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An algorithm for quasi-linear control problems in the economics of renewable resources: The steady state and end state for the infinite and long-term horizon

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A B S T R A C T

This paper presents the problem of finding the optimal harvesting strategy, maximizing the expected present value of total revenues. The problem is formulated as an optimal control problem. Combining the techniques of Pontryagin's Maximum Principle and the shooting method, an algorithm has been developed that is not affected by the values of the parameter. The algorithm is able to solve conventional problems as well as cases in which the optimal solution is shown to be bang–bang with singular arcs. In addition, we present a result that characterizes the optimal steady-state in infinite-horizon, autonomous models (except in the discount factor) and does not require the solution of the dynamic optimization problem. We also present a result that, under certain additional conditions, allows us to know a priori the final state solution when the optimization interval is finite. Finally, several numerical examples are presented to illustrate the different possibilities of the method.

1. Introduction

Renewable resources [1] include those resources that exhibit growth, maintenance, and recovery from exploitation over an economic planning horizon. The economics of this resources traditionally considers stocks of fish, forests, or freshwater. The principal question in the management of renewable resources is [2]: How much of a resource should be harvested during the present bearing in mind the future time periods? Time is usually considered over the horizon of a single manager or economic operation. Moreover, from an economic point of view, the economic value has been traditionally discounted to take into account a positive time preference. In this paper, we consider the particular case of harvesting of fish [3], although its extension to other problems of renewable resources is straightforward.

Harvesting of fish is both an ecological and economic issue. In the economic context, the renewable resource problem is stated as a maximization of profit over a future time horizon, subject to the natural dynamics of the harvested resource, an initial stock size, a target for the end of the optimization interval (or a limit in the case of an infinite-time horizon), and other technology constraints. A large number of papers has focused on the optimal harvesting policies when the resource stock follows deterministic models (see an excellent summary in [4]). In this paper, we also consider a deterministic environment.

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When formulating a physical problem, the first step is to obtain a suitable mathematical model of it. Simplifications are frequently used to make the problem tractable. The previous literature has typically focused on the case of linear harvest functions. In this paper, however, we avoid the errors associated of using a linear model or the inconvenience of estimating the values of more complex models for the harvest function. To this end, and in line with [5,6], we model the dynamics of the fish stock biomass considering the harvest as a independent variable.

Furthermore, what usually occurs is that the model parameters present a range of variation and this variation can lead to problems of different mathematical nature. This is the case of the parameters associated with marine fishing [5,6]. Faced with the complication of having to use different techniques when the functional is linear or nonlinear in the control variable, our method presents the contribution of being valid in cases that range between quasi-linearity to singular arcs. It is also valid, of course, in conventional solutions. As we shall see later, the stated problem can be considered as an Optimal Control (OC) problem, which has some noteworthy features. First, the optimization interval is infinite. Second, the time t is not explicitly present in the problem (time-autonomous problem), except in the discount factor. Third, we impose constraints on the control and, four, it constitutes a quasi-linear problem when considering real values for the parameters. To solve the problem, we have designed an algorithm capable of jointly handling all these features. In short, we may state that our method makes use of a basic result of OC: Pontryagin's Maximum Principle (PMP) [7] combined with the shooting method [8] to build this optimization algorithm.

Another important property of intertemporal economic problems is related with their long-term stability, i.e., when the problem converges to a steady-state. The method developed in [9] characterizes the optimal steady-state in single-state, infinite-horizon, autonomous models (except in the discount factor) by means of a function, called the evolution function, and defined only in terms of the model's parameters. The method does not require the solution of the dynamic optimization problem. This method was applied in [10] for a very simple particular case. We now follow the same procedure, but for our more general functional. To the best of our knowledge, the stated problem has never been addressed using this approach. With respect to the previous result, we have also obtained a result that allows us, employing certain simplifications, to know a priori the final state solution when the optimization interval is finite, but of sufficient duration to have previously reached the steady state. Once again, it is not necessary to solve the dynamic problem, but it is necessary for the system to be autonomous.

Section 2 of the paper presents a complete overview of the different modellings needed for the problem being addressed. Furthermore, the main mathematical results that will be used in its solutions are presented. The optimization algorithm is developed in Section 3, with particular attention being paid to the theorem on which this algorithm is based, namely Theorem 3: a necessary and sufficient maximum condition. The most cumbersome part of the algorithm, where the concatenation of extremal arcs to impose constraints on the control is described in detail, is summarized in the Appendix. Section 4 presents several numerical examples to illustrate the algorithm's performance under different conditions. Thus, not only the steady state is analysed, but also the dynamics of the solution, the influence of the chosen model, its sensitivity to relevant parameters such as the initial stock and the discount rate, and a comparison with the case of a long-term finite time horizon. An example of a quasi-linear model is also presented in this section, thus illustrating how the algorithm is able to solve both conventional problems and problems which, by their nature, tend towards bang-singular solutions.

