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Economic Study of Problems of Depletion of Several Interrelated Non-renewable Resources

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Abstract In this paper we generalize the classic problem of the economic study of the extraction of non-renewable resources. The most notable generalizations presented here are the presence of constraints on the depletion rate, multiple resources and variable prices and costs over time. To solve the problem, we first use the theory of optimal control combined with a modification of the classic shooting method and an algorithm inspired by the cyclic coordinate descent algorithm. Numerous examples are presented to illustrate the possibilities the method offers.

Keywords Non-renewable resources · Gray's problem · Optimal control · Pontryagin's maximum principle · Transversality condition

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1 Introduction

The first economists to address the economics of extractive resources included W. S. Jevons, L. C. Gray, and H. Hotelling. Renewed interest in these papers arose following the first oil price shock in the 1970s. In Hotelling (1931), the premise is that global exhaustion is manifested in increasing prices of increasingly scarce extractive resources. In Gray (1914), exhaustion is introduced by the incorporation of increasing costs of extraction occurring over time. In the problem introduced in Gray (1914), the variables of the problem are:

$$\begin{array}{ll} S(t): & \text{the amount of resource in the ground at time } t \\ R(t): & \text{the amount depleted (per unit of time) at time } t \\ b(R(t)): & \text{the cost of extraction} \\ p: & \text{the price of the extracted resource} \\ r: & \text{the rate of discount} \end{array} \tag{1}$$

with:

$$\dot{S}(t) = -R(t). \tag{2}$$

For a more comprehensive exposition about the theory of non-renewable resources, we refer the reader to Dasgupta and Heal (1979), Fisher (1981), Hartwick (1989) and Perman et al. (2003). A well-known variant of the classic problem consists in considering that the cost of extraction, b , depends on the depletion rate $R(t)$, besides on the stock $S(t)$, i.e.: $b(R(t), S(t))$. This is a more realistic description of the extraction technology used for many exhaustible natural resources, and implies increasing extraction costs as the stock is depleted. We also consider the additional possibility of constraints on the depletion rates, $R(t)$, the possibility of a residual value for the remaining resource after a time T, and the possibility of a penalty for depletion time: $B[T, S(T)]$. We likewise consider the cost of depletion, $b(R(t), S(t))$, to be inversely proportional to the stock, $S(t)$, and directly proportional to the square of the depletion rate, $R(t)$, being k the constant of proportionality. The most widespread models simplify the problem by considering the price, and the constant of proportionality to be constant. In this paper, however, we also consider the possibility of these functions varying over time. In short, we shall employ with the following model:

$$\max_R \int_0^T \left[p(t)R(t) - \frac{k(t)R(t)^2}{S(t)} \right] e^{-rt} dt + B[T, S(T)]. \tag{3}$$

Furthermore, it is evident that in problems of non-renewable resources, the depletion time, T, is a factor to be taken very much into account. If the owner extracts the resource too fast, that will make the price fall, possibly to zero. If the owner extracts too slowly, although profits will be higher, they may be postponed to a too distant future. Is it more profitable to extract all the resources in a short time or to exhaust them slowly, prolonging the process over time? We shall thus address the problem of profit maximization in the extraction of non-renewable resources not only in terms of depletion rates, $R(t)$, but also in terms of time, considering T to be unknown:

$$\max_{R,T} \int_0^T \left[p(t)R(t) - \frac{k(t)R(t)^2}{S(t)} \right] e^{-rt} dt + B[T, S(T)]. \quad (4)$$

We believe that the joint analysis of these two problems makes the interpretation of the optimal solution more realistic.

One of the most common simplifications is to consider the model with a single, known, finite stock of a non-renewable resource. The aim of this paper is to go beyond this simplification, proposing, as shall be seen in Sect. 2, a multidimensional functional with N variables. To solve the problem, we first use the theory of optimal control. Optimal control theory is a well-known framework within which numerous economic problems have been addressed (see, for example, Sydsæter et al. 2000). We recommend reading Chaps. 14 and 15 in Perman et al. (2003), where the reader will find a comprehensive description of dynamic optimization problems using the maximum principle. Section 3 of the present paper provides a summary of the main mathematical results required. Given the complexity of the problem proposed here, it is necessary to use optimal control theory in combination with an adaptation of the classic shooting method and a variant of the cyclic coordinate descent algorithm. This will lead to the optimization algorithm presented in Sect. 4. which also presents the necessary and sufficient conditions guaranteeing the optimal nature of the solution thus obtained. Numerous examples are presented in Sect. 5 to illustrate the possibilities of the method. Finally, Sect. 6 outlines the main contributions of the paper.

2 Statement of the Multidimensional Variational Problem

In this section, we shall generalize problems (3) and (4) considering the problem of profit-maximizing depletion of several non-renewable resources from which different products with different market prices can be obtained, each of which has an extraction/production cost that is inversely proportional to the square of the depletion rate.

One possibility would be to consider N different resources independent of each other, such that the multidimensional problem will be additively-separable of the form:

$$\max_{R_1,\ldots,R_N} \int_0^T \sum_{i=1}^{N} \left[p_i(t)R_i(t) - \frac{k_i(t)R_i(t)^2}{S_i(t)} \right] e^{-rt} dt + \sum_{i=1}^{N} B_i[T, S_i(T)]. \quad (5)$$

