - m > 0$. This gives first result.

One also has

$$\bigcap_{\epsilon>0} R_{x,\epsilon} = \bigcap_{n\geq 1} R_{x,\frac{1}{n}} = f^{-1}(x)$$

and, as all the sets have finite measure and because $R_{x,\epsilon}$ is included in $R_{x,\epsilon'}$, if $\epsilon < \epsilon'$, then the Monotone Convergence Theorem states that when $\epsilon \to 0$,

$$I(x+\epsilon) = \int_{R_{x,\epsilon}} m\,dt + \int_{R'_{x,\epsilon}} g_x(t)\,dt$$

converges to

$$I(x+\epsilon) \overset{\epsilon\to 0}{\longrightarrow} m\mu(f^{-1}(x)) + \int_S g_x(t)\,dt$$

where $S = [0, T] \setminus f^{-1}(x)$, so that

$$I(x+\epsilon) \to m\mu(f^{-1}(x)) + (I(x) - M\mu(f^{-1}(x))).$$

The fact that $I(x^-) = I(x)$ is proved in the same way.

Definition 3. Given $\mathbf{q} \in \mathbb{D}$, define the following functions:

$$S^i_{\mathbf{q}}(k,t) := \begin{cases} M_i & \text{if } k < \mathbb{Y}^i_{\mathbf{q}}(t) \\ m_i & \text{if } k \geq \mathbb{Y}^i_{\mathbf{q}}(t); \end{cases} \qquad \mathfrak{S}^i_{\mathbf{q},k}(t) = \mathfrak{S}^i_{\mathbf{q}}(k,t) := \int_0^t S^i_{\mathbf{q}}(k,s)ds; \qquad \mathrm{T}^i_{\mathbf{q}}(k) := \mathfrak{S}^i_{\mathbf{q}}(k,T).$$

Corollary 1. *If $\mathbb{Y}^i_{\mathbf{q}}(t)$ is not constant on any interval, then for any $b_i \in [m_iT, M_iT]$ there exists k_i such that the maximum of the functional $J^i_{\mathbf{q}}$ on D_i is given by $\mathfrak{S}^i_{\mathbf{q}}(k_i, t)$ and there are no singular arcs.*

Proof. As $\mathbb{Y}^i_{\mathbf{q}}(t)$ is not constant on any interval, by Proposition 1, the maximum of $J^i_{\mathbf{q}}$ on D_i has no singular arcs and can be defined by construction using Theorem 1. We only need to show that for any $b_i \in [m_iT, M_iT]$, there is some k_i such that $T^i_{\mathbf{q}}(k) = b_i$. To this end, notice that:

- If $k_i = \min \mathbb{Y}^i_{\mathbf{q}}([0,T])$, then $\mathrm{T}^i_{\mathbf{q}}(k) = M_iT$.
- If $k_i = \max \mathbb{Y}^i_{\mathbf{q}}([0,T])$, then $\mathrm{T}^i_{\mathbf{q}}(k) = m_iT$.

As, by Lemma 1, $\mathrm{T}^i_{\mathbf{q}}$ is continuous and decreasing, for any $b_i \in [m_iT, M_iT]$ there exists k_i such that $\mathrm{T}^i_{\mathbf{q}}(k) = b_i$.

3.3. Construction of the solution: adaptation of the shooting method

Given $\mathbf{q} \in \mathbb{D}$, in order to maximize $J^i_{\mathbf{q}}$, one needs to find the ith critical efficiency level k such that there exists $q_i \in \mathbb{D}_i$ satisfying the hypotheses of Theorem 1. The following results establish a constructive way for this as long as $m_iT \leq b_i \leq M_iT$. Notations are those of Definition 3.

Lemma 2. *The function $\mathrm{T}^i_{\mathbf{q}}$ is strictly decreasing on the interval $\mathbb{Y}^i_{\mathbf{q}}([0,T])$ (it is an interval because $\mathbb{Y}^i_{\mathbf{q}}$ is continuous) and it is discontinuous at k if and only if the set $D(k) := \{t : \mathbb{Y}^i_{\mathbf{q}}(t) = k\}$ has strictly positive measure. If k is a discontinuity, then $\mathrm{T}^i_{\mathbf{q}}(k^-) - \mathrm{T}^i_{\mathbf{q}}(k^+) = (M_i - m_i) \cdot \mu(Dis(k))$.*

Proof. Apply Lemma 1 to $f(t) = \mathbb{Y}^i_{\mathbf{q}}(t)$.