2. Statement of the problem

2.1. Biological models

For the study of the economics of a renewable resource [1], we shall first see the pattern of biological growth of the resource. In this paper, we consider the growth function for a population of some species of fish. We assume that this fishery has a *intrinsic growth rate* denoted by r, which represents the difference between the population's birth and natural mortality rates. Let us assume that the *population stock* is x. Then, in the absence of human harvesting, the *rate of change of the population*, $\dot{x}$, over time is given by:

$$\dot{x}(t) = rx(t). \tag{1}$$

Integrating this equation, and for a positive value of r, we can straightforwardly see that the population grows exponentially:

$$x(t) = x_0 e^{rt} \tag{2}$$

where x_0 is the initial stock level. This result is only plausible over a short interval of time. It is well known that any population of fish has a finite *carrying capacity*, k, which limits the population's growth. To include this effect, the literature presents several models in which the *actual growth rate* depends on the stock size. A commonly used functional form is the *simple logistic function (or Verhulst equation)*:

$$\dot{x}(t) = f_{sl}(x) = rx(t)\left(1 - \frac{x(t)}{k}\right) \tag{3}$$

where r denotes the intrinsic growth rate and k, the carrying capacity of the species. This model is a good approximation to the natural growth processes of many fish populations, because it has two properties: (1) *compensation*, i.e. the proportionate

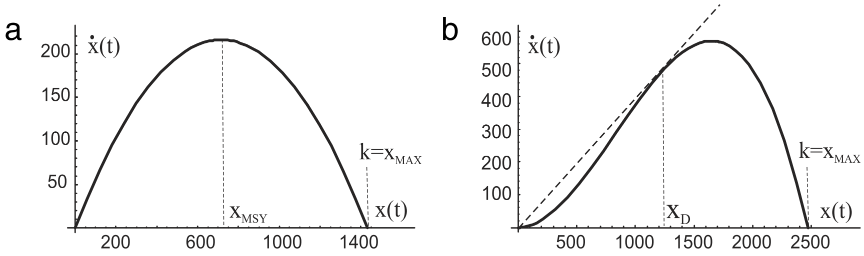

Fig. 1. Biological models.

growth rate of the stock ($\dot{x}/x$) declines as the stock size rises, and (2) a *maximum stock size k*. An example of the simple logistic function is shown in Fig. 1(a) using the data for Denmark cod [5] that we shall present in detail in Section 4.

The graph represents the relationship between the stock size, $x(t)$, and the rate of change, $\dot{x}(t)$, of the population due to biological growth. For the simple logistic function, $\dot{x}(t)$ is a quadratic function of the resource stock size, $x(t)$. The maximum amount of growth, x_{MSY}, will occur when the stock size is equal to half of k, where MSY are the initials for *maximum sustainable yield*. The Verhulst equation belongs to a more general class of models, the so-called Bernoulli equation:

$$\dot{x}(t) = r(t)x(t) - d(t)N^{\theta+1}; \quad r(t) = b(t) - d_0(t) \tag{4}$$

where $r(t)$ is, once again, the intrinsic growth rate.

Several generalizations of the logistic growth model exist and, among them, we will use the *modified logistic model* [1]:

$$\dot{x}(t) = f_{ml}(x) = rx^{\gamma}(t)\left(1 - \frac{x(t)}{k}\right). \tag{5}$$

This model has for $\gamma > 1$ the property of *depensation* (as opposed to compensation), i.e., at low stock levels, the proportionate growth rate ($\dot{x}/x$) is an increasing function of the stock size. A function showing depensation at stock levels below x_D (and compensation thereafter) is shown in Fig. 1(b), the data corresponding to Norway cod [5], which we shall use in Section 4. Other approaches are the Gompertz equation:

$$\dot{x}(t) = rx(t)\ln\frac{k}{x(t)} \tag{6}$$

the Beverton–Holt model, the Ricker equation, etc.

2.2. Harvesting and economic models

When human harvesting is included in the problem, the dynamics of the fish stock biomass (x) is modelled as:

$$\dot{x}(t) = f_l(x) - h(t) \tag{7}$$

$\dot{x}$ being the instantaneous change in stock biomass, h, the rate of biomass harvest, and $f_l(x)$, the logistic growth function. As already stated, we shall use the simple (3) and the modified (5) logistic growth function in this paper.

Many factors determine the size of the production function, i.e. the harvest, h. First of all, the harvest will depend on the amount of resources devoted to fishing and, for the sake of simplicity, many authors (see, for example, [1,10–12]) assume that all the different dimensions of the activity of harvesting can be aggregated into one magnitude called effort, $E(t)$. Second, and broadly speaking, the harvest will depend on the size of the resource stock. Hence, we have:

$$h(t) = H(E(t), x(t)). \tag{8}$$

Thus, in this model, h depends on the fishing effort, E (for example, size of nets, number of trawlers, number of fishing days) as well as the population level. This relationship can take a variety of particular forms. One very simple form [10,13], which appears to be a relatively good approximation, is given by:

$$H(t) = qE(t)x(t) \tag{9}$$

where q is a constant number, often called the catchability coefficient, $E(t)$ is fishing effort, and $x(t)$ is the fish stock level at time t. The proportionality constant, q, describes, how "easy" the fish can be harvested. This approach has obvious advantages in terms of mathematical tractability, but constitutes a relatively simple approach. A more general production function can be written in the Cobb–Douglas form:

$$H(t) = qE^{\alpha}(t)x^{\beta}(t) \tag{10}$$

where α and β are two positive parameters such that $\alpha + \beta \geq 1$ ((9) is the particular case with $\alpha = \beta = 1$). This is a widely used model in economics; however, its use in fishery models has been very uncommon [11].