This problem is much more simple than the problem we are proposing. In our work, conversely, the N different resources are not completely independent. We can consider, for example, the extraction of a product (say, quarry material) from which different resources N (different types of aggregates) can be obtained and exploited. Thus, the formula we propose is:

$$\max_{R_1,\ldots,R_N} \int_0^T \left[\sum_{i=1}^{N} p_i(t)R_i(t) - \frac{\sum_{i=1}^{N} k_i(t)R_i(t)^2}{\sum_{i=1}^{N} S_i(t)} \right] e^{-rt} dt + \sum_{i=1}^{N} B_i[T, S_i(T)], \quad (6)$$

and hence the presence of the term of the sum of stocks $\sum_{i=1}^{N} S_i(t)$ of the denominator which means that the production of any one of them influences the production cost of the others. In the general case of N resources, with T fixed, the optimization problem facing the producer is:

$$\max_{R_1,\ldots,R_N} \int_0^T \left[\sum_{i=1}^{N} p_i(t)R_i(t) - \frac{\sum_{i=1}^{N} k_i(t)R_i(t)^2}{\sum_{i=1}^{N} S_i(t)} \right] e^{-rt} dt + \sum_{i=1}^{N} B_i[T, S_i(T)],$$
$$\text{over the set: } \Theta := \{\mathbf{S} \in \left(\widehat{C}^1[0,T]\right)^N / \mathbf{S}(0) = \mathbf{S}_0, \mathbf{0} \le \mathbf{R}(t) \le \mathbf{M}\}, \tag{7}$$

where $\mathbf{S} = (S_1, \cdots, S_N)$ is the vector of admissible resources, $S_i(t)$ being the amount of resource "i" still to be extracted at time t, with:

$$\dot{S}_i(t) = -R_i(t),\ S_i(0) = S_{i0},\ S_i(T) \ge 0,\ i = 1, \ldots, N, \tag{8}$$

$p_i(t)$, the price of extracted resource "i", $\mathbf{R} = (R_1, \cdots, R_N)$ is the vector of admissible rates (of depletion of the resource), $R_i(t)$ being the amount depleted (per unit of time) of resource "i" at time t and $\mathbf{S}_0 = (S_{10}, \cdots, S_{N0})$ is the vector of admissible resources at time $t = 0$, S_{i0} being the amount of resource "i" in the ground at time $t = 0$, $\mathbf{M} = (M_1, \cdots, M_N)$ is the vector of the maximum constraints, M_i represents the maximum constraints for the depleted amount of resource "i", r is the rate of discount and $k_i(t)$ is a factor that represents the cost of extraction. The optimization problem (7) is to find the depletion profile, $\mathbf{R}^*$, maximizing the total profit.

Second, we address the multidimensional problem of profit maximization, leaving T free :

$$\max_{T,R_1,\ldots,R_N} \int_0^T \left[\sum_{i=1}^{N} p_i(t)R_i(t) - \frac{\sum_{i=1}^{N} k_i(t)R_i(t)^2}{\sum_{i=1}^{N} S_i(t)} \right] e^{-rt} dt + \sum_{i=1}^{N} B_i[T, S_i(T)],$$
$$\text{over the set: } \Theta := \{\mathbf{S} \in \left(\widehat{C}^1[0,T]\right)^N / \mathbf{S}(0) = \mathbf{S}_0, \mathbf{0} \le \mathbf{R}(t) \le \mathbf{M}\}. \tag{9}$$

The optimization problem (9) is to find the depletion profile, $\mathbf{R}^*$, and the time, T^*, that maximizes the total profit. We thus have two problems in this case: How much is it profitable to extract per unit of time? and How to determine the length of the depletion period?

Denoting:

$$L(t, \mathbf{S}(t), \mathbf{R}(t)) = \left[\sum_{i=1}^{N} p_i(t)R_i(t) - \frac{\sum_{i=1}^{N} k_i(t)R_i(t)^2}{\sum_{i=1}^{N} S_i(t)} \right] e^{-rt}, \tag{10}$$

and

$$B[T, \mathbf{S}(T)] = \sum_{i=1}^{N} B_i[T, S_i(T)]. \tag{11}$$

Both problems, (7) and (9) can be stated as optimal control problems. The state variable, $\mathbf{x}$, in these problems is the amount of remaining resource, $\mathbf{S}$, while the control variable, $\mathbf{u}$, is the rate of depletion, $\mathbf{R}$.

$$\mathbf{x}(t) = \mathbf{S}(t); \mathbf{u}(t) = \mathbf{R}(t). \tag{12}$$

In the next section, we shall see the mathematical foundations that allow us to solve the problem of maximizing the multidimensional functional:

$$J = \int_0^T L(\mathbf{x}(t), \mathbf{u}(t), t)dt + B[T, \mathbf{x}(T)], \text{ over } \Theta. \tag{13}$$

3 Mathematical Foundations

This section provides a summary of the mathematical bases we shall subsequently use to obtain the optimal solution of the problems. Due to the nature of problems (7) and (9) presented above, we believe that optimal control theory, and more specifically Pontryagin's Maximum principle (PMP), is the ideal tool. Recall that the optimal control approach allows solving problems in which the control variable has constraints. Let us begin by recapping this approach. The simplest optimal control problem, which is posed moreover in the unidimensional case, is that of a fixed end-time, T, and an free end state, $x(T)$:

$$\max_{u(t)} J = \int_0^T F(x(t), u(t), t)dt + B[T, x(T)], \tag{14}$$

subject to:

$$\dot{x}(t) = f(x(t), u(t), t), \tag{15}$$
$$u(t) \in U(t), 0 \leq t \leq T, \tag{16}$$
$$x(0) = x_0, \tag{17}$$

where x_0, and T are fixed. The following hypotheses are assumed to be verified: (i) F and f are continuous; (ii) F and f have partial first derivatives with respect to continuous t and x. They may not have a continuous derivative in u; (iii) The control variable, $u(t)$, may not be continuous, it only needs to be piecewise continuous; (iv) The state variable, $x(t)$, is continuous, but its derivative only needs to be piecewise continuous ($x(t)$ admits corner points); and (v) B has continuous partial first derivatives. The set of admissible controls, U, is often compact and convex.

A functional of the type considered above is said to be Bolza form. The first term of the functional, J, is an integral that depends on the values that $x(t)$ and $u(t)$ take over the time horizon and therefore evaluates the behaviour of the system over time. The second term, B, evaluates the state of the system at the end of the time interval. The Hamiltonian is defined as:

$$H(x(t), u(t), \lambda(t), t) = F(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t), \tag{18}$$

where $\lambda(t)$ is the costate variable. The following theorem is the fundamental result and establishes the necessary conditions of optimality for the problem being addressed here (Pontryagin et al. 1962) or (Macki and Strauss 1982).

