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The operation of infimal/supremal convolution in mathematical economics

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The infimal convolution operation arises in mathematical economics in the analysis of several problems. In this paper we first present a survey and summarize two previous papers by the authors on the classic firm's cost-minimization problem. Moreover, we present a new application: the analytical solution of the utility maximization problem which we shall obtain applying the supremal convolution operation.

Keywords: mathematical economics; infimal/supremal convolution; firm's cost-minimization problem; utility maximization problem

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1. Introduction

The infimal convolution (IC) operation is a fundamental fact in discrete convex analysis that is often usefully applied in mathematical economics. In [16], the continuous IC operation arises in the analysis of problems involving transferable utilities. In [7], the value of the optimal allocation problem is given by the IC. [2] investigates basic properties of ICs defined between two convex risk measures. [1] formulates the optimal risk allocation problem as their IC. An excellent review of the literature on IC within the context of optimal risk exchange and optimal allocation problems can be found in [10].

This operator is well known within the context of convex analysis [11,15,17,18]. We shall commence by reviewing its definitions and basic properties.

Definition 1 *Let* $F, G : \mathbb{R} \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ *be two functions. We denote as the IC of* F *and* G *the operation defined as follows*:

$$(F \bigodot G)(x) := \inf_{y \in \mathbb{R}} \{F(x) + G(y - x)\}.$$

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It is well known that $(\mathcal{F}(\mathbb{R},\bar{\mathbb{R}}),\bigodot)$ is a commutative semigroup. Furthermore, for every finite set $E\subset\mathbb{N}$ and $F_i:\mathbb{R}\to\bar{\mathbb{R}}$, $\forall i\in E$, it is verified that

$$(\bigodot_{i\in E}F_i)(K)=\inf_{\sum_{i\in E}x_i=K}\sum_{i\in E}F_i(x_i).$$

When the functions are considered constrained to a certain domain, $\mathrm{Dom}(F_i)=[m_i,M_i]$, the above definition continues to be perfectly valid redefining $F_i(x)=+\infty$ if $x\notin\mathrm{Dom}(F_i)$. In this case, the definition may be expressed as follows:

$$(F_1\bigodot F_2)(\xi):=\min_{\substack{x_1+x_2=\xi\\ m_i\le x_i\le M_i}}(F_1(x_1)+F_2(x_2))=\min_{\substack{m_1\le x\le M_1\\ m_2\le\xi-x\le M_2}}((F_1(x)+F_2(\xi-x)).$$

Another problem of mathematical economics in which the IC operation offers major advantages is in the classic firm's cost-minimization (FCM) problem. In two previous papers [3,4], the authors of the present paper presented two new applications of the IC: in [3], the FCM problem with the Cobb–Douglas production function, and in [4], the FCM problem with the linear production function in economies of scale. The approach and fundamental results obtained are summarized in Section 2. One advantage of our technique is that it allows the analytical solution to be obtained in the general case, considering constraints for the inputs.

In Section 3, we present a new application to demonstrate the enormous potential of this mathematical tool in the field of economics: the analytical solution of the utility maximization problem. We shall address this problem in an exact way in this paper, transforming it into the constrained supremal convolution (SC) problem of the log-concave functions. It should be stressed that classical optimization techniques solve problems of this type for specific budget levels. In this paper, we provide, in a simple way, closed formulae for solving the family of problems resulting from ξ covering the entire range of possible budget levels. Therefore, the principal advantage of the method proposed here compared with traditional techniques, in which a different problem needs to be solved each budget level, is that we obtain the solution for any value of the budget level.

We present an academic example with $m=20$ inputs in Section 4 to demonstrate the potential of our method in large systems. Finally, Section 5 presents a realistic case based on a classic example [8,9], to which we have added inequality constraints on the inputs. Our method is able to incorporate these constraints, thus increasing the possibilities of classical models.

2. Applications of IC

We shall see some applications of the IC operator in this section, all of which are variants of the classic FCM problem.

2.1 *FCM problem with the Cobb–Douglas production function*

One of the most well-known problems in the field of Microeconomics is the FCM Problem (see, e.g. [12,14,20]). In [3], we established the analytical expression for the cost function $c(\mathbf{w},y)$ using the Cobb–Douglas model, considering maximum constraints for the inputs. Moreover, we proved that this solution belongs to the class C^1.