In this paper, we avoid the inconvenience of using (9) or estimating the values of the Cobb–Douglas model for the harvest function. To this end, and in line with [5,6], we model the dynamics of the fish stock biomass (x) in the more general form as:

$$\dot{x}(t) = f_l(x) - h(t) \tag{11}$$

where $h(t)$ will be considered as a independent variable.

Let us now see how to model the cost functions. Let $\pi(x, h)$ be the instantaneous net revenue from the harvest of the stock biomass, given as [14]:

$$\pi(x, h) = p(h)h - c(x, h) \tag{12}$$

where $p(h)$ is the inverse demand function and $c(x, h)$, the cost function associated with the harvest. We assume an economic model with downward sloping demand and stock-dependent costs verifying:

$$\frac{\partial p(.)}{\partial h} < 0; \qquad \frac{\partial c(.)}{\partial h} > 0; \qquad \frac{\partial c(.)}{\partial x} < 0. \tag{13}$$

These conditions are accomplished in real-world fisheries. The functional forms for the demand and cost functions adopted in this paper are:

$$p(h) = p_0 - p_1 h \tag{14}$$

$$c(x, h) = \frac{ch^{\alpha}}{x} \tag{15}$$

where h represents landings of fish and p_0 and p_1 are coefficients. The cost $c(x, h)$ is defined as total costs less depreciation and interest payments, i.e., an approximation to total variable costs. These models correspond to a real-world fishery, where the price of the harvest depends on the amount harvested and the cost of harvesting depends on the stock biomass. Substituting (14) and (15) in (12), the profit function is:

$$\pi(x, h) = p_0 h - p_1 h^2 - \frac{ch^{\alpha}}{x} \tag{16}$$

where the meaning of the parameters is: p_0 is the price of the stock, p_1 is the strength of demand, c is the cost of exploitation and α is the harvest cost parameter.

2.3. Objective functional

Our model of renewable resource exploitation is an open-access fishery model, in which each firm takes the market price of landed fish as given. The firm's objective is to maximize profits from the harvest schedule over an infinite time horizon, subject to the dynamic constraint Eq. (7) and other natural and policy restrictions that involve limits (or bounds) for the harvest, $h(t)$, and stock, $x(t)$. Hence, our objective is to maximize profit from the harvest schedule over an infinite time horizon:

$$\max_{h(t)} \int_0^{\infty} \pi(x, h) e^{-\delta t} dt = \max_{h(t)} \int_0^{\infty} \left(p_0 h - p_1 h^2 - \frac{ch^{\alpha}}{x} \right) e^{-\delta t} dt \tag{17}$$

subject to:

$$\dot{x}(t) = f_l(x) - h(t); \qquad x(0) = x_0 \tag{18}$$

$$h(t) \in H(t); \quad x \in [0, k] \tag{19}$$

where $\delta > 0$ is the discount rate, i.e. the marginal returns on capital for the company, and x_0 is the initial stock level. Regarding this functional, we are interested in two kinds of problems or solutions in this paper:

(i) The first is the equilibrium or *steady-state solution*. In this solution, the resource stock size is unchanging over time (a biological equilibrium) and the fishing harvest is constant. Once equilibrium is achieved, it would remain unchanged provided that relevant economic or biological conditions remain constant.

(ii) The second kind of solution is devoted to the *dynamics* of renewable resource harvesting, i.e., the adjustment path towards the equilibrium, or from one equilibrium to another as conditions change. In this case, we study the fishery response over time to disturbances, and how a system would reach a steady state if it were not already in one.

As can be seen, the stated problem (17)–(19) is one of Optimal Control (OC) that presents a number of noteworthy features. First, the optimization interval is infinite. Second, the time t is not explicitly present in the problem (time-autonomous problem), except in the discount factor. Third, we impose constraints on the control and, fourth, it constitutes a problem which is quasi-linear when real values are considered for the parameters. In fact, in many real cases [5], p_1 (strength of demand) is close to 0, and α (harvest cost parameter) is near to 1. Hence, we are dealing with a quasi-linear model.

Faced with the complication of having to use different techniques when the functional is linear or nonlinear in the control variable, the contribution of our method is that it is valid in cases ranging between quasi-linearity and singular arcs. We have used the combined techniques of Pontryagin's Maximum Principle (PMP) [7] and the shooting method to build this optimization algorithm.

2.4. Optimal control theory

Due to the nature of problem (17), (18), (19) presented above, in which the control variable has constraints, we believe that optimal control theory, and more specifically Pontryagin's Maximum Principle, is the ideal tool. Let us begin by presenting the simplest optimal control problem, which is posed in the unidimensional case.