Theorem 1 (Pontryagin's Maximum Principle (PMP)) *Let $u^*(t)$ be the optimal piecewise control path, and $x^*(t)$, the optimal associated state path, defined in the interval $[0, T]$. There is hence a continuous function, $\lambda^*(t)$, which has piecewise continuous first derivatives, such that for each $t \in [0, T]$, the following conditions are verified:*

$$\begin{aligned} &\text{(i)} \quad \dot{\lambda}^*(t) = -\frac{\partial H(x^*(t), u^*(t), \lambda^*(t), t)}{\partial x}; \lambda^*(T) = \frac{\partial B[T, x^*(T)]}{\partial x}(TC) \\ &\text{(ii)} \quad H(x^*(t), u^*(t), \lambda^*(t), t) \geq H(x^*(t), u(t), \lambda^*(t), t); u(t) \in U(t) \\ &\text{(iii)} \ \dot{x}^*(t) = f(x^*(t), u^*(t), t); x^*(0) = x_0. \end{aligned} \tag{19}$$

These are necessary but not sufficient conditions. Furthermore, the solution may not be interior and hence maximizing the Hamiltonian does not necessarily imply $\partial H/\partial u = 0$. The transversality condition (TC) is modified (see Pontryagin, 1962)) depending on the conditions of the problem. The above case corresponds to problem (7). To study problem (9), we must also consider the case in which the end-time, T, and the end state, $x(T)$, are free (the optimal T^* being unknown and to be determined). In this case, it is known that a further condition must be met in addition to conditions (i), (ii) and (iii), namely:

$$\text{(iv)} \quad H(x^*(T^*), u^*(T^*), \lambda^*(T^*), T^*) + \frac{\partial B[T^*, x^*(T^*)]}{\partial T} = 0. \tag{20}$$

If the dynamic function, f, and the integrand, F, have no explicit time-dependence, the problem is said to be autonomous. Then $H_t \equiv 0$, which implies that the Hamiltonian is constant throughout said solution:

$$H(x^*(t), u^*(t), \lambda^*(t)) = const. \tag{21}$$

We shall now present a sufficient condition for the optimum (Chiang 2000) or (Clarke 1983).

Theorem 2 (Mangasarian's Theorem) *Let $u^*(t)$, $x^*(t)$, $\lambda^*(t)$ be the results obtained when applying PMP, $\forall t \in [0, T]$, to the optimum control problem. If it is verified that: a) F and f are concave in x, u, for each $t \in [0, T]$; b) B is concave in x; and c) $\lambda^*(t) \geq 0$, for each $t \in [0, T]$, if $f(x(t), u(t), t)$ is nonlinear in x, u, then u^* is the optimal control problem, with x^* being the optimal state path and λ^*, the optimal path of the costate variables.*

There is no constraint on the sign of λ if f is linear in x and in u. We shall later see the verification of this theorem to guarantee the maximality of the solution obtained by PMP.

3.1 An Economic Interpretation of PMP

Economic interpretations for PMP are commonly based on the classic paper by Dorfman (1969), which is widely cited in books on optimal control in economics. The decision-making problem of a company is considered, the aim of which is to maximize the profits obtained over a time horizon. At each instant, t, we have that:

- $x(t)$ represents the company's stock. The initial stock is assumed to be known, $x(0)$.
- $u(t)$ represents the decisions the company makes; for example, regarding the production rate. These are the control variables. Such decisions will be subject to certain constraints, $u(t) \in U(t)$.
- $F(x(t), u(t), t)$ is the instantaneous rate of profit per unit of time.
- $B[T, x(T)]$ is the value (or cost) of the company at the end-time, T.
- $\lambda(t)$ is the marginal variation in optimum profits generated from t until the end, produced by a change in the stock at t. It is hence the shadow price.

Thus, the transversality condition (TC) of PMP is evident in view of the interpretation given to $\lambda(t)$. At T, no decision has to be made, the value of the company is S and hence the shadow price is $\partial S/\partial x|_{t=T}$.

Given the advantages in terms of resolution (21) that autonomous (explicitly time-independent) problems present, it is always desirable to deal with problems of this kind. Moreover, the only explicit appearance of time in many economic applications is via a discount factor in the integrand:

$$\max_{u(t)} J = \int_0^T F(x(t), u(t)) e^{-rt} dt. \tag{22}$$

We therefore sought how to transform the original problem into an alternative autonomous problem. This transformation results in what is called the current value Hamiltonian, which we summarize next (Barro and Sala-i-Martin 2004). We define a new Hamiltonian and a new costate variable:

$$\begin{aligned} H_C &= H \cdot e^{rt} = F(x(t), u(t)) + m \cdot f(x(t), u(t)) \\ m &= \lambda \cdot e^{rt}. \end{aligned} \tag{23}$$

It is straightforward to show (revised PMP) that the conditions of the optimum in terms of the Hamiltonian, H_C, and the costate variable, m, in current values are:

$$\begin{aligned} &\text{(i)} \quad \dot{m}^*(t) = -\frac{\partial H_C}{\partial x} + r \cdot m \text{ with: } m^*(T)e^{-rT} = 0 \\ &\text{(ii)} \quad \max_{u(t) \in U(t)} H_C \\ &\text{(iii)} \ \dot{x}^*(t) = \frac{\partial H_C}{\partial m} \qquad\qquad \text{with: } x^*(0) = x_0. \end{aligned} \tag{24}$$

We shall subsequently see that our method is able to work with more complex problems, which are explicitly dependent on the time, t, of both price and cost, and that, despite the fact that the previous current value of the Hamiltonian is no longer applicable, our technique solves problems of this type without any difficulty.