Proposition 2. *Let $b_i \in [m_iT, M_iT]$. If $b_i \in [\mathrm{T}^i_{\mathbf{q}}(k^+), \mathrm{T}^i_{\mathbf{q}}(k^-)]$, define*

$$\overline{S}^i_{\mathbf{q}}(k,t) := \begin{cases} M_i & \text{if } k < \mathbb{Y}^i_{\mathbf{q}}(t) \\ m_i + \dfrac{b_i - \mathrm{T}^i_{\mathbf{q}}(k^+)}{\mathrm{T}^i_{\mathbf{q}}(k^-) - \mathrm{T}^i_{\mathbf{q}}(k^+)} & \text{if } k = \mathbb{Y}^i_{\mathbf{q}}(t) \\ m_i & \text{if } k > \mathbb{Y}^i_{\mathbf{q}}(t). \end{cases}$$

Otherwise, for $b_i = \mathrm{T}^i_{\mathbf{q}}(k^+) = \mathrm{T}^i_{\mathbf{q}}(k^-)$*, define* $\overline{S}^i_{\mathbf{q}}(k,t) := S^i_{\mathbf{q}}(k,t)$ *as above. Then* $\overline{\mathfrak{S}}^i_{\mathbf{q}}(k,t) := \int_0^t \overline{S}^i_{\mathbf{q}}(k,s)ds$ *maximizes the functional* $J^i_{\mathbf{q}}$ *on* $\mathbb{D}_i$*.*

Proof. The function $\overline{\mathfrak{S}}^i_{\mathbf{q}}(k,t)$ is defined by requiring that Theorem 1 holds.

Let us verify that for any $b_i \in [\mathrm{T}^i_{\mathbf{q}}(k^+), \mathrm{T}^i_{\mathbf{q}}(k^-)]$, there is k such that $\overline{\mathfrak{S}}^i_{\mathbf{q}}(k,T) = b_i$.

Let $D^+(k) := \{t : k < \mathbb{Y}^i_{\mathbf{q}}(t)\}$ and $D^-(k) := \{t : k > \mathbb{Y}^i_{\mathbf{q}}(t)\}$. If $b_i \in [\mathrm{T}^i_{\mathbf{q}}(k^+), \mathrm{T}^i_{\mathbf{q}}(k^-)]$ then certainly

$$\begin{aligned}\overline{\mathfrak{S}}^i_{\mathbf{q}}(k,T) &= \int_0^T \overline{S}^i_{\mathbf{q}}(k,s)ds \\ &= M_i \cdot \mu(D^+(k)) + m_i \cdot \mu(D^-(k)) + \left(m_i + \frac{b_i - \mathrm{T}^i_{\mathbf{q}}(k^+)}{\mathrm{T}^i_{\mathbf{q}}(k^-) - \mathrm{T}^i_{\mathbf{q}}(k^+)}\right) \cdot \mu(Dis(k)) \\ &= M_i \cdot \mu(D^+(k)) + m_i \cdot \mu(D^-(k)) + m_i \cdot \mu(D(k)) + \frac{b_i - \mathrm{T}^i_{\mathbf{q}}(k^+)}{\mathrm{T}^i_{\mathbf{q}}(k^-) - \mathrm{T}^i_{\mathbf{q}}(k^+)}\left(\mathrm{T}^i_{\mathbf{q}}(k^-) - \mathrm{T}^i_{\mathbf{q}}(k^+)\right).\end{aligned}$$

As

$$\mathrm{T}^i_{\mathbf{q}}(k^+) = M_i \cdot \mu(D^+(k)) + m_i \cdot \mu(D^-(k)) + m_i \cdot \mu(D(k))$$

then

$$\overline{\mathfrak{S}}^i_{\mathbf{q}}(k,T) = b_i.$$

If $b_i = \mathrm{T}^i_{\mathbf{q}}(k^+) = \mathrm{T}^i_{\mathbf{q}}(k^-)$ then $\overline{\mathfrak{S}}^i_{\mathbf{q}}(k,T) = b_i$, simply taking $\overline{S}^i_{\mathbf{q}}(k,t) := S^i_{\mathbf{q}}(k,t)$.

In Section 5 we shall solve an example illustrating Proposition 2.

Notice that for $b_i \in [\mathrm{T}^i_{\mathbf{q}}(k^+), \mathrm{T}^i_{\mathbf{q}}(k^-)]$, the solutions to the corresponding problems only differ on those points on which $k = \mathbb{Y}^i_{\mathbf{q}}(t)$.

The construction of the solution has two stages:

1. Finding k such that $b_i \in [\mathrm{T}^i_{\mathbf{q}}(k^+), \mathrm{T}^i_{\mathbf{q}}(k^-)]$.
2. Computing $\overline{\mathfrak{S}}^i_{\mathbf{q}}(k,t)$.