Case 1. *Fixed end-time*, T, *and free end state*, $x(T)$:

$$\max_{u(t)} J = \int_0^T F(x(t), u(t), t)dt + B[T, x(T)] \tag{20}$$

subject to:

$$\dot{x}(t) = f(x(t), u(t), t); \qquad x(0) = x_0 \tag{21}$$

$$u(t) \in U(t), \quad 0 \le t \le T \tag{22}$$

where x_0 and T are fixed. The following hypotheses are assumed to be verified: (i) F and f are continuous; (ii) F and f have partial first derivatives with respect to continuous t and x; (iii) The control variable, $u(t)$ needs to be piecewise continuous; (iv) The state variable, $x(t)$, is continuous, and its derivative needs to be piecewise continuous (i.e. $x(t)$ admits corner points); and (v) B has continuous partial first derivatives. The set of admissible controls, U, is often compact and convex. A functional of the type considered above is said to be Bolza form. The Hamiltonian is defined as:

$$H(x(t), u(t), \lambda(t), t) = F(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t) \tag{23}$$

where $\lambda(t)$ is the costate variable. The following theorem (see, for example, [7,15]) establishes the necessary (but not sufficient) conditions of optimality for the problem being addressed here (20), (21), (22).

Theorem 1 (*Pontryagin's Maximum Principle (PMP)*). *Let* $u^*(t)$ *be the optimal piecewise control path and* $x^*(t)$, *the optimal associated state path, defined in the interval* $[0, T]$. *There is hence a continuous function* $\lambda^*(t)$ *which has piecewise continuous first derivatives, such that for each* $t \in [0, T]$, *the following conditions are verified:*

$$\begin{array}{ll}
\text{(i)} & \dot{\lambda}^*(t) = -\dfrac{\partial H(x^*(t), u^*(t), \lambda^*(t), t)}{\partial x} \\
\text{(TC)} & \lambda^*(T) = \dfrac{\partial B[T, x^*(T)]}{\partial x} \\
\text{(ii)} & H(x^*(t), u^*(t), \lambda^*(t), t) \ge H(x^*(t), u(t), \lambda^*(t), t); \quad u(t) \in U(t) \\
\text{(iii)} & \dot{x}^*(t) = f(x^*(t), u^*(t), t); \qquad x^*(0) = x_0.
\end{array} \tag{24}$$

The solution may not be interior and hence maximizing the Hamiltonian (ii) does not necessarily imply $\partial H/\partial u = 0$. Moreover, the transversality condition (TC) is modified depending on the final conditions of the problem.

Case 2. *Fixed end-time*, T, *and fixed end state*, $x(T)$. The final condition for the state variable, $x(T)$, replaces the final condition for the co-state variable, $\lambda^*(T)$ (or transversality condition) that was obtained in the previous case.

Case 3. *Fixed end-time*, T, *and end state lower bounds*, $x(T) \ge \bar{x}$. In this case, TC is replaced by:

$$\lambda^*(T) - \frac{\partial B[T, x^*(T)]}{\partial x} \ge 0 \quad (=0, \text{ if } x(T) \ge \bar{x}). \tag{25}$$

Case 4. *Free end-time*, T, *and free end state*, $x(T)$. In this case, the optimal time, T^*, is unknown and to be determined. Moreover, it is known that a further condition must be met in addition to conditions (i), (TC), (ii) and (iii), namely:

$$\text{(iv)} \quad H(x^*(T^*), u^*(T^*), \lambda^*(T^*), T^*) + \frac{\partial B[T^*, x^*(T^*)]}{\partial T} = 0. \tag{26}$$

2.5. Infinite horizon problems

In this paper, we consider a functional with a very important additional feature: an infinite time horizon. Moreover, our functional will be stated in the Lagrangian form; i.e., using the term $B[T, x(T)] \equiv 0$. We thus have:

$$\max_{u(t)} J = \int_0^\infty F(x(t), u(t), t)dt. \tag{27}$$

When addressing problems in which the end-time, T, is not finite, drawbacks can arise when ensuring the convergence of the integral, which is the goal. An interesting case, and one which often occurs in economics, is that in which the integrand takes the form:

$$F(x(t), u(t), t) = G(x(t), u(t), t)e^{-\delta t} \tag{28}$$

where δ is some positive rate of discount common in economic analysis, and G is an upper bounded function. Under these conditions, the integral is found to be convergent for each admissible control. The idea is that the discount factor and the exponential function cause the function F to tend towards 0, when G has an upper bound, say $\overline{G}$:

$$\int_0^\infty G(x(t), u(t), t)e^{-\delta t}dt \le \int_0^\infty \overline{G}e^{-\delta t}dt = \frac{\overline{G}}{\delta}. \tag{29}$$

Another point to keep in mind is how to generalize transversality conditions suitably to the case of an infinite time horizon. If we focus on the case being addressed in this paper, with a free end state, $x(\infty)$, it seems logical to expect that it will be necessary to verify the following TC:

$$\lim_{t\to\infty} \lambda(t) = 0. \tag{30}$$

However, this condition is not always necessary for problems with an infinite time horizon, as can be seen in some well-known counterexamples. Nonetheless, it can be shown [16] that the condition (30) is necessary for problems with a time discount, as in our case.