4 Optimization Algorithm: Necessary and Sufficient Conditions

We shall consider $\Theta := \prod_{i=1}^{N} \Theta_i$, being:

$$\Theta_i := \{S_i \in (\widehat{C}^1[0,T]) \,|S_i(0) = S_{i0} \text{ and } 0 \leq R_i(t) \leq M_i\}. \tag{25}$$

Let $\mathbf{s} = (s_1, \cdots, s_{i-1}, S_i, s_{i+1}, \cdots, s_N) \in \Theta$. The solution algorithm that we present here, inspired by the cyclic coordinate descent method, is based on the solution of a succession of N problems in each of which we consider the i-th component to be free ($i = 1, \cdots, N$), and the others fixed. We shall therefore first analyze the problem that consists in assuming all the components to be fixed except for one; e.g., the i-th component. The functional we wish to maximize $J^i_{\mathbf{s}}:\Theta_i \longrightarrow \mathbb{R}$:

$$\max_{R_i} J^i_{\mathbf{s}}(S_i) := \max_{R_i} \int_0^T \Bigg[-\sum_{\substack{j=1\\ j\neq i}}^N p_j(t)\dot{s}_j(t) + p_i(t)R_i(t) - \frac{\sum_{\substack{j=1\\ j\neq i}}^N k_j(t)\dot{s}_j(t)^2 + k_i(t)R_i(t)^2}{\sum_{\substack{j=1\\ j\neq i}}^N s_j(t) + S_i(t)} e^{-rt} \Bigg] dt + B_i[T, S_i(T)]. \tag{26}$$

Definition 1 We define the i-th maximizing map as the map $\Phi_i : \Theta \longrightarrow \Theta$ that satisfies:

$$\begin{aligned} &\text{(i) } \Phi_i(\mathbf{s}) - \mathbf{s} = (d_1, \cdots, d_N) \text{ with } d_j(\cdot) = 0, \forall j \neq i \\ &\text{(ii) } J(\Phi_i(\mathbf{s})) \geq J(\mathbf{z}), \forall \mathbf{z} \in \{\mathbf{z} \in \Theta | z_j(\cdot) = s_j(\cdot), \forall j \neq i\}. \end{aligned} \tag{27}$$

We shall denote by Φ the map associated with the ascent algorithm, which will be the composition of the i-th maximizing map:

$$\Phi = \phi_N \circ \cdots \circ \phi_1. \tag{28}$$

In every k-th iteration of the algorithm, through the i-th maximizing applications in the established order, thus obtaining the new, admissible element:

$$\mathbf{s}_k = \Phi(\mathbf{s}_{k-1}) = (\phi_N \circ \phi_{N-1} \circ \cdots \circ \phi_2 \circ \phi_1)(\mathbf{s}_{k-1}). \tag{29}$$

The limit of this ascending succession will be provided by the sought after maximum.

In the following section, we shall test a result that will allow us to characterize the maximum candidates of the proposed problem, $J^i_{\mathbf{s}}(S_i)$.

4.1 Maximum Necessary Condition

Definition 2 We define the i-th coordination function of $\mathbf{s} = (s_1, \cdots, s_{i-1}, S_i, s_{i+1}, \cdots, s_N) \in \Theta$ as:

$$\mathbb{Y}^i_{\mathbf{s}}(t) := \int_0^t \left(\frac{\sum_{\substack{j=1\\ j\neq i}}^N k_j(t)\dot{s}_j(s)^2 + k_i(t)R_i(s)^2}{\left(\sum_{\substack{j=1\\ j\neq i}}^N s_j(s) + S_i(s)\right)^2} \right) e^{-rs} ds$$
$$+ \left(p_i(t) - \frac{2k_i(t)R_i(t)}{\sum_{\substack{j=1\\ j\neq i}}^N s_j(t) + S_i(t)} \right) e^{-rt}. \quad (30)$$

Theorem 3 *If* $\mathbf{s} = (s_1, \cdots, s_{i-1}, S_i, s_{i+1}, \cdots, s_N) \in \Theta$ *is solution of the problem* (26)*, then there exists* $\{C_i\}_{i=1}^N \subset \mathbb{R}$ *satisfying:*

$$\mathbb{Y}^i_{\mathbf{s}}(t) \text{ is } \begin{cases} \leq C_i & \text{if } R_i(t) = 0 \\ = C_i & \text{if } 0 < R_i(t) < M_i \\ \geq C_i & \text{if } R_i(t) = M_i \end{cases} \quad (31)$$

Proof To prove the above result, we present the problem considering the state variable to be $x_i(t) = S_i(t)$, the control variable, $u_i(t) = R_i(t)$ and the state equation, $\dot{x}_i(t) = -u_i(t)$. The optimal control problem is thus:

$$\max_{u_i} \int_0^T \left[-\sum_{\substack{j=1\\ j\neq i}}^N p_j(t)\dot{s}_j(t) + p_i(t)u_i(t) - \frac{\sum_{\substack{j=1\\ j\neq i}}^N k_j(t)\dot{s}_j(t)^2 + k_i(t)u_i(t)^2}{\sum_{\substack{j=1\\ j\neq i}}^N s_j(t) + x_i(t)} \right] e^{-rt} dt$$
$$+ B_i[T, x_i(T)], \quad (32)$$

with:

$$\begin{cases} \dot{x}_i(t) = -u_i(t) \\ x_i(0) = S_{i0},\ x_i(T) \geq 0 \\ 0 \leq u_i(t) \leq M_i \end{cases} \quad (33)$$

We shall term the optimal control $u_i^*(t)$, therefore the optimal state will be $x_i(t)$. Let H be the Hamiltonian associated with the problem:

$$H(t, x_i, u_i, \lambda_i) = \left[-\sum_{\substack{j=1\\ j\neq i}}^{N} p_j(t)\dot{s}_j(t) + p_i(t)u_i(t) \right.$$
$$\left. - \frac{\sum_{\substack{j=1\\ j\neq i}}^{N} k_j(t)\dot{s}_j(t)^2 + k_i(t)u_i(t)^2}{\sum_{\substack{j=1\\ j\neq i}}^{N} s_j(t) + x_i(t)} \right] e^{-rt} - \lambda_i \cdot u_i(t). \quad (34)$$