We first present the classic FCM problem. This problem can be expressed as follows: produce a given output y, and choose inputs to minimize its cost:

$$\begin{aligned} c(\mathbf{w},y) = \min_{\mathbf{x}\geq \mathbf{0}} \quad & \mathbf{wx} \\ \text{s.t.} \quad & f(\mathbf{x}) = y, \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^m$ are the inputs and $\mathbf{w} \in \mathbb{R}^m$ are the factor prices. We consider the general Cobb–Douglas production function

$$y = f(\mathbf{x}) = A \prod_{i=1}^{m} x_i^{\alpha_i}$$

and we shall usually measure units so that the total factor productivity $A = 1$. The sum of α_i determines the returns to scale. Our cost-minimization problem will be:

$$\begin{aligned} c(\mathbf{w},y) = \min \quad & \sum_{i=1}^{m} w_i x_i \\ \text{s.t.} \quad & y = \prod_{i=1}^{m} x_i^{\alpha_i} \\ & 0 \leq x_i \leq M_i; \quad i = 1,\ldots,m. \end{aligned} \tag{2}$$

We shall address this problem in an exact way in this paper, transforming it into a nonlinear (exponential) separable programming problem, which we state as a constrained IC problem. Taking into account the following changes in the variables:

$$\ln y = q,$$
$$\alpha_i \ln x_i = z_i,\ i = 1,\ldots,m$$

the cost-minimization problem (2) is equivalent to the IC problem:

$$\begin{aligned} \tilde{c}(\mathbf{w},q) = \min \quad & \sum_{i=1}^{m} w_i\, e^{(1/\alpha_i)z_i} \\ \text{s.t.} \quad & \sum_{i=1}^{m} z_i = q \\ & -\infty < z_i \leq \alpha_i \ln M_i = P_i^{\max}; \quad i = 1,\ldots,m. \end{aligned} \tag{3}$$

The function $\tilde{c}(\mathbf{w},\cdot)$ is, in fact, the IC of the exponential functions

$$F_i(z_i) := w_i\, e^{(1/\alpha_i)z_i}.$$

In [3] the solution of the cost-minimization problem is shown to be the following.

THEOREM 1 *The conditional demand function for the kth input is*

$$x_k(\mathbf{w},y) = \begin{cases} \displaystyle\prod_{i=1}^{m} \left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i/\tilde{\alpha}_1} \cdot y^{1/\tilde{\alpha}_1} & \text{if } y < e^{\theta_1}, \\ \exp\left[\dfrac{(-\sum_{i=1}^{j} P_i^{\max})}{\tilde{\alpha}_{j+1}}\right] \displaystyle\prod_{i=j+1}^{m} \left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)^{\alpha_i/\tilde{\alpha}_{j+1}} \cdot y^{1/\tilde{\alpha}_{j+1}} & \text{if } e^{\theta_j} \leq y < e^{\theta_{j+1}} \leq e^{\theta_k}, \\ e^{P_k^{\max}/\alpha_k} & \text{if } y \geq e^{\theta_k} \end{cases}$$

with the coefficients

$$\theta_k = \sum_{i=k}^{m} \frac{\alpha_i}{\alpha_k} P_k^{\max} + \sum_{i=k}^{m} \ln\left(\frac{\alpha_i w_k}{\alpha_k w_i}\right)^{\alpha_i} + \sum_{i=1}^{k-1} P_i^{\max}$$

and the cost function is a piecewise potential (*plus constant*)

$$c(\mathbf{w}, y) = \begin{cases} \tilde{w}_1 y^{1/\tilde{\alpha}_1} & \text{if } y < e^{\theta_1}, \\ \tilde{\mu}_k + \tilde{w}_k y^{1/\tilde{\alpha}_k} & \text{if } e^{\theta_{k-1}} \le y < e^{\theta_k} \end{cases}$$

with the coefficients

$$\tilde{\mu}_k = \sum_{i=1}^{k-1} w_i \, e^{P_i^{\max}/\alpha_i}; \quad \tilde{\alpha}_k = \sum_{i=k}^{m} \alpha_i,$$

$$\tilde{w}_k = \exp\left[\frac{(-\sum_{i=1}^{k-1} P_i^{\max})}{\tilde{\alpha}_k}\right]\left[\tilde{\alpha}_k \prod_{j=k}^{m}\left(\frac{w_j}{\alpha_j}\right)^{\alpha_j/\tilde{\alpha}_k}\right].$$

2.2 *FCM problem with a linear production function and economies of scale*

Problems involving economies of scale (in production and sales) (see, e.g. [6,13,19]) can often be formulated as concave quadratic programming problems [5]. In [4], we consider a case in which n products are being produced, with x_i being the number of units of product i and w_i being the unit production cost of product i. As the number of units produced increases, the unit cost usually decreases. This can often be correlated by a linear functional

$$w_i(x_i) = b_i - c_i x_i, \tag{4}$$

where $c_i > 0$. Thus, given constraints on production demands and the availability of each product and using the classic linear production function model, the FCM problem can be written as

$$\begin{aligned} C(y) = \min_{\mathbf{x}} \quad & \sum_{i=1}^{n} x_i w_i(x_i), \\ \text{s.t.} \quad & \sum_{i=1}^{n} a_i x_i = y; \quad a_i \neq 0, \; i = 1, \ldots, n, \\ & 0 \le x_i \le U_i; \quad i = 1, \ldots, n. \end{aligned} \tag{5}$$

where y is the output and U_i are the maximum constraints for the inputs. This is a concave minimization problem. As well as representing a situation in which the inputs are acquired with a discount proportional to the amount, the affine function model for the prices (4) can also be interpreted as dealing with inputs which are in turn outputs of a prior production process of economies of scale with a quadratic cost: $x_i b_i - c_i x_i^2$. On the other hand, the linear production function is presented in a natural way when the output is the result of the sum of the inputs ($a_i = 1$) or, in general, a specific fraction of each of these.