Stage 1 can be approached adapting the shooting method: one varies the efficiency level until finding the one for which the total usage of the ith input on $[0,T]$ is b_i (or more precisely, a discontinuity of $\mathrm{T}^i_{\mathbf{q}}$ such that $b_i \in [\mathrm{T}^i_{\mathbf{q}}(k^+), \mathrm{T}^i_{\mathbf{q}}(k^-)]$). This can be done using any numerical method for finding changes of sign of a discontinuous function. Stage 2 can be performed either symbolically, when $\mathbb{Y}$ is known and integrable or using any method of numerical integration otherwise.

4. Cyclic coordinate descent

The solution of the optimization problem in m dimensions satisfies Theorem 1 in all its components:

Theorem 2. *An admissible element* $\mathbf{q} = (q_1, \ldots, q_m) \in \mathbb{D}$*, is a solution of problem* (1)*, if and only if there exist* $\{k_i\}_{i=1}^m \subset \mathbb{R}$ *satisfying:*

$$\mathbb{Y}^i_{\mathbf{q}}(t) \text{ is } \begin{cases} \leq k_i & \text{if } \dot{q}_i(t) = m_i \\ = k_i & \text{if } m_i < \dot{q}_i(t) < M_i \\ \geq k_i & \text{if } \dot{q}_i(t) = M_i. \end{cases}$$

Let $\mathbf{q} = (q_1, \ldots, q_m) \in \mathbb{D}$. Recall that

$$L^i_{\mathbf{q}}(t, z_i, \dot{z}_i) := L(q_1(t), \ldots, q_{i-1}(t), z_i, q_{i+1}(t), \ldots, q_m(t), \dot{q}_1(t), \ldots, \dot{z}_i, \ldots, \dot{q}_m(t))$$

and the functional $J^i_{\mathbf{q}} : \mathbb{D}_i \longrightarrow \mathbb{R}$ is

$$J^i_{\mathbf{q}}(z_i) := J(q_1, \ldots, q_{i-1}, z_i, q_{i+1}, \ldots, q_m) = \int_0^T \left(F^i_{\mathbf{q}}(t) + G^i_{\mathbf{q}}(t)z_i(t) + \left(H^i_{\mathbf{q}}(t) + P^i_{\mathbf{q}}(z_i(t))\right)\dot{z}_i(t)\right)dt.$$

With the notations of Definition 3,

Definition 4. We define ith maximizing map as the map $\Phi_i : \mathbb{D} \longrightarrow \mathbb{D}$ given by

$$\Phi_i(q_1, \ldots, q_m) := (q_1, \ldots, q_{i-1}, \overline{\mathfrak{S}}^i_{\mathbf{q}}(k^i_{\mathbf{q}}, t), q_{i+1}, \ldots, q_m)$$

where $k^i_{\mathbf{q}}$ is such that $b_i \in [\bar{\mathrm{T}}^i_{\mathbf{q}}(k^{i\,+}_{\mathbf{q}}), \bar{\mathrm{T}}^i_{\mathbf{q}}(k^{i\,-}_{\mathbf{q}})]$ and corresponds to the k_i of Theorem 1 when $(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m)$ are fixed.

Roughly, one could say that Φ_i acts on each $\mathbf{q} \in \mathbb{D}$ keeping all but the ith components constant and changing the ith one by the function which maximizes $J^i_{\mathbf{q}}$.

We shall denote by Φ the map associated with the descent algorithm, which will be the composition of the mi-th maximizing maps:

$$\Phi = \Phi_m \circ \cdots \circ \Phi_1.$$

The m-dimensional optimization algorithm consists in a sequence of steps in each of which one "maximizes the m components one by one" by means of composing the ith maximizing maps in the established order, to obtain a new admissible element, q_n, at step n:

$$\mathbf{q}_n = \Phi(\mathbf{q}_{n-1}) = (\Phi_m \circ \Phi_{m-1} \circ \cdots \circ \Phi_2 \circ \Phi_1)(\mathbf{q}_{n-1}).$$

A solution of the problem is a fixed point of the descent map Φ, and conversely.