2.6. Bang-singular solutions

To conclude this section, we also present the theory underlying the particular case of the control appearing linearly [15]:

$$\begin{aligned} &\max \int_0^T [f_1(t,x) + uf_2(t,x)]dt \\ &\dot{x}(t) = g_1(t,x) + ug_2(t,x); \qquad x(0) = x_0 \\ &u(t) \in U(t), \quad 0 \le t \le T \end{aligned} \tag{31}$$

because, as mentioned in the Introduction, the functional is very close to this case in real numerical examples. The Hamiltonian is now linear in u and can be written as:

$$H(t,x,u,t) := f_1(t,x) + \lambda g_1(t,x) + [f_2(t,x) + \lambda g_2(t,x)]u. \tag{32}$$

The optimality condition (ii), maximize H w.r.t. u, leads to:

$$u^*(t) = \begin{cases} u_{\max} & \text{if } H_u > 0 \\ u_{\text{sing}} & \text{if } H_u = 0 \\ u_{\min} & \text{if } H_u < 0 \end{cases} \tag{33}$$

and u^* is undetermined if:

$$\Phi(x,\lambda) \equiv H_u = f_2(t,x) + \lambda g_2(t,x) = 0. \tag{34}$$

The function Φ is called the *switching function*. If $\Phi = 0$ only at isolated points in time, then the optimal control switches between its upper and lower bounds, which is known as a *bang–bang control*. The times when the OC switches from $u_{\max}$ to $u_{\min}$ or vice-versa are called *switching times*. If $\Phi = 0$ for every t in some subinterval, then the original problem is called a *singular control problem* and the corresponding trajectory, a singular arc.

Finally, the following theorem [15,17] establishes a sufficient condition for the optimum.

Theorem 2 (*Mangasarian's Theorem*)**.** *Let $u^*(t)$, $x^*(t)$, $\lambda^*(t)$ be the results obtained when applying PMP, $\forall t \in [0,T]$, to the optimum control problem. If it is verified that:* (a) *F and f are concave in x, u, for each $t \in [0,T]$;* (b) *B is concave in x; and* (c) *$\lambda^*(t) \ge 0$, for each $t \in [0,T]$, if $f(x(t),u(t),t)$ is nonlinear in x, u, then u^* is the optimal control problem, with x^* being the optimal state path and λ^*, the optimal path of the costate variables.*

There is no constraint on the sign of λ if f is linear in x and in u. We shall see the verification of this theorem to guarantee the maximality of the solution obtained by PMP later on in the paper.

3. Optimization algorithm

Our problem (17)–(19) can be straightforwardly stated as an OC problem in Lagrange form simply considering $u(t) = h(t)$ as the control. Let us consider the following problem:

$$\max_{u(t)} J = \int_0^\infty F\left(t, x(t), u(t)\right) dt \tag{35}$$

subject to satisfying:

$$\dot{x}(t) = f\left(t, x(t), u(t)\right), \quad 0 \le t \le \infty \tag{36}$$

$$x(0) = x_0 \tag{37}$$

$$u(t) \in U(t), \quad 0 \le t \le \infty \tag{38}$$

where J is the functional, $F = \pi(x, h)e^{-\delta t}$ is the objective function, x, the stock, is the state variable, with initial condition x_0, the harvest $h \equiv u$ is the control variable, $U = [u_{\min}, u_{\max}]$ denotes the set of admissible control values, and t is the operation time, which starts from 0 and goes to ∞. The state variable must satisfy the state Eq. (36) with given initial conditions (37). In this statement, the final state is to be free. Let H be the Hamiltonian function associated with the problem

$$H(t, x, u, \lambda) = F(t, x, u) + \lambda \cdot f(t, x, u) \tag{39}$$

where λ is the costate variable. Using PMP, the optimal solution must be obtained from a two-point boundary value problem. In order for $u^* \in U$ to be optimal, a nontrivial function λ must necessarily exist, such that for almost every $t \in [0, \infty)$:

$$\dot{x} = H_\lambda = f; \qquad x(0) = x_0 \tag{40}$$

$$\dot{\lambda} = -H_{\mathbf{x}}; \qquad \lim_{t\to\infty} \lambda(t) = 0 \tag{41}$$

$$H(t, x, u^*, \lambda) = \max_{u(t)\in U} H(t, x, u, \lambda). \tag{42}$$

In virtue of PMP and Eq. (41), there exists a piecewise C^1 function λ that satisfies:

$$\dot{\lambda}(t) = -H_x = -F_x - \lambda(t) \cdot f_x \tag{43}$$

and hence:

$$\lambda(t) = \left[K - \int_0^t F_x e^{\int_0^s f_x dz} ds\right] e^{-\int_0^t f_x ds} \tag{44}$$

denoting $K = \lambda(0)$. From (42), it follows that for each t, $u(t)$ maximizes H. Hence, in accordance with the Kuhn–Tucker Theorem, for each t, there exists two real non negative numbers, β_1 and β_2, such that $u(t)$ is a critical point of:

$$\mathbb{H}(u) = F + \lambda(t) \cdot f + \beta_1 \cdot (u_{\min} - u) + \beta_2 \cdot (u - u_{\max}) \tag{45}$$

it being verified that if $u^* > u_{\min}$, then $\beta_1 = 0$ and if $u^* < u_{\max}$, then $\beta_2 = 0$. We thus have $\dot{\mathbb{H}} = 0$ and the following cases: Case (1) $u_{\min} < u^* < u_{\max}$. In this case, $\beta_1 = \beta_2 = 0$ and hence

$$F_u + \lambda(t) \cdot f_u = 0. \tag{46}$$

From (44) and (46), we have:

$$K = -\frac{F_u}{f_u} \cdot e^{\int_0^t f_x ds} + \int_0^t F_x \cdot e^{\int_0^s f_x dz} ds. \tag{47}$$