Using Equation (4) and making these changes in the variables

$$a_i x_i = z_i; \quad a_i U_i = M_i,$$
$$\frac{b_i}{a_i} = \beta_i; \quad \frac{c_i}{a_i^2} = \gamma_i$$

the FCM problem may be re-written as follows:

$$\begin{aligned} C(y) = \min_{\mathbf{z}} \quad & \sum_{i=1}^{n} (\beta_i z_i - \gamma_i z_i^2) \\ \text{s.t.} \quad & \sum_{i=1}^{n} z_i = y \\ & 0 \leq z_i \leq M_i; \quad i = 1, \ldots, n, \end{aligned} \tag{6}$$

which makes $C(y)$ the IC of the quadratic functions:

$$F_i(z_i) := \beta_i z_i - \gamma_i z_i^2$$

respectively constrained to the domains $[0, M_i]$; i.e.

$$C = F_1 \bigodot F_2 \bigodot \ldots \bigodot F_n.$$

In [4], we studied the IC of two concave functions, which is crucial as the basis for the optimization algorithm. Unfortunately, the IC operator does not preserve the concave nature of the functions. In general, the result is a piecewise concave function. This means that the IC of more than two functions cannot be obtained by means of a simple reiteration of the aforestated lemma, but requires resorting to calculating the IC of several piecewise concave functions. To perform this calculation, we shall interpret a piecewise concave function as the minimum function of several concave functions, preceding as shown in the following obvious lemma.

LEMMA 1 *Let the function*

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in [m_1, M_1] \\ \cdots & \cdots \quad \cdots \\ F_k(x) & \text{if } x \in [m_k, M_k] \end{cases}$$

be piecewise concave (*concave in each interval* $[m_k, M_k]$). *Thus,*

$$F(x) = \min_{i \in \{1, \ldots k\}} F_i(x),$$

where we have redefined each function $F_i(x)$ *as*

$$F_i(x) := \begin{cases} F_i(x) & \text{if } x \in [m_i, M_i] \\ \infty & \text{if } x \notin [m_i, M_i] \end{cases}, \quad i = 1, \ldots k.$$

Once redefined in this way, the calculation of the IC of two piecewise concave functions requires a combinatorial exploration that is reflected in the following proposition.

PROPOSITION 1 *Let* $F(x) := \min_{i\in A}(F_i(x))$ *and* $G(x) := \min_{i\in B}(G_i(x))$, *then*:

$$(F\bigodot G)(t) = \min_{(i,j)\in A\times B}(F_i\bigodot G_j)(t).$$

This proposition justifies the construction of the IC of the two functions defined piecewise as the minimum function of all the possible ICs of *pairs of pieces*. Now, bearing in mind the associative nature of the IC operation, the IC may be calculated by means of a recursive process, carrying out n operations of IC considering the following recurrence:

$$H_1\bigodot H_2\bigodot\cdots\bigodot H_n = (H_1\bigodot H_2\bigodot\cdots\bigodot H_{n-1})\bigodot H_n.$$

The calculation of the IC

$$(F\bigodot G)(x) = \min_{i\in\{1,\ldots,N\}}((F_i\bigodot G)(x))$$

is performed in two phases:

Phase (1) Calculation of $F_i\bigodot G$ for each $i \in \{1,\ldots,N\}$.

Phase (2) Calculate $\min_{i\in\{1,\ldots,N\}}(F_i\bigodot G)(x)$.

Once again, we stress the fact that the previously proposed recursive algorithm allows us to calculate the analytical solution for the piecewise concave quadratic functions.

3. The utility maximization problem

In this section, we shall consider the utility maximization problem for the case of the utility function following the Cobb–Douglas model, $\prod_{i=1}^{m} x_i^{\alpha_i}$, and the availability of the commodities having upper constraints. We shall obtain the analytical solution of the problem with the aid of the SC operator. This operator is defined in a similar way to that of the IC:

DEFINITION 2 *For every finite set* $E\subset\mathbb{N}$ *and* $F_i:\mathbb{R}\to\bar{\mathbb{R}}$, $\forall i\in E$ *functions, we denote as the SC of* F_i *the operation defined as follows:*

$$(\underset{i\in E}{\circledast} F_i)(K) = \sup_{\sum_{i\in E}x_i=K}\sum_{i\in E}F_i(x_i).$$

When the functions are considered constrained to a certain domain, the definition may be expressed as follows:

$$(F_1\circledast F_2)(\xi) := \max_{\substack{x_1+x_2=\xi\\ m_i\le x_i\le M_i}}(F_1(x_1)+F_2(x_2)) = \max_{\substack{m_1\le x\le M_1\\ m_2\le\xi-x\le M_2}}((F_1(x)+F_2(\xi-x)).$$