In virtue of Pontryagin's Principle, there exists a piecewise C^1 function (costate variable), $\lambda_i^*(t)$ that satisfies the two following conditions:

$$\dot{\lambda}_i^*(t) = -\frac{\partial H(t, x_i(t), u_i^*(t), \lambda_i^*(t))}{\partial x_i} = -\left(\frac{\sum_{\substack{j=1\\ j\neq i}}^{N} k_j(t)\dot{s}_j(t)^2 + k_i(t)u_i(t)^2}{\left(\sum_{\substack{j=1\\ j\neq i}}^{N} s_j(t) + x_i(t) \right)^2} \right) e^{-rt}, \quad (35)$$

from which it follows that

$$\lambda_i^*(t) = -\int_0^t \left(\frac{\sum_{\substack{j=1\\ j\neq i}}^{N} k_j(t)\dot{s}_j(s)^2 + k_i(t)u_i(s)^2}{\left(\sum_{\substack{j=1\\ j\neq i}}^{N} s_j(s) + x_i(s) \right)^2} \right) e^{-rs} ds + C_i. \quad (36)$$

On the other hand, the following must be verified:

$$H(t, x_i^*(t), u_i^*(t), \lambda_i^*(t)) \geq H(t, x_i^*(t), u_i, \lambda_i^*(t)), \forall u_i \in [0, M_i]. \quad (37)$$

We therefore have the three following possibilities:

$$(1)\, 0 < u_i(t) < M_i \Longrightarrow 0 = \frac{\partial H(t, x_i(t), u_i^*(t), \lambda_i^*(t))}{\partial u_i} \quad (38)$$

$$\frac{\partial H(t, x_i(t), u_i^*(t), \lambda_i^*(t))}{\partial u_i} = \left(p_i(t) - \frac{2k_i(t)u_i(t)}{\sum_{\substack{j=1\\ j\neq i}}^{N} s_j(t) + x_i(t)} \right) e^{-rt} - \lambda_i^*(t) \Longrightarrow \quad (39)$$

$$0 = -\lambda_i^*(t) + \left(p_i(t) - \frac{2k_i(t)u_i(t)}{\sum_{\substack{j=1\\ j\neq i}}^{N} s_j(t) + x_i(t)} \right) e^{-rt} = \mathbb{Y}_{\mathbf{s}}^i(t) - C_i \Longrightarrow \mathbb{Y}_{\mathbf{s}}^i(t) = C_i. \quad (40)$$

$$(2)\, 0 = u_i(t) \Longrightarrow 0 \geq \frac{\partial H(t, x_i(t), u_i^*(t), \lambda_i^*(t))}{\partial u_i} \Longrightarrow \quad (41)$$

$$0 \geq -\lambda_i^*(t) + \left(p_i(t) - \frac{2k_i(t)u_i(t)}{\sum_{\substack{j=1\\j\neq i}}^N s_j(t) + x_i(t)} \right) e^{-rt} = \mathbb{Y}_{\mathbf{s}}^i(t) - C_i \Longrightarrow \mathbb{Y}_{\mathbf{s}}^i(t) \leq C_i. \tag{42}$$

$$(3)\; M_i = u_i(t) \Longrightarrow 0 \leq \frac{\partial H(t, x_i(t), u_i^*(t), \lambda_i^*(t))}{\partial u_i} \Longrightarrow \tag{43}$$

$$0 \leq -\lambda_i^*(t) + \left(p_i(t) - \frac{2k_i(t)u_i(t)}{\sum_{\substack{j=1\\j\neq i}}^N s_j(t) + x_i(t)} \right) e^{-rt} = \mathbb{Y}_{\mathbf{s}}^i(t) - C_i \Longrightarrow \mathbb{Y}_{\mathbf{s}}^i(t) \geq C_i. \tag{44}$$

□

Note The solutions to the problems considered here (7) and (9) verify Theorem 3 in each of its components, as well as the transversality condition that derives from Pontryagin's Principle:

$$\lambda_i^*(T) = \frac{\partial B_i[T, x_i^*(T)]}{\partial x_i}. \tag{45}$$

Furthermore, the problem (9) has to satisfy the additional condition (iv):

$$H(T^*, x_i^*(T^*), u_i^*(T^*), \lambda_i^*(T^*)) + \frac{\partial B_i[T^*, x_i^*(T^*)]}{\partial T} = 0. \tag{46}$$

The following proposition, whose demonstration is identical to that of Theorem 3, is verified, thus guaranteeing that, via maximizing applications, the algorithm presented here leads to the maximum.

Proposition 1 *For any* $\mathbf{s} = (s_1, \ldots, s_N) \in \Theta$*,there exists* $C_i \in \mathbb{R}$, *such that if:*

$$\phi_i(\mathbf{s}) = (s_1, \cdots, s_{i-1}, S_i, s_{i+1}, \cdots, s_N), \tag{47}$$

then:

$$\mathbb{Y}_{\Phi_i(\mathbf{s})}^i(t)\; is \begin{cases} \leq C_i & if \quad R_i(t) = 0 \\ = C_i & if \quad 0 < R_i(t) < M_i \\ \geq C_i & if \quad R_i(t) = M_i \end{cases} \tag{48}$$

The peculiar form of $\phi_i(\mathbf{s})$ expressed in this Proposition allows us to undertake its approximate calculation using similar numerical methods to those used to solve differential equations in combination with an appropriate adaptation of the classic shooting method. The problem (7) will consist in applying Φ, the map associated with the ascent algorithm. In every k-th iteration of the algorithm, the N components will have been maximized through the i-th maximizing applications, ϕ_i, in the established order, thus obtaining the new admissible element. To construct $\phi_i(s)$, we shall find for each C_i the function S_i which satisfies $S_i(0) = S_{i0}$ and the conditions of Proposition 1, and from among these functions, the one for which the transversality condition is satisfied.

For problem (9), we must perform a double shooting method in which not only the transversality condition is satisfied, but additional condition (iv) of PMP Theorem must also be fulfilled (19).