Consider $\mathbb{D}$ equipped with the topology induced by the norm

$$\|\mathbf{p}\|^* := \max\{\|\mathbf{p}\|_\infty, \|\dot{\mathbf{p}}\|_\infty\} = \max\{\max_{i=1,\dots,m} \|p_i\|_\infty, \max_{i=1,\dots,m} \|\dot{p}_i\|_\infty\}.$$

Given an admissible function $\mathbf{z}_0 \in \mathbb{D}$, consider the sequence defined by $\mathbf{z}_{n+1} = \Phi(\mathbf{z}_n)$. In [13] is shown that, for strictly convex functionals with respect to z_i', one can adapt Zangwill's Global Convergence Theorem [16] to obtain the above convergence result. The key hypothesis is that the derivatives of the admissible functions are uniformly bounded—in our case, the controls, which are between the real values m_i and M_i. The following is a precise statement of the result:

Theorem 3. *For every $\mathbf{q}_0 \in \mathbb{D}$, the sequence generated by the algorithm $\{\mathbf{q}_n = \Phi(\mathbf{q}_{n-1})\}_{n\in\mathbb{N}}$ possesses a subsequence that converges in $(\mathbb{D}, \|\,\|^*)$ and the limit is a fixed point of Φ. Moreover, any convergent subsequence of $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ will converge to a fixed point of Φ.*

5. Examples

We are going to show two examples in this section: the first one, of a theoretical nature, will be useful to illustrate the appearing of singular arcs. The second one is based on the optimization of hydraulic power systems. This example serves a double aim: on one hand, it shows the properties of convergence (number of iterations, errors, etc.) of the cyclic coordinate descent algorithm. On the other, it shows how the conditions imposed on the optimization problem are natural. We show how some complex real models verify them and hence, how our method allows optimizing systems of great dimensions.

5.1. Example 1

This is a one-dimensional problem which verifies the conditions for the existence of singular arcs. It is also solvable analytically. We shall obviate the superindex i and the subindex $\mathbf{q}$. Hence, we consider

$$L(t, z(t), \dot{z}(t)) := F(t) + G(t)z(t) + (H(t) + P(z(t)))\dot{z}(t)$$

with $H(t) := t$, and $P(\cdot)$ and $F(\cdot)$ arbitrary continuous functions. Consider

$$G(t) := \begin{cases} t & \text{if } t \le 1 \\ 1 & \text{if } 1 \le t \le 2 \\ 3 - t & \text{if } 2 \le t \le 3. \end{cases}$$

And let $T = 3$, $m_i = 0$ and $M_i = 1$. The problem is, then, maximizing the functional

$$\int_0^3 (F(t) + G(t)z(t) + (H(t) + P(z(t)))\dot{z}(t))dt$$

on the set:

$$\mathbb{D} := \{z \in \widehat{C}^1[0,3] : z(0) = 0, z(3) = b, 0 \le \dot{z}(t) \le 1\}$$

which is equivalent to maximizing

$$\max_{z\in\mathbb{D}} \int_0^3 (G(t)z(t) + (H(t))\dot{z}(t))dt$$

because

$$\int_0^3 (F(t) + P(z(t))\dot{z}(t))\, dt$$

is a function of $z(3)$ and $z(0)$ and hence non-optimizable (see Fig. 1).

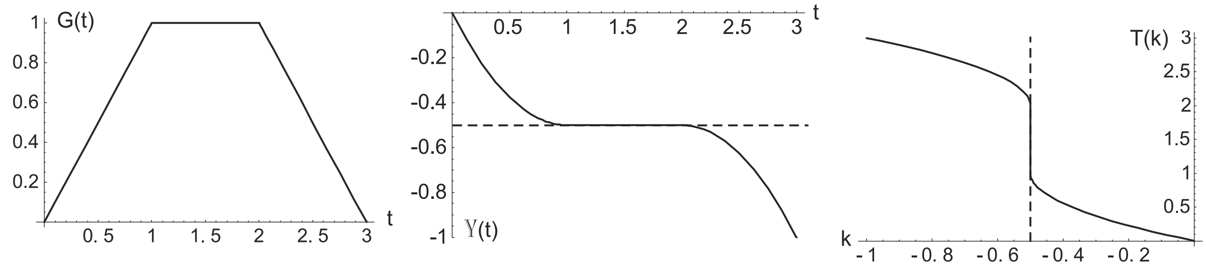

Fig. 1. Functions $G(t)$, $\mathbb{Y}(t)$ and $T(k)$.

The efficiency function is given by:

$$\mathbb{Y}(t)=\begin{cases}\dfrac{t^2}{2}-t & \text{if } 0\le t\le 1\\ -\dfrac{1}{2} & \text{if } 1\le t\le 2\\ -\dfrac{t^2}{2}+2t-\dfrac{5}{2} & \text{if } 2\le t\le 3.\end{cases}$$