In the utility maximization problem, the aim is to choose the best among all possible options subject to the budget constraint: $\sum_{i=1}^{m}p_ix_i=\xi$ and to the available amount of commodities: $0\le x_i\le N_i$ such that the utility is maximized, where $\mathbf{p}=(p_1,\ldots,p_m)$ is the price vector of the different commodities, i.e.:

$$\begin{aligned} u(\mathbf{p},\xi) = \max \quad & \prod_{i=1}^{m} x_i^{\alpha_i}, \\ \text{s.t.} \quad & \sum_{i=1}^{m} p_i x_i = \xi, \\ & 0 \le x_i \le N_i. \end{aligned} \tag{7}$$

Problems of this kind, with box constraints, become complicated in the presence of boundary solutions. We shall address this problem in an exact way in this paper, transforming it into a nonlinear separable programming problem, which we state as a constrained SC problem. The problem (7) is equivalent to a new problem:

$$\begin{aligned} \tilde{u}(\mathbf{p},\xi) = \max \quad & \sum_{i=1}^{m} \alpha_i \ln\left(\frac{y_i}{p_i}\right), \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i = \xi, \\ & 0 < y_i \le p_i N_i = M_i \end{aligned} \tag{8}$$

with $\alpha_i, p_i > 0$, $i = 1, \ldots, m$, in which only the following change in the variables needs to be taken into account:

$$p_i x_i = y_i.$$

The function $\tilde{u}(\mathbf{p},\cdot)$ is in fact the SC of the log-concave functions

$$F_i(y_i) := \alpha_i \ln\left(\frac{y_i}{p_i}\right).$$

In this paper, we develop the necessary mathematical tools to justify the proposed method for solving the stated problem.

3.1 *Solution of the problem*

In this section, we shall calculate the SC for the functions $F_i(y_i)$ and then go on to prove that it belongs to the class C^1. The demonstration of the results not shown will be analogous to those developed in a previous paper [9] for exponential functions.

Let us calculate the SC of the concave functions $F_i(y_i)$ considering their domain to be constrained to $(0, M_i]$. Let us assume throughout the paper, without loss of generality, that

$$F_i'(M_i) \le F_{i+1}'(M_{i+1}), \quad \forall i = 1 \ldots m. \tag{9}$$

Let the function $F : (0, M_1] \times \cdots \times (0, M_m] \longrightarrow \mathbb{R}$ given by

$$F(y_1, \ldots, y_m) := \sum_{i=1}^{m} F_i(y_i).$$

Let C_ξ be the set

$$C_\xi := \left\{ (y_1, \ldots, y_m) \in (0, M_1] \times \cdots \times (0, M_m] \Big/ \sum_{i=1}^{m} y_i = \xi \right\}.$$

The SC of the $\{F_i\}_{i=1}^m$ is

$$(F_1 \circledast \cdots \circledast F_m)(\xi) := \max_{C_\xi} \sum_{i=1}^{m} F_i(y_i).$$

Let us now see the definitions of the elements which are present in our problem.

Definition 3 *Let us call the function* $\Psi_i : (0, \sum_{j=1}^m M_j] \longrightarrow (-\infty, M_i]$ the ith distribution function, *defined by*

$$\Psi_i(\xi) = y_i, \quad \forall i = 1, \ldots, m,$$

where $(y_1, \ldots, y_m)$ *is the unique maximum of* F *on the set* C_ξ, *i.e.*

$$\sum_{i=1}^{m} \Psi_i(\xi) = \xi \quad \text{and} \quad \sum_{i=1}^{m} F_i(\Psi_i(\xi)) = (F_1 \circledast \cdots \circledast F_m)(\xi).$$

The following lemma guarantees that if y_i reaches its maximum value, all those y_k for which the derivative of F_k at its maximum value is greater than or equal to the derivative corresponding to F_i will likewise have already reached their maximum.

Lemma 2 *If the function* F *reaches at* $(a_1, \ldots, a_m)$ *the maximum on the set* C_ξ *and, for a certain* $i \in \{1, \ldots, m\}$, $a_i = M_i$, *then*

$$\forall k \in \{1, \ldots, m\} / F_i'(M_i) \leq F_k'(M_k) \Longrightarrow a_k = M_k.$$

The following lemma establishes the order of the points where the variables reach their maximum value.

Lemma 3 *The parameters*

$$\theta_k := \sum_{i=k}^{m} M_i + \frac{M_k}{\alpha_k} \sum_{i=1}^{k-1} \alpha_i$$

satisfy

$$\theta_m \leq \theta_{m-1} \leq \cdots \leq \theta_1 = \sum_{i=1}^{m} M_i.$$

Proof

$$\begin{aligned}
\theta_{k+1} &= \sum_{i=k+1}^{m} M_i + \frac{M_{k+1}}{\alpha_{k+1}} \sum_{i=1}^{k} \alpha_i \leq \sum_{i=k+1}^{m} M_i + \frac{M_k}{\alpha_k} \sum_{i=1}^{k} \alpha_i \\
&= \sum_{i=k}^{m} M_i + \frac{M_k}{\alpha_k} \sum_{i=1}^{k-1} \alpha_i = \theta_k.
\end{aligned}$$

■

The following theorem establishes a necessary and sufficient condition to obtain the interior solution.