4.2 Maximum Sufficient Condition

We shall now see that the conditions that the functions present in our model (7) and (9) must verify in order to ensure that the solution of Theorem 3 obtained by the algorithm (31) is optimal are satisfied. To do so, we shall now present the sufficient condition of optimal control (Chiang 2000) or (Clarke 1983), known as Mangasarian's Theorem, which we presented previously (Theorem 2).

The concavity of L with respect to S and R is evident. We shall assume the concavity of B with respect to x (the models of the examples used show this assumption to be true). Furthermore, in the problems proposed here, $f(x(t), u(t), t) = -u(t)$, therefore f is linear and hence there is no constraint on the sign of the costate variable, λ.

5 Examples

In this section, we shall demonstrate the versatility of our method via several examples. In effect, we shall see that we can impose bounds on the control, work simultaneously with N non-renewable resources and consider non-constant prices and costs which are variable over time.

5.1 Example 1

Let us begin by comparing the two problems presented previously: (7) and (9) analyzing the influence of the control limits on the optimal solution. To do so, we shall consider the simplest case, with only one resource $N = 1$ and constant prices and costs. For problem (7) with a fixed end-time , we have considered $T = 1$, obviously leaving T free for problem (9). The remaining data, shown in Table 1, correspond to the following model:

$$J = \int_0^T \left[pR(t) - \frac{kR(t)^2}{S(t)} \right] e^{-rt} dt - T. \tag{49}$$

As can be seen, we have chosen a model with a final cost:

$$B[T, x(T)] = -T, \tag{50}$$

Table 1 Parameters of the model

p	k	r	$S(0)$	M
1.5	1	0.1	10	5

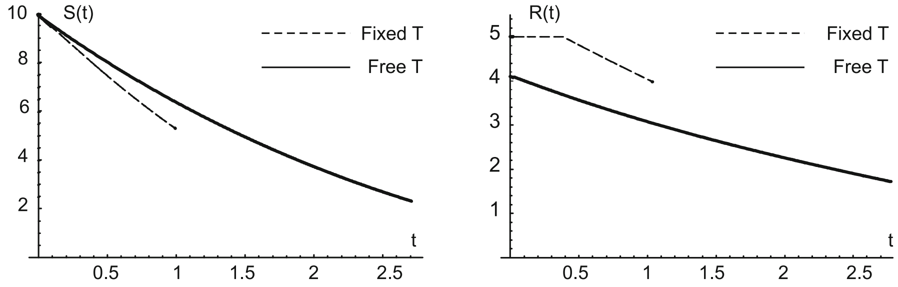

Fig. 1 Comparison of both problems

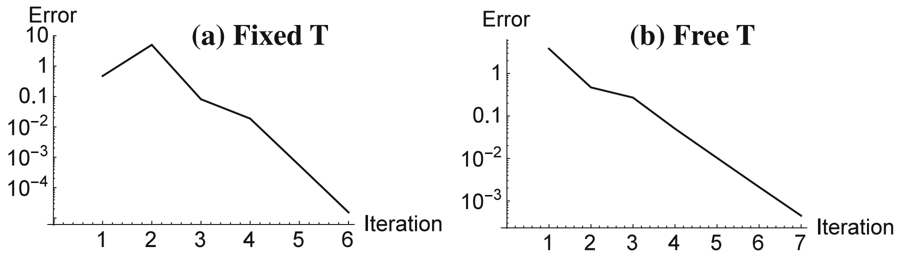

Fig. 2 Convergence of the algorithm. **a** Fixed *T*. **b** Free *T*

which penalizes the time taken to exhaust the resource. The optimal results obtained for the depletion profile, $R(t)$, and the remaining resource, $S(t)$, can be seen in Fig. 1 for both problems.

In problem (9) (with T free), we obtain an optimal value of $T^* = 2.72$, with a profit $\max J = 4.074$, a final value of stock, $S(T^*) = 2.3367$, and a depletion profile, $R(t)$, which reaches the maximum value of $R(0) = 4.1019$ without reaching the maximum bound at any time.

The approach to the two problems is clearly different. By limiting the depletion time in problem (7), we see that the depletion profile, $R(t)$, now reaches the maximum permissible value of $M = 5$ for values of $t \in [0, 0.36]$, a notably lower profit of only $\max J = 2.914$ is obtained and a more than double final value of stock, $S(T) = 5.3203$, for a depletion time (with fixed $T = 1$) also considerably smaller than in problem (9). It is therefore evident that knowing the time it will take to deplete a non-renewable resource beforehand means that the approach to extracting the resource to maximize profits is completely different.

The algorithm converges rapidly for both problems. The algorithm corresponding to problem (7) needs only adjust the transversality condition (i) of PMP Theorem (1). This example should therefore fulfil the following condition:

$$\lambda^*(T) = 0. \tag{51}$$

As previously stated, we use a variant of the shooting method to adjust this value.

Employing a discretization of 500 subintervals, the algorithm runs very quickly (see Fig. 2a). In the example, we achieve the prescribed tolerance in (51): $tol = 5 \times 10^{-4}$,

Table 2 Parameters of the model

p_1	k_1	r	$S_1(0)$	M_1
1	1	0.1	10	10

in only 6 iterations, the CPU time required by the program being 0.84 sec on a personal computer (Intel Core 2/2.66 GHz).

When we consider a free T, the algorithm corresponding to problem (9) must not only adjust the transversality condition (i) of PMP (1), but also fulfil the additional condition (iv) of PMP Theorem (1), which in this example leads to:

$$H(x^*(T^*), u^*(T^*), \lambda^*(T^*), T^*) - 1 = 0. \tag{52}$$

Accordingly, we must use a double shooting method in this case.

With the same discretization of 500 subintervals, the algorithm also runs very quickly (see Fig. 2b) despite the increase in complexity. In the example, we achieve the two prescribed tolerances: in (51), $tol_\lambda = 5 \cdot 10^{-4}$, and in (52), $tol_H = 5 \cdot 10^{-4}$, in only 7 iterations, the CPU time required by the program being 33.61 sec on a personal computer (Intel Core 2/2.66 GHz).