Imposing $\mathbb{Y}(t)=k$ one gets immediately $T(k)$:

$$T(k)=\begin{cases}2+\sqrt{-1-2k} & \text{if } -1\le k\le -1/2\\ 1-\sqrt{1+2k} & \text{if } -1/2<k\le 0\end{cases}$$

with

$$T(-1/2^-)=2,\qquad T(-1/2^+)=1.$$

So that, for each b, the value of k for the optimal solution is:

$$k_b=\begin{cases}\dfrac{-2b+b^2}{2} & \text{if } b\le 1\\ -1/2 & \text{if } 1<b<2\\ \dfrac{-5+4b-b^2}{2} & \text{if } b\ge 2.\end{cases}$$

For $b\notin (T(-1/2^+),T(-1/2^-))$, the optimal $\dot z(t)$ is of bang–bang type:

$$\dot z(t)=\begin{cases}1 & \text{if } k_b<\mathbb{Y}(t)\\ 0 & \text{if } k_b>\mathbb{Y}(t).\end{cases}$$

However, if $1<b<2$, then there is a singular arc and in this case, our chosen solution is given by the following formula:

$$\frac{b-T(k_b^+)}{T(k_b^-)-T(k_b^+)}=\frac{b-T(-1/2^+)}{T(-1/2^-)-T(-1/2^+)}=\frac{b-1}{2-1}.$$

So that, if $b\in[1,2]$:

$$\dot z(t)=\begin{cases}1 & \text{if } -1/2<\mathbb{Y}(t)\\ b-1 & \text{if } -1/2=\mathbb{Y}(t)\\ 0 & \text{if } -1/2>\mathbb{Y}(t).\end{cases}$$

For example, for $b=1.5$, the solution is:

$$\dot z(t)=\begin{cases}1 & \text{if } 0<t\le 1\\ 0.5 & \text{if } 1<t<2\\ 0 & \text{if } 2\le t\le 3.\end{cases}$$

6. Example 2

One of the most important problems in the context of optimization of complex real systems is the optimization of hydro-power systems, with hundreds of papers being published each year on this topic. In this example, the optimization problem of one company is described, the objective function of which can be defined as its profit maximization. Let us assume that our hydro-power system accounts for m hydro-plants that do not have pumping capacity.

The mapping $H : \Omega_H \subset [0,T] \times (\mathbb{R}^+)^m \times (\mathbb{R}^+)^m \longrightarrow \mathbb{R}^+$,

$$H(t, z_1(t), \dots, z_i(t), \dots, z_m(t), \dot{z}_1(t), \dots, \dot{z}_i(t), \dots, \dot{z}_m(t)) = H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$$

is called the function of effective hydraulic contribution and is the power contributed to the system at instant t by the set of hydro-plants, $z_i(t)$ being the volume that is discharged up to the instant t by the i-th hydro-plant and $\dot{z}_i(t)$ the rate of water discharged at instant t by the ith hydro-plant. We say that $\dot{\mathbf{z}} = (z_1, \dots, z_m)$ is admissible for H if z_i belongs to the class $\widehat{C}^1[0,T]$, and $(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) \in \Omega_H, \forall t \in [0,T]$. The volume b_i that must be discharged up to the instant T is called the admissible volume of the ith hydro-plant. Let $\mathbf{b} = (b_1, \dots, b_m) \in (\mathbb{R}^+)^m$ be the vector of admissible volumes.

In a general model, with hydraulic coupling between the m hydro-plants, we call $H_i(t, z_i(t), \dot{z}_i(t)) : \Omega_{H_i} = [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ the function of effective hydraulic contribution by the ith hydro-plant, being:

$$H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = \sum_{i=1}^{m} H_i(t, z_i(t), \dot{z}_i(t)).$$

Let us assume that the function H_i is strictly increasing with respect to the rate of water discharge, $\dot{z}_i$, i.e. the higher the rate of water discharge, the greater the power generated, and that $[\partial H_i / \partial z_i]_{\dot{z}_i=0} = 0$. Real models meet these constraints. Besides, we consider $\dot{z}_i(t)$ to be bounded by technical constraints:

$$m_i \le \dot{z}_i(t) \le M_i; \quad i = 1, \dots, m, \forall t \in [0,T].$$

The objective function is given by revenue during the optimization interval $[0,T]$. Revenue is obtained by multiplying the total production of the company by the clearing price $p(t)$ in each hour, t. Our model of the spot market explicitly represents the price of electricity as a known exogenous variable. With this statement, our objective functional is:

$$\max_{\mathbf{z}} F(\mathbf{z}) = \max_{\mathbf{z}} \int_0^T L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))dt$$

with:

$$L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = p(t)H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$$

on the set:

$$\Omega = \left\{ \mathbf{z} \in \left(\widehat{C}^1[0,T]\right)^m | z_i(0) = 0, z_i(T) = b_i; m_i \le \dot{z}_i(t) \le M_i; \ \forall i = 1, \dots, m, \ \forall t \in [0,T] \right\}.$$