THEOREM 2 *The function* $F(y_1, \ldots, y_m)$ *attains the maximum over the set* C_ξ *at the point* $(a_1, \ldots, a_m) \in \overset{\circ}{C}_\xi$ *iff*

$$\xi < \theta_m = \frac{M_m}{\alpha_m} \sum_{i=1}^{m} \alpha_i.$$

Proof Note that in problem (7) the x_i belong to a compact set, which clearly guarantees the existence of the maximum. Furthermore, the concavity of the functions F_i guarantees its uniqueness. ∎

Having proven the above results, we are now in a position to obtain the Distribution Functions:

THEOREM 3 *For every* $k = 1, \ldots, m$, *the* kth *distribution function is*

$$\Psi_k(\xi) = \begin{cases} \dfrac{\alpha_k}{\sum_{i=1}^{m} \alpha_i} \xi & \text{if } \xi < \theta_m, \\ \dfrac{\alpha_k}{\sum_{i=1}^{j-1} \alpha_i} \left[\xi - \displaystyle\sum_{i=j}^{m} M_i \right] & \text{if } \theta_j \le \xi < \theta_{j-1} \le \theta_k, \\ M_k & \text{if } \xi \ge \theta_k, \end{cases}$$

with the coefficients

$$\theta_k = \sum_{i=k}^{m} M_i + \frac{M_k}{\alpha_k} \sum_{i=1}^{k-1} \alpha_i.$$

Suggestion for the Proof. In view of Theorem 1, if $\xi < \theta_m$, then the distribution functions $\Psi_k(\xi) < M_k$ for all k and it remains to derive the expression for y_k. If $\theta_m \le \xi < \theta_{m-1}$, then the maximum of $\sum_{i=1}^{m} F_i(y_i)$ cannot be attained in the interior. According to Lemma 2, at least $y_m = M_m$. Thus, $\Psi_m(\xi) = M_m$. The same argument applies to the remaining problem of dimension $m - 1$. Finally, repeating the argument once again, we obtain the results shown for the kth distribution function.

Now, following a similar reasoning to that used in [3], it is straightforwardly proven that the SC of the functions $\{F_i\}_{i=1}^m$ belongs to the class C^1 :

THEOREM 4 *Let* $\{F_i\}_{i=1}^m \subset C^1(\mathbb{R})$. *Let us consider*

$$(F_1 \circledast F_2 \circledast \cdots \circledast F_m)(\xi) := \max_{C_\xi} \sum_{i=1}^{m} F_i(y_i)$$

$$\text{with } C_\xi = \left\{ (y_1, \cdots y_m) \in \prod_{i=1}^{m} (0, M_i] \,\middle|\, \sum_{i=1}^{m} y_i = \xi \right\}.$$

Then

$$(F_1 \circledast F_2 \circledast \cdots \circledast F_m) \in C^1 \left(0, \sum_{i=1}^{m} M_i \right]$$

THEOREM 5 *The SC of the log functions $F_i(y_i)$ is a logarithmic piecewise function*:

$$(F_1 \circledast F_2 \circledast \cdots \circledast F_m)(\xi) = \begin{cases} \tilde{\beta}_{m+1} + \displaystyle\sum_{i=1}^{m} \alpha_i \ln\left(\frac{\xi}{p_i}\right) & \text{if } \xi < \theta_m, \\ \tilde{\beta}_k + \displaystyle\sum_{i=1}^{k-1} \alpha_i \ln\left(\xi - \sum_{i=k}^{m} M_i\right) & \text{if } \theta_k \le \xi < \theta_{k-1}, \end{cases}$$

with the coefficients

$$\tilde{\alpha}_k = \sum_{i=1}^{k} \alpha_i; \quad \tilde{\beta}_k = \sum_{i=1}^{k-1} \alpha_i \ln\left(\frac{\alpha_i}{\tilde{\alpha}_{k-1}}\right) + \sum_{i=k}^{m} \alpha_i \ln\left(\frac{M_i}{p_i}\right).$$

Moreover, it belongs to the class C^1.

3.2 *Solution of the utility maximization problem*

Having calculated the function $\tilde{u}(\mathbf{p}, \xi)$ and having established its character, C^1, considering the fact that $u(\mathbf{p}, \xi) = \mathbf{e}^{\tilde{u}(\mathbf{p}, \xi)}$, the following theorem provides us with the analytical solution of the utility maximization problem:

THEOREM 6 *The demand for the* kth *commodities is*

$$x_k(\xi) = \begin{cases} \dfrac{\alpha_k}{\tilde{\alpha}_m \cdot p_k} \xi & \text{if } \xi < \theta_m, \\ \dfrac{\alpha_k}{\tilde{\alpha}_{j-1} \cdot p_k} \left(\xi - \displaystyle\sum_{i=j}^{m} M_i\right) & \text{if } \theta_j \le \xi < \theta_{j-1} \le \theta_k, \\ \dfrac{M_k}{p_k} & \text{if } \xi \ge \theta_k, \end{cases}$$

and the maximum utility obtained for each budget level ξ is

$$u(\mathbf{p}, \xi) = \begin{cases} \displaystyle\prod_{i=1}^{m} \left(\frac{\alpha_i}{p_i \cdot \tilde{\alpha}_m}\right)^{\alpha_i} \cdot \xi^{\alpha_i} & \text{if } \xi < \theta_m, \\ \tilde{\mu}_k \displaystyle\prod_{i=1}^{k-1} \left(\frac{\alpha_i}{p_i \cdot \tilde{\alpha}_{k-1}}\right)^{\alpha_i} \cdot \left(\xi - \sum_{j=k}^{m} M_j\right)^{\alpha_i} & \text{if } \theta_k \le \xi < \theta_{k-1}, \end{cases}$$

with the coefficients

$$\tilde{\alpha}_k = \sum_{i=1}^{k} \alpha_i; \quad \tilde{\mu}_k = \prod_{i=k}^{m} \left(\frac{M_i}{p_i}\right)^{\alpha_i}.$$

Proof Taking into account the change in the variable $p_i x_i = y_i$ and Theorem 3, we obtain the expression of the demand for the kth *commodities*, $x_k(\xi)$.

Similarly, from Theorem 5, we obtain the expression of the *maximum utility for each budget level* ξ, $u(\mathbf{p}, \xi)$. ■

4. Example I

We shall now consider the following example:

$$u(\mathbf{p},\xi) = \max \quad \prod_{i=1}^{m} x_i^{\alpha_i},$$

$$\text{s.t.} \quad \sum_{i=1}^{m} p_i x_i = \xi,$$

$$0 \le x_i \le N_i,$$

with $m = 20$ commodities, and with the data presented in Table 1. This example has also been used in [3].

Figure 1 shows the graph of the utility function, $u(\mathbf{p},\xi)$, obtained for each budget level ξ.

In addition, the area where $\tilde{\alpha}_k > 1$ and where the utility function is hence a concave function, is highlighted in grey.

The values $\{\theta_k\}_{k=1}^m = \{142.78, 138.798, 127.86, 125.94, 122.04, 118.717, 111.115, 97.92, 97.2707, 90.7267, 88.51, 83.8387, 83.7333, 71.6176, 54.3, 52.4364, 45.42, 36.6, 31.3105, 27.4083\}$ constitute the different levels of budget level ξ at which the parameters of the utility function expression change. These correspond to the levels at which the different commodities reach their maximum value, which, according to the theoretical development, they do so in this example in the following order: $\{14, 4, 16, 20, 11, 5, 7, 3, 6, 13, 19, 9, 12, 15, 18, 8, 2, 17, 10, 1\}$. The analytical expression of the piecewise utility function is presented in Table 2, being obtained as shown in Theorem 6.

Table 1. Example data.

i	1	2	3	4	5	6	7	8	9	10
α_i	0.12	0.25	0.1	0.11	0.13	0.14	0.24	0.22	0.15	0.19
N_i	1	2	1.5	3	2.4	3.9	3	1	1.9	1
p_i	1.1	2	3.2	6.1	4	1.7	5	4.2	2.9	2
i	11	12	13	14	15	16	17	18	19	20
α_i	0.15	0.09	0.18	0.05	0.17	0.08	0.16	0.21	0.1	0.15
N_i	2	3	2.5	3	1	2.8	2	1.4	2	3
p_i	6	1.1	3	4.1	5	2.8	1	3	2	4.5

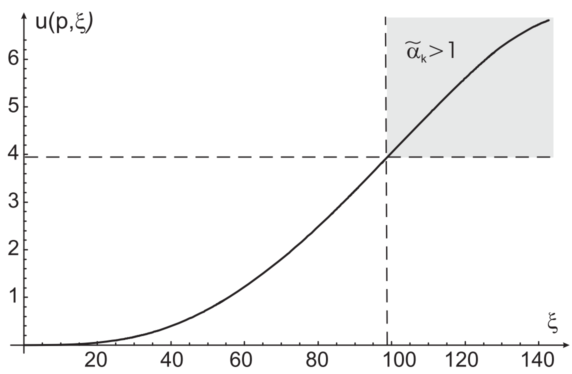

Figure 1. Utility function, $u(\mathbf{p},\xi)$.

Table 2. Piecewise utility function.