If we denote by $\mathbb{Y}_x(t)$ (the *coordination function*) the second member of the above equation (the *coordination equation*), the following relation is fulfilled:

$$\mathbb{Y}_x(t) = K. \tag{48}$$

Case (2) $u^* = u_{\max}$, then $\beta_2 \ge 0$ and $\beta_1 = 0$. By analogous reasoning, and bearing in mind that $f_u = -1 < 0$, we have:

$$\mathbb{Y}_x(t) \le K. \tag{49}$$

Case (3) $u^* = u_{\min}$, then $\beta_1 \ge 0$ and $\beta_2 = 0$. By analogous reasoning, we have:

$$\mathbb{Y}_x(t) \ge K. \tag{50}$$

The theoretical development carried out allows us to present a necessary maximum condition.

Theorem 3 (*A Necessary Maximum Condition*). *Let u^* be the optimal control, let $x^* \in \widehat{C}^1$ be a solution of the above problem. Then there exists a constant $K \in \mathbb{R}$ such that:*

$$\begin{array}{lll} \text{If } u_{\min} < u^* < u_{\max} & \Longrightarrow & \mathbb{Y}_{x^*}(t) = K \\ \text{If } u^* = u_{\max} & \Longrightarrow & \mathbb{Y}_{x^*}(t) \le K \\ \text{If } u^* = u_{\min} & \Longrightarrow & \mathbb{Y}_{x^*}(t) \ge K. \end{array} \tag{51}$$

Thus, the problem consists in finding for each K the function x_K that satisfies (37):

$$x_K(0) = x_0 \tag{52}$$

the conditions of Theorem 3 and, from among these functions, the one that satisfies the transversality condition:

$$\lim_{t\to\infty} \lambda(t) = 0. \tag{53}$$

From the computational point of view, the algorithm consists of two fundamental steps:

Step (1) *The construction of* x_K. The construction of x_K can be performed using a discretized version of the coordination equation (47). For each K, we construct the x_K, using (48) and when the values obtained do not obey the constraints (38), we force the solution to belong to the boundary until the moment established by conditions (49) and (50). Appendix summarizes this process of concatenating extremal arcs.

Step (2) *The calculation of the optimal* K. The calculation of the optimal K could be achieved by means of an adaptation of the shooting method. Varying the coordination constant, K, we search for the extremal that fulfils the second boundary condition (53). The procedure is similar to the shooting method, used to resolve second-order differential equations with boundary conditions, which may be performed approximately using elemental procedures. Starting out from two values for the coordination constant, K: $K_{\min}$ and $K_{\max}$ and using a conventional method such as the secant method, our algorithm converges satisfactorily, as we shall see in the section with numerical examples.

The transversality condition (53) can be easily imposed because, when $x(t)$ reaches the steady state, $\lambda(t)$ begins to converge asymptotically towards zero. The stopping criterion for the algorithm is based on the desired tolerance. We shall see this fact in greater detail in Example 1 in Section 4.

A sufficient minimum condition

We shall now verify the following conditions of Mangasarian's theorem. First, we see that F is concave in x, u, for each $t \in [0, T]$, given that in our case:

$$F(x, u, t) = \left(p_0 u - p_1 u^2 - \frac{cu^\alpha}{x}\right) e^{-\delta t}. \tag{54}$$

Second, we need to verify that f is also concave in x, u, for each $t \in [0, T]$. In our case, we have that:

$$\dot{x}(t) = f(x(t), u(t), t) = f_l(x) - u(t). \tag{55}$$

It is straightforward to verify that the condition is only fulfilled for the simple logistic growth function (3):

$$f_l(x) = f_{sl}(x) = rx(t)\left(1 - \frac{x(t)}{k}\right) \tag{56}$$

but not for the modified logistic growth function, $f_{ml}(x)$ (5). Third, given that $B \equiv 0$ in our model, B is concave in x. Finally, we only need to verify that $\lambda^*(t) \geq 0$, *for each* $t \in [0, T]$. Let us recall that from (46):

$$F_u + \lambda(t) \cdot f_u = 0. \tag{57}$$

It is therefore straightforward to obtain:

$$\lambda(t) = -\frac{F_u}{f_u} = F_u = \left(p_0 - 2p_1 u - \alpha \frac{cu^{\alpha-1}}{x}\right) e^{-\delta t} > 0 \tag{58}$$

in the case of real models where, as we shall see, it is always verified that $p_1 > 0$ and $\alpha > 1$.

With all the above, it has been demonstrated that, when the single logistic growth function is used, the conditions of Theorem 3 are also sufficient.