The hydro-network is assumed to have several chains of hydro-plants on different rivers. We assume that the rate of discharge at the upstream plant affects the behavior at the downstream plants. We say that the hydraulic system has hydraulic coupling. We use the two most widely-used models of hydro-plants [17]: fixed-head and variable-head. In the fixed-head model, the ith function of effective hydraulic generation, H_i, is only a function of $\dot{z}(t)$. It is the usual model employed in large-scale hydro-plants and reflects the fact that the effective head of water is constant during operation:

$$H_i(t, z_i(t), \dot{z}_i(t)) = A_i(t)\dot{z}_i(t).$$

In a variable-head model, however, the ith function of effective hydraulic generation, H_i, is a function of $z(t)$ and $\dot{z}(t)$ and is given by:

$$H_i(t, z_i(t), \dot{z}_i(t)) = A_i(t)\dot{z}_i(t) - B_i\dot{z}_i(t)\left[z_i(t) - Coup_i(t)\right] \tag{4}$$

where $Coup_i(t)$ represents the hydraulic coupling between plants. In the variable-head model, the second term represents the negative influence of the consumed volume and reflects the fact that consuming water lowers the effective height and hence the performance of the plant. The coefficients $A_i(t)$ and B_i are:

$$A_i(t) = \frac{1}{G_i} B_{y_i}(S_{0i} + t \cdot i_i); \qquad B_i = \frac{B_{y_i}}{G_i}$$

the parameters that appear in this formula being the efficiency, G, in (m^4/h Mw), the natural inflow, i, in (m^3/h), the initial volume, S_0, in (m^3), and the coefficient B_y, in (m^{-2}), a parameter that depends on the geometry of the reservoir. For our example, the data are based on the hydro-plants owned by the company EDP in Asturias (Spain). The data of the hydro-plants are summarized in Table 1, where the minimum (m_i) and maximum (M_i) rate of discharge, $\dot{z}(t)$, are in (m^3/h) and the available volume, b, in (m^3).

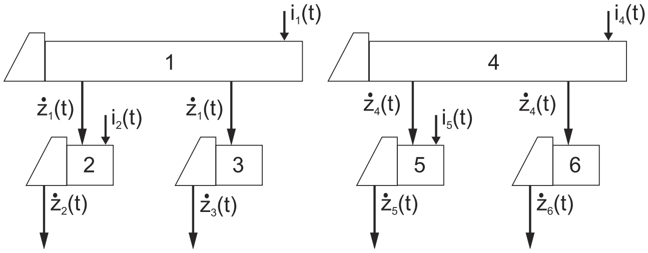

Fig. 2. The hydraulic system.

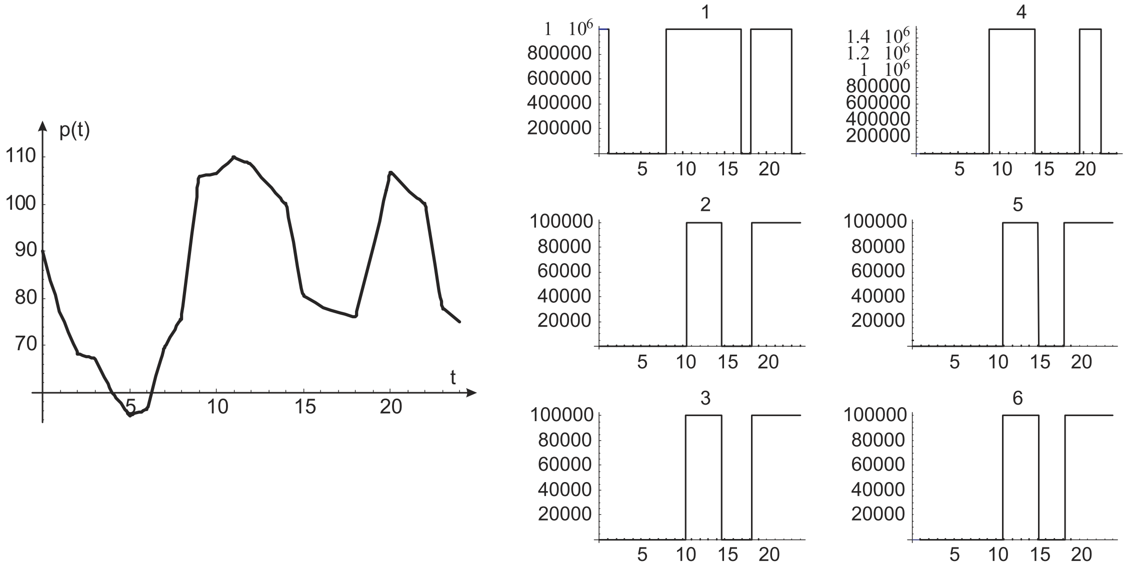

Fig. 3. The clearing price $p(t)$ and the optimal control $\dot{z}_i(t)$, $(i = 1, \ldots, 6)$.