$u(\mathbf{p}, \xi)$	$\xi \in [a, b)$
$6.53045 \cdot 10^{-6} \cdot \xi^{2.99}$	$[0, 27.4083)$
$0.0000109281 \cdot (-1.1 + \xi)^{2.87}$	$[27.4083, 31.3105)$
$0.0000250898 \cdot (-3.1 + \xi)^{2.68}$	$[31.3105, 36.6)$
$0.0000513895 \cdot (-5.1 + \xi)^{2.52}$	$[36.6, 45.42)$
$0.000164155 \cdot (-9.1 + \xi)^{2.27}$	$[45.42, 52.4364)$
$0.000463575 \cdot (-13.3 + \xi)^{2.05}$	$[52.4364, 54.3)$
$0.00123358 \cdot (-17.5 + \xi)^{1.84}$	$[54.3, 71.6176)$
$0.00285854 \cdot (-22.5 + \xi)^{1.67}$	$[71.6176, 83.7333)$
$0.00451842 \cdot (-25.8 + \xi)^{1.58}$	$[83.7333, 83.8387)$
$0.00958282 \cdot (-31.31 + \xi)^{1.43}$	$[83.8387, 88.51)$
$0.0158164 \cdot (-35.31 + \xi)^{1.33}$	$[88.51, 90.7267)$
$0.0385114 \cdot (-42.81 + \xi)^{1.15}$	$[90.7267, 97.2707)$
$0.0768389 \cdot (-49.44 + \xi)^{1.01}$	$[97.2707, 97.92)$
$0.12455 \cdot (-54.24 + \xi)^{0.91}$	$[97.92, 111.115)$
$0.403284 \cdot (-69.24 + \xi)^{0.67}$	$[111.115, 118.717)$
$0.752439 \cdot (-78.84 + \xi)^{0.54}$	$[118.717, 122.04)$
$1.50284 \cdot (-90.84 + \xi)^{0.39}$	$[122.04, 125.94)$
$2.87947 \cdot (-104.34 + \xi)^{0.24}$	$[125.94, 127.86)$
$3.95551 \cdot (-112.18 + \xi)^{0.16}$	$[127.86, 138.798)$
$6.01492 \cdot (-130.48 + \xi)^{0.05}$	$[138.798, 142.78)$

These results allow us to check the C^1 character of the utility function.

5. Example II

Finally, in this section we present a realistic case based on a classic example, to which we have added inequality constraints on the inputs. We consider the micro-funded model presented in [8] which defines lifestyle and explains the relationship between health and income and the effects of income on health. The model proposed in [9] is augmented in [8] to produce a model with two equations: (1) the consumer utility function and (2) the health production function (HPF).

First, let us assume that an economy produces two commodities for consumption, x and z. The consumer utility function is assumed to be a Cobb–Douglas function in which health, h, is an input, and, for this reason, health affects the consumer utility function. The other two inputs are the commodities x and z. The utility function can be written as

$$U(h, x, z) = h^{\alpha} x^{\beta} z^{\delta}, \tag{10}$$

where α, β and δ are the elasticities of h, x and z, respectively. The HPF is

$$h(x, z, h_0, \psi, t, \varepsilon) = x^{\rho} z^{-\gamma} h_0 \psi\, e^{\phi t}\, e^{\varepsilon}$$

where health, in addition to the commodities x and z, also depends on the initial level of health (h_0), public health (ψ), time (t) and a stochastic component, ε. The function can be split into two parts: $x^{\rho} z^{-\gamma}$ can be interpreted as a consumer's activity, and $h_0 \psi\, e^{\phi t}\, e^{\varepsilon}$ can be attributed to other factors. To simplify the model, [8] use the relation: $\Omega = h_0 \psi\, \mathrm{e}^{\phi t}\, e^{\varepsilon}$ and assume $\Omega = 1$. The HPF is hence:

$$h(x, z) = x^{\rho} z^{-\gamma} \tag{11}$$

To maximize the consumer utility function, Equation (11) is substituted into Equation (10) obtaining:

$$U(x, z) = x^{\alpha\rho + \beta} z^{\delta - \alpha\gamma} \tag{12}$$

The elasticity with respect to x becomes $\alpha\rho + \beta$, and the elasticity with respect to z becomes $\delta - \alpha\gamma$. The consumer's budget constraint is

$$p_x \cdot x + p_z \cdot z = cy \tag{13}$$

where p_x and p_z are the prices of the goods, y is the per capita income used for consumption, and c is the average propensity to consume ($0 < c < 1$). Furthermore, we shall consider limitations on the maximum values of commodities x and z :

$$0 < x \leq M_x; \;\; 0 < z \leq M_z. \tag{14}$$

To maximize (12) w.r.t Equations (13) and (14), simply applying Theorem 6, we are able to present the closed formulae for the solution. Assuming, without loss of generality that $F'_x(M_x) \leq F'_z(M_z)$, the demand for the commodities is

$$x(y) = \begin{cases} \dfrac{\alpha\rho + \beta}{\beta + \delta + \alpha(\rho - \gamma)} \dfrac{cy}{p_x} & \textit{if } cy < \theta_2, \\ \dfrac{cy - p_z M_z}{p_x} & \textit{if } \theta_2 \leq cy \leq \theta_1, \end{cases}$$

$$z(y) = \begin{cases} \dfrac{\delta - \alpha\gamma}{\beta + \delta + \alpha(\rho - \gamma)} \dfrac{cy}{p_z} & \textit{if } cy < \theta_2, \\ M_z & \textit{if } \theta_2 \leq cy \leq \theta_1, \end{cases}$$