3.1. Steady-state solution

As presented in [9], when the terminal time is infinite, it is not always necessary to study the evolution to the optimal solution. For time-autonomous problems, where the time, t, is not explicitly present in the problem, except in the discount factor, the optimal solution is time invariant in the long term and converges to an equilibrium state. The method developed in [9] characterizes the optimal steady-state in single-state, infinite-horizon problems, by means of a simple function of the state variable, called the *evolution function*, defined in terms of the model's parameters. The method does not require the solution of the dynamic optimization problem. The method considers the one-dimensional, infinite-horizon problems of the form:

$$\max_{u(t)} J = \int_0^\infty G(x(t), u(t)) e^{-\delta t} dt \tag{59}$$

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad x(0) = x_0. \tag{60}$$

For a steady-state solution, $u = R(x)$, the *evolution function*, is defined by:

$$L(x) = \delta\left(\frac{G_u(x, R(x))}{f_u(x, R(x))} + \dot{W}(x)\right) \tag{61}$$

with:

$$W(x)=\frac{1}{\delta}G(x,R(x)). \tag{62}$$

The function $L(x)$ serves to formulate the following necessary condition for the location of the optimal steady state x_s:

$$L(x_s)=0 \tag{63}$$

that is valid for internal states (i.e., without the influence of the bounds, $U(t)$). The idea underlying the method can be illustrated considering the classic problem of Calculus of Variations of the following form:

$$\max\int_0^\infty F\left(t,x(t),\dot{x}(t)\right)dt=\max\int_0^\infty G(x(t),\dot{x}(t))e^{-\delta t}dt,\qquad x(0)=x_0. \tag{64}$$

The Euler equation corresponding to this problem:

$$F_x-\frac{d}{dt}F_{\dot{x}}=0 \tag{65}$$

computing the total derivative of the time partial $\frac{d}{dt}F_{\dot{x}}$, can be rewritten into an alternative version that is often more convenient:

$$F_x=F_{\dot{x}t}+F_{\dot{x}x}\dot{x}+F_{\dot{x}\dot{x}}\ddot{x}. \tag{66}$$

When the solution $x(t)$ tends towards a steady state, x_s, by definition:

$$\dot{x}=\ddot{x}=0 \tag{67}$$

and, for this particular functional (64):

$$F_{\dot{x}t}=-\delta F_{\dot{x}}. \tag{68}$$

We solve for x_s by setting $\dot{x}=\ddot{x}=0$ in the above Euler equation and obtain the implicit condition:

$$F_x=F_{\dot{x}t}\rightarrow G_x(x_s,0)+\delta G_{\dot{x}}(x_s,0)=0 \tag{69}$$

which corresponds to (63), since, according to the chain rule:

$$G_{\dot{x}}=\frac{dG}{du}\frac{du}{d\dot{x}}=\frac{G_u}{d\dot{x}/du}=\frac{G_u}{f_u}. \tag{70}$$

This method was applied in [10] for a particular case of considering the linear model for harvesting in terms of effort:

$$h(t)=qE(t)x(t) \tag{71}$$

and a very simple profit function of the form:

$$\pi(x(t),E(t))=ph(t)-c_1E(t)-\frac{c_2E^2(t)}{2}. \tag{72}$$

We now follow the same procedure, but for our more general functional. To the best of our knowledge, the stated problem has never been addressed using this approach. Considering:

$$\max_{u(t)}\int_0^\infty G(x(t),u(t))e^{-\delta t}dt=\max_{h(t)}\int_0^\infty \pi(x(t),h(t))e^{-\delta t}dt \tag{73}$$

$$\dot{x}(t)=f\left(x(t),u(t)\right)=f_l(x)-h(t);\qquad x(0)=x_0. \tag{74}$$

At the equilibrium state, x_s, $\dot{x}=0$, therefore the equilibrium harvest is obtained as:

$$u_s=R(x_s)\rightarrow h_s=f_l(x_s). \tag{75}$$

Differentiating the functions G and f with respect to h, we have:

$$\frac{G_u}{f_u}=\frac{\pi_h}{f_h}=-\pi_h. \tag{76}$$

Now from (62), we have:

$$W(x_s)=\frac{1}{\delta}G(x_s,R(x_s))=\frac{1}{\delta}\pi(x_s,f_l(x_s)). \tag{77}$$

Differentiating with respect to x_s gives $\dot{W}(x_s)$, and substituting in the evolution function (61) and equating to zero according to the necessary condition (63):

$$L(x_s)=\delta\left(-\pi_h(x_s,f_l(x_s))+\dot{W}(x_s)\right)=0. \tag{78}$$

The above equation can be solved for x_s, allowing us to obtain the harvest equilibrium, h_s. We can thus know the steady-state solution a priori, without solving the dynamic problem. This important result will be verified in the section containing numerical examples.

Table 1
Parameters for cod.

	Denmark	Iceland	Norway
r	0.603	0.6699	0.000665
k	1433	1988	2473
p_0	18.66	20.96	12.65
p_1	0.006344	0.0426	0.00839
c	3886.426	5363.179	5848.1
α	1.069	1.1	1.1

Where: r is the potential (or intrinsic) growth rate, k is the carrying capacity of the stock, p_0 is the price of the stock, p_1 is the strength of demand, c is the cost of exploitation and α is the harvest cost parameter.