5.2 Example 2

Case 2A.

We shall now see the effect of including N resources simultaneously. As a basic model, we shall use the case with only one resource, $N = 1$, a fixed end-time, $T = 1$, constant prices and costs, and the data that appear in Table 2.

The model considered for $N = 1$ is once again (49). The optimal solution thus obtained is: a profit max $J = 0.9134$, a final value of stock $S_1(T) = 6.3837$, and a depletion profile, $R_1(t)$, which reaches a maximum at $R_1(0) = 4.0438$, without reaching the permitted maximum of $M_1 = 10$.

In the general case with N resources, the model takes the following form:

$$J = \int_0^T \left[\sum_{i=1}^{N} p_i R_i(t) - \frac{\sum_{i=1}^{N} k_i R_i(t)^2}{\sum_{i=1}^{N} S_i(t)} \right] e^{-rt} dt - T. \tag{53}$$

Let us assume that all resources have the same parameters: p_i, k_i, $S_i(0)$, and M_i, and that resource $i = 1$. Thus, as N increases, the comparison will be more reliable, as it will not be influenced by the particular characteristics of each one of the resources (Fig. 3).

In Table 3, we compare the number of iterations, It, and the time needed, $t(s)$, to reach the prescribed tolerance in (51): $tol = 5 \cdot 10^{-4}$. We use the same discretization of 500 subintervals as in Example 1. The table also shows the profit, max J, the final value of stock, $S_i(T)$, and the maximum depletion rate, $R_i(t) \equiv R_i(0)$, which reaches the permitted maximum, M_i, in some cases.

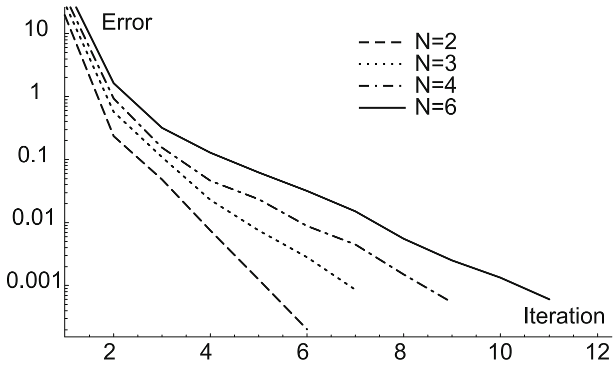

Fig. 3 Convergence of the algorithm in the multidimensional case

Table 3 Influence of the number of resources

N	It	$t(s)$	max J	$\frac{\max J}{N}$	$S_i(T)$	$R_i(0)$
1			0.9134	0.9134	6.3837	4.0438
2	6	12.35	5.3919	2.6959	4.4127	6.8069
3	7	17.42	11.346	3.7821	3.2197	8.8390
4	9	37.28	18.227	4.5568	2.4558	10
6	11	80.93	33.560	5.5934	1.6397	10

We do not present iterations or time for the case $N = 1$ as the concept of the algorithm is completely different in the multidimensional case and hence the comparison is invalid. Recall that for $N > 1$, we now use a variant of the cyclic coordinate descent method. We thus carry out successive runs of the algorithm ($j = 1, 2, \ldots$), cyclically adjusting the resources ($i = 1, 2, \ldots, N$) one-by-one until the prescribed tolerance is reached. In this case, for the convergence of the algorithm, the error has been considered as the sum of the differences (in absolute value) of the values of the constant, C_i, between two consecutive iterations, $j - 1$ and j, of the previous algorithm, C_i^{j-1} and C_i^j:

$$Error(j) = \sum_{i=1}^{N} \left| C_i^j - C_i^{j-1} \right| < tol_N = 1 \cdot 10^{-3}. \tag{54}$$

The algorithm is seen to perform magnificently with increasing N, eventually displaying a quadratic convergence: $O(N^2)$, i.e. the processing time of the algorithm grows in proportion to square of number of resources N.

Table 3 also shows that the constraint on the control affects the optimal solution in some cases: specifically, for $N = 4$, $R_4(t) = M_4 = 10$, for $t \in [0, 0.064]$, and for $N = 6$, $R_6(t) = M_6 = 10$, for $t \in [0, 0.364]$. Figure 4 shows the behavior of the depletion profile, $R(t)$, and the remaining resource, $S(t)$, in greater detail.

On the other hand, the results for the profit obtained by the optimal solution should not surprise us. The profit, max J, increases markedly overall; however, what we might call the profit per resource also increases substantially: $\frac{\max J}{N}$. The explanation for this can be found in the model used (53). Increasing N leads to an increase in the denom-

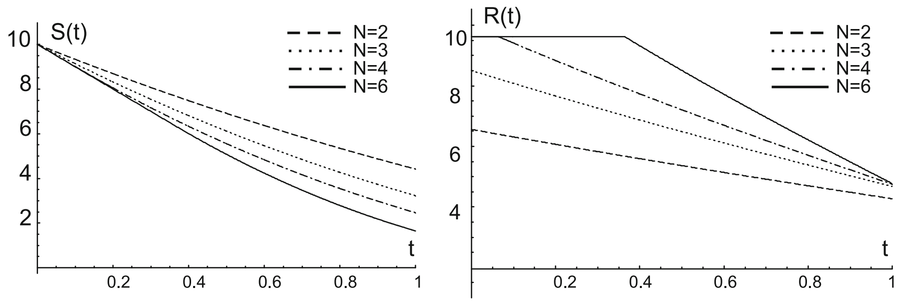

Fig. 4 Optimal solution in the multidimensional case

Table 4 Influence of the number of resources

N	$S_i(0)$	max J	$\frac{\max J}{N}$	$S_i(T)$	$R_i(0)$
1	10	0.9134	0.9134	6.3837	4.0438
2	$\frac{10}{2}$	2.19596	1.09798	2.2063	3.40345
3	$\frac{10}{3}$	3.11544	1.03848	1.0737	2.94552
4	$\frac{10}{4}$	3.80688	0.95172	0.6136	2.5982

inator of the term representing cost, $b(R(t), S(t))$, which is inversely proportional to the stock we have considered in all the resources: $S_i(0) = 10, \forall i$.