where

$$\theta_2 = p_z M_z \frac{\beta + \delta + \alpha(\rho - \gamma)}{\delta - \alpha\gamma}; \quad \theta_1 = p_x M_x + p_z M_z$$

and the maximum utility $u(p_x, p_z, y)$ obtained for each budget level y is

$$u = \begin{cases} \left(\dfrac{\alpha\rho + \beta}{\beta + \delta + \alpha(\rho - \gamma)} \dfrac{cy}{p_x} \right)^{\alpha\rho+\beta} \left(\dfrac{\delta - \alpha\gamma}{\beta + \delta + \alpha(\rho - \gamma)} \dfrac{cy}{p_z} \right)^{\delta-\alpha\gamma} & \textit{if } cy < \theta_2, \\ \left(\dfrac{cy - p_z M_z}{p_x} \right)^{\alpha\rho+\beta} (M_z)^{\delta-\alpha\gamma} & \textit{if } \theta_2 \leq cy \leq \theta_1. \end{cases}$$

6. Conclusions

In this paper, we have presented some of the mathematical economics problems in which the IC operator is a very useful tool. We have presented a summary of two applications to the FCM problem: the FCM problem with the Cobb–Douglas production function, and the FCM problem with the linear production function in economies of scale. Moreover, we have presented a new application: the analytical solution of the utility maximization problem.

IC is a mathematical tool with an enormous potential in the field of economics. It should be noted that this technique allows the analytical solution to be established in the general case with m inputs and considering constraints for the inputs. Our study presents a number of advantages with respect to other methods: the exact boundary solution is obtained and the method is not affected by the dimension of the problem. At the same time, it is easy to generalize to other studies, such as the classic profit maximization problem, including maximum constraints for the inputs.

Finally, we stress the fact that our paper does not solve a single concrete problem, but rather a uniparametric family of problems resulting from varying the budget level ξ in the equality constraint.

References

[1] B. Acciaio, *Short note on inf-convolution preserving the Fatou property*, Ann. Financ. 5(2) (2009), pp. 281–287.
[2] T. Arai, *Convex risk measures on Orlicz spaces: Inf-convolution and shortfall*, Math. Financ. Econ. 3(2) (2010), pp. 73–88.
[3] L. Bayon, J.M. Grau, M.M. Ruiz, and P.M. Suarez, *The explicit solution of the profit maximization problem with box-constrained inputs*, Appl. Math. Comput. 217(21) (2011), pp. 8705–8715.
[4] L. Bayon, J.A. Otero, M.M. Ruiz, P.M. Suarez, and C. Tasis, *The profit maximization problem in economies of scale*, J. Comput. Appl. Math. 236(12) (2012), pp. 3065–3072.
[5] H.P. Benson, Concave minimization: Theory, applications and algorithms, in *Handbook of Global Optimization*, R. Horst, and P.M. Pardalos, eds., Kluwer, Dordrecht, 1995, pp. 43–148.
[6] W.J. Boyes and M. Melvin, *Microeconomics*, 8th ed., Cengage Learning, Mason, OH, 2010, pp. 146–157.
[7] C. Burgert and L. Rüschendorf, *On the optimal risk allocation problem*, Stat. Decis. 24 (2006), pp. 153–171.
[8] C. Coppola, *Health, Lifestyle and Growth, Social Exclusion*, AIEL Series in Labour Economics, Springer-Verlag, Berlin Heidelberg, 2012, pp. 17–34.
[9] P. Contoyannis and A.M. Jones, *Socio-economic status, health and lifestyle*, J. Health Econ. 23 (2004), pp. 965–995.
[10] D. Filipovic and M. Kupper, *Monotone and cash-invariant convex functions and hulls*, Insur.: Math. Econ. 41(1) (2007), pp. 1–16.
[11] J.B. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms I*, Springer-Verlag, Berlin, 1996.
[12] G.A. Jehle and P.J. Reny, *Advanced Microeconomic Theory*, 2nd ed., Addison-Wesley, Boston, 2001.
[13] S.T. Liu, *Profit maximization with quantity discount: An application of geometric program*, Appl. Math. Comput. 190 (2007), pp. 1723–1729.
[14] D.G. Luenberger, *Microeconomic Theory*, McGraw-Hill, New York, 1995.
[15] J.J. Moreau, *Inf-convolution, sous-additivit e, convexit eriques*, J. Math. Pures et Appl. 49 (1970), pp. 109–154.
[16] N.S. Papageorgiou, *Convex integral functionals*, Trans. AMS 349(4) (1997), pp. 1421–1436.
[17] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.
[18] T. Stromberg, *The operation of infimal convolution*, Diss. Math. 352 (1996), pp. 1–58.
[19] J.B. Taylor and A. Weerapana, *Microeconomics*, 7th ed., Cengage Learning, Mason, OH, 2011, pp. 211–214.
[20] H.R. Varian, *Intermediate Microeconomics: A Modern Approach*, 7th ed., W.W. Norton & Company, New York, 2005.