3.2. Long-term horizon: the end state

Inspired by the previous section, in this section we present a new result that allows us to calculate a priori the final value that is reached when the optimization interval is not infinite (once again, without the need to solve the dynamic problem). To do so, we shall use the same model as above, but with an optimization interval $[0, T]$, assuming that it is long enough for the steady state to be reached in its development. In order to obtain the result, we need to make an additional assumption: the system must be autonomous, and hence we must consider $\delta = 0$. As in [9], the solution must be interior, unconstrained by bounds. Let us consider the following problem:

$$\max \int_0^T F\left(x(t), \dot{x}(t)\right) dt; \quad \dot{x}(t) = f_l(x) - h; \qquad x(0) = x_0 \tag{79}$$

$$\max_{h(t)} \int_0^T \pi(x, h) dt; \qquad \dot{x}(t) = f_l(x) - h; \qquad x(0) = x_0. \tag{80}$$

Based on the preceding hypotheses, the method starts out from the well-known result of the calculus of variations [15] which states that Euler's equation can be rewritten for autonomous systems as follows:

$$F - \dot{x}F_{\dot{x}} = cte. \tag{81}$$

Given that the solution for the steady state, (x_s, h_s) (with $\dot{x} = 0$), may be known a priori by means of the method explained in the previous section, the value of the constant, cte, present in (81) can be obtained straightforwardly:

$$cte = F(x_s) = \pi(x_s, h_s). \tag{82}$$

If we now consider the final moment, T, the two following conditions must be simultaneously verified at that moment, given that the end state is free:

$$\begin{cases} F_{\dot{x}}(x(T), h(T)) = 0 \\ F(x(T), h(T)) = cte. \end{cases} \tag{83}$$

The first is the transversality condition corresponding to the free end state, and the second the simplified Euler equation (81). Simply solving this system, the end state $(x(T), h(T))$ can be obtained straightforwardly.

4. Numerical examples

In this section, we shall see the excellent behaviour of our approach via several examples. We use the parameters estimated by [5] based on Northeast Arctic cod, capelin and herring. The model in Denmark and Iceland for the three species is the simple logistic function:

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{k}\right) \tag{84}$$

whereas in Norway, the model obtained is a modified logistic function:

$$\dot{x}(t) = rx^2(t)\left(1 - \frac{x(t)}{k}\right). \tag{85}$$

The data for cod can be seen in Table 1, and for capelin and herring, in Table 2. In these tables, growth is measured in 10^6 kg and time in year for the biological parameter values, prices are measured in NOK/kg for the economic parameter values and costs are measured in 10^6 NOK for the parameter values of the cost function. All the results presented in what follows were obtained using a program written by the authors and implemented in Mathematica 10.0 ©. For example, the equations are easily solved using the FindRoot command.

Table 2
Parameters for capelin and herring.

	Denmark	Iceland	Norway
r	0.5442	1.1008	0.00021781
k	4896	3669	8293
p_0	4.0104	1.211	1.0
p_1	0.0007511	0.0001	0.0
c	0.02198	0.000175	0.07
α	1.33	2	1.4

Where: r is the potential (or intrinsic) growth rate, k is the carrying capacity of the stock, p_0 is the price of the stock, p_1 is the strength of demand, c is the cost of exploitation and α is the harvest cost parameter.

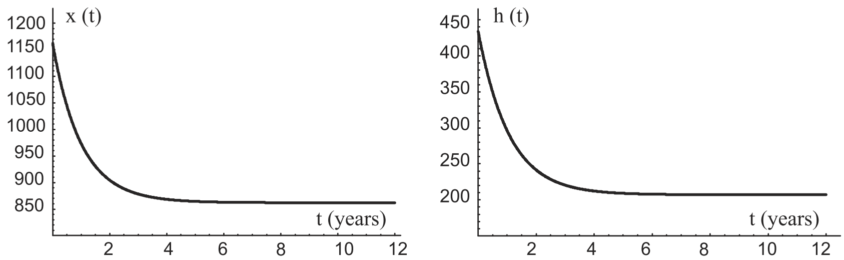

Fig. 2. Steady-state solution.

4.1. Example 1: steady-state solution

To begin, we present the basic case for Denmark cod, in which we shall obtain the steady-state solution. The results obtained for the optimal stock profile, $x(t)$ in (10^6 kg), and the optimal harvest path, $h(t)$ in (10^6 kg yr^{-1}), are shown in Fig. 2. We have assumed an initial stock $x_0 = 1162$ (10^6 kg), $x_0 \in [0, k]$, with $k = 1433$ (10^6 kg), these optimal paths being obtained using a 5% discount rate, δ. We shall maintain this value of δ in all the examples (unless otherwise noted).

The steady state in this model is characterized by:

$$x^* = 862.06\ 10^6\ (\text{kg}) \tag{86}$$

$$h^* = 207.14\ 10^6\ (\text{kg yr}^{-1}) \tag{87}$$

$$t^* = 4.4\ (\text{yr}) \tag{88}$$

where t^* is a parameter we have introduced to characterize the speed at which the steady state, x^*, is reached. It is defined as the time required for the solution to vary less than 0.5% from x^*. From that moment on, harvesting is adjusted to compensate for the natural growth of fish. This value of h^* can easily be verified using the simple logistic function $f_{sl}(x^*)$ for the case of Denmark cod. In this case, we have assumed that the constraints do not affect the harvest, $h(t)$, and we have considered $H(t) = [h_{\min}, h_{