Case 2B.

For this reason, it also makes sense to posit a different modelling, in which we impose the following condition:

$$\sum_{i=1}^{N} S_i(0) = S_1(0) = 10, \tag{55}$$

in order for the comparison to be more effective. The rest of the model remains the same as before:

$$J = \int_0^T \left[\sum_{i=1}^{N} p_i R_i(t) - \frac{\sum_{i=1}^{N} k_i R_i(t)^2}{\sum_{i=1}^{N} S_i(t)} \right] e^{-rt} dt - T. \tag{56}$$

Table 4 presents the results thus obtained, considering $N = 1, 2, 3, 4$, but in such a way that condition (55) is verified and assuming for the sake of simplicity that the resources are equal. Hence: $p_i = 1$, $k_i = 1$, and $M_i = 10$.

Comparing the results in Table 4 with those obtained in Case 2A in Table 3, it can be seen that the values of $R_i(0)$, for $N = 2, 3$, where the solution is unconstrained, in Case 2B are just $1/N$ those obtained in Case 2A, while the opposite result in obtained in $S_i(T)$. This result is obviously due to having now taken $S_i(0) = \frac{10}{N}$.

As for profit, it is seen to behave in a specific way. While the total profit, max J, grows with N, the profit per resource, $\frac{\max J}{N}$, does not increase steadily, but presents a maximum for $N = 2$, and then decreases. This result obtained with model (56), along with constraint (55), is very interesting with a view to planning the optimization of non-renewable resources and diversifying overall stock in terms of both depletion and selling it on the market, thus enabling improved planning of tasks and optimization of profits.

5.3 Example 3

We conclude by presenting the case of considering non-constant prices $p(t)$ and costs $k(t)$, which are variable over time. This example leads to non-autonomous problems and hence the well-known technique of the current value of the Hamiltonian (24) is no longer applicable. However, as our approach is more general, the solution is obtained without any modification of the algorithm. As an example, we present the following case. Consider one single resource $N = 1$, with a free end-time T, while the remaining data, shown in Table 5, correspond to the following model:

$$J = \int_0^T \left[p(t)R(t) - \frac{k(t)R(t)^2}{S(t)} \right] e^{-rt} dt - T. \qquad (57)$$

We have thus considered 3 cases, successively increasing the cost of depletion, $k(t)$. The optimal results obtained can be seen in Fig. 5, while significant values are also presented in Table 5. The computation times of the 3 cases are of a similar order, but are significantly higher than those obtained in Sect. 5.1, with T being free, though with constant prices and costs.

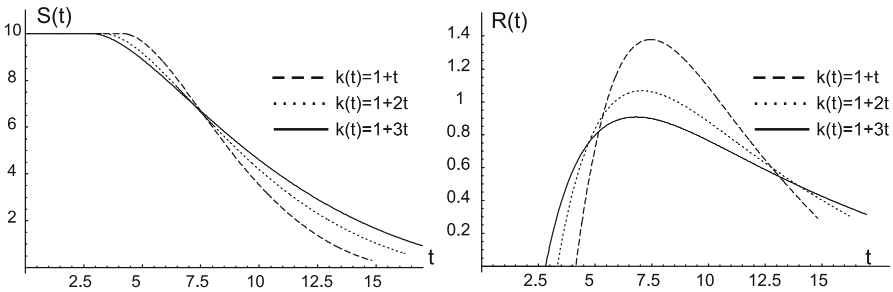

Fig. 5 Non-constant prices, $p(t)$, and costs, $k(t)$

Table 5 Parameters of the model and optimal solution

Case	$p(t)$	$k(t)$	r	$S(0)$	M	max J	T^*	It	$t(s)$
(i)	$1+2t$	$1+t$	0.1	10	10	46.5605	14.8176	10	269.52
(ii)	$1+2t$	$1+2t$	0.1	10	10	38.718	16.2174	10	195.53
(iii)	$1+2t$	$1+3t$	0.1	10	10	33.0745	16.975	9	153.86

The very different behavior of the optimal solution can now be appreciated, as the prices, $p(t)$, and costs, $k(t)$, are no longer constant. Whereas before the depletion profile, $R(t)$, was always decreasing, its behavior is now totally different. It is curious to see that as the cost, $k(t)$, increases more and more over time, the profit, max J, logically decreases. However, the result we obtain is not so intuitive; we find that the optimal time, T^*, also increases at the cost of notably reducing the depletion profile, $R(t)$. The reason for this can be found by carefully analyzing the model employed (57).

6 Conclusions

In this paper, we have presented a generalization of the classic problem of optimizing profits when working with non-renewable resources. Our mathematical approach goes beyond the classic use of optimal control theory, as we address problems of greater complexity than those previously reported in the literature. We summarize below the main highlights of our work.

- Possibility of working with n non-renewable resources from which different products can be obtained. The solution algorithm that we have presented, inspired by the cyclic coordinate descent model, allows multidimensional problems to be solved easily with reasonable computation times, as the algorithm exhibits quadratic convergence.
- We are able to impose bounds on each of the proposed controls. It is very important to consider constraints of this kind for the problem to be realistic. Optimal control is the perfect mathematical tool for this purpose.
- Prices and costs can be explicit functions of time, t. In this case, the classical technique of the current value Hamiltonian is no longer applicable. However, our technique solves these problems without any difficulty.
- We present the analysis set of two optimization problems: with a fixed depletion time in the first and another in which the end-time is also the object of optimization.

We believe that all the above results in a study of a general nature, with a broad range of future applications, given the highly versatile modelling presented here. Furthermore, the numerous solved examples can serve as a source of comparison for future studies that aim to address the same type of problem, though using other techniques.

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