Table 1
Hydro-plant coefficients.

Plant i	G	b	i	S_0	B_y	m_i	M_i
1	337 542	$15\ 10^6$	216 000	$270.3\ 10^6$	$3.06555\ 10^{-7}$	0	10^6
2	342 003	10^6	28 500	$15.3\ 10^6$	$2.1875\ 10^{-6}$	0	10^5
3	342 003	10^6	0	$15.3\ 10^6$	$2.1875\ 10^{-6}$	0	10^5
4	337 542	$12\ 10^6$	216 000	$270.3\ 10^6$	$3.06555\ 10^{-7}$	0	$1.5\ 10^6$
5	363 950	10^6	127 300	$10.2\ 10^6$	$2.3448\ 10^{-6}$	0	10^5
6	363 950	10^6	0	$10.2\ 10^6$	$2.3448\ 10^{-6}$	0	10^5

For a more complete interpretation of the results, we have simulated two basins (which do not correspond to reality), as shown in Fig. 2. Plants 1 and 4 are both fixed-head and have identical coefficients except for the available volume, b, and the maximum rate of discharge, M_i. The downstream plants 2 and 3 are variable-head and thus smaller in size. They too are identical to each other except for the fact that 3 has no natural inflow. The same configuration is applied to the second basin, containing plants 5 and 6, though these have different coefficients to plants 2 and 3. Moreover, given the size of the downstream plants with respect to the upstream plants, we can dispense with the term $-B_i\dot{z}_i(t)z_i(t)$ in (4) as $z_i(t) \ll Coup_i(t)$. As can be seen in Fig. 2, the hydraulic coupling between our plants is:

$$Coup_2(t) = Coup_3(t) = z_1(t); \qquad Coup_5(t) = Coup_6(t) = z_4(t).$$

A computer program was written (using the Mathematica package) to apply the results obtained in this paper to a hydro-power system. The solution may be constructed in a simple way by taking into account the aforementioned algorithm. The optimal bang–bang control, $\dot{z}_i(t)(\mathrm{m}^3)$, and the clearing price, $p(t)$ (euro/h Mw), are shown in Fig. 3 and the optimal hydro-power, $H_i(t)$, in Fig. 4. The optimization interval is $T = 24(h)$ and a discretization of 2400 subintervals was used.

The switching times $t_i(h)$ of the hydro-plants are presented in Table 2. The results have a clear-cut interpretation. As can be seen, plant 1 distributes its water in 3 bang–bang intervals. In contrast, plant 4 can make better use of the water available during the times when the price is higher, as its M_i is higher than that of plant 1, and it only needs two bang–bang intervals.

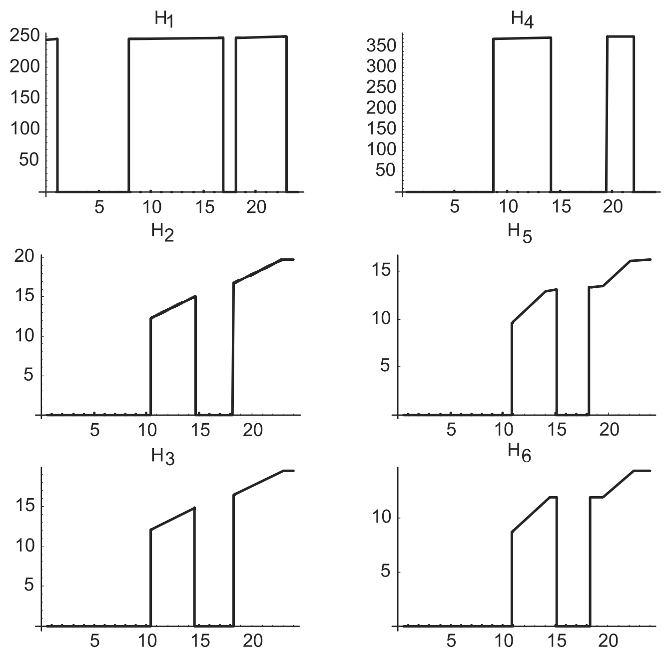

Fig. 4. Optimal hydro-power $H_i(t)$.

Table 2
Switching times.

Plant	t_1	t_2	t_3	t_4	t_5
1	1.14	7.93	16.91	18.08	22.96
2	10.39	14.63	18.24		
3	10.36	14.62	18.26		
4	8.68	14.13	19.49	22.04	
5	10.88	15.09	18.21		
6	10.85	15.16	18.31		

Plants 2 and 3 perform very similarly, which demonstrates the limited influence of the natural inflow, i. As can be seen in Table 2, the switching times of both plants are almost equal. The same behavior is observed between plants 5 and 6.

As far as effective hydraulic generation, H_i, is concerned, Fig. 4 shows that plants 1 and 4 also display bang–bang values due to being fixed-head plants. In the other plants, however, due to the fact that they are variable-head, H_i is not constant despite $\dot{z}_i$ being so. The reason for these power plants presenting an increasing H_i is the influence of the coupling of the upstream plants 1 and 4. The volume discharged by these plants (z_1 and z_4) increases the effective head of water of the downstream plants, which are thus able to produce a notably higher H_i. This leads to another interesting feature: the last operating range runs up to the instant $T = 24$, in all the plants, despite the lowest prices being at instants 23 and 24, precisely in order to benefit from the better utilization of the flow. The intervals of constant operation of plants 3 and 6 coincide with areas where there is no discharge upstream, plus the fact of not having any natural inflow, i. At plants 2 and 5, the intervals are essentially constant, even though the natural inflow, i, causes a minimal increase. The optimal values of C_i that satisfy Theorem 2 are:

$$C_1 = -0.018602633818129266 \qquad C_4 = -0.023763928225530719$$
$$C_2 = -0.013233038539526272 \qquad C_5 = -0.010500785409674454$$
$$C_3 = -0.013005892657789548 \qquad C_6 = -0.009489377688517156.$$

The algorithm runs very quickly (see Fig. 5). In the example, only 3 iterations were needed and the CPU time required by the program was 93.6 sec on a personal computer (Intel Core 2/2.66 GHz).

7. Conclusions

In this paper, we first constructively solve the one-dimensional optimal control problem with a Lagrangian of the form:

$$L(t, z, \dot{z}) = F(t) + G(t)z(t) + (H(t) + P(z(t)))\,\dot{z}(t).$$

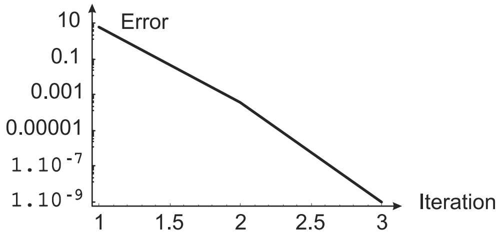

Fig. 5. Convergence of the algorithm.

The shooting method in combination with Pontryagin's maximum principle and the theory of singular control provides the theoretical basis that has enabled us to construct an algorithm for solving the problem approximately and, in some cases, even analytically. This is an interesting contribution to the theory of bang–singular–bang type problems whose singular arches are of infinite order. Second, this study also allows us to solve multidimensional problems whose Lagrangian is of the form:

$$L(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = \mathbf{z}^t A(t)\mathbf{z} + \mathbf{z}^t B(t)\dot{\mathbf{z}} + \dot{\mathbf{z}}^t C(t)\dot{\mathbf{z}} + \mathbf{s}(t)^t \cdot \dot{\mathbf{z}} + \mathbf{z}^t P\dot{\mathbf{z}} + \mathbf{r}(t)^t \cdot \mathbf{z}$$

using the cyclical coordinate descent method to do so. The paper presents two examples. In the first example, we show how to obtain the analytical solution of a one-dimensional bang–singular–bang problem. In the second, a very complex problem is addressed in the context of the optimization of hydro-power systems. In this example, our method can be seen to converge rapidly for multidimensional problems, confirming its potential use in real engineering problems. The mathematical framework of application of the theory presented here is very broad. It encompasses the class of functionals whose Lagrangian, L, for each t is a quadratic form on $\widetilde{\mathbf{z}} = (z_1, \ldots, z_m, \dot{z}_1, \ldots .\dot{z}_m, 1)$, i.e.:

$$f(\widetilde{\mathbf{z}}) = \widetilde{\mathbf{z}}^T \mathbf{A}\widetilde{\mathbf{z}}$$

with $A = a_{ij}(t)$ being a symmetrical matrix, with null diagonal entrances and where its coefficients $a_{k,k+m}(t)$ are constant. That is, the coefficients of z_i^2 and $\dot{z}_i^2$ are null and the coefficients of $z_i \cdot \dot{z}_i$ are constant. A specific possible future line of research involves the study of the problem in which terms of the form $f(t) \cdot z_i \cdot \dot{z}_i$, i.e. with the only constraints of $a_{ii}(t) = 0$, may appear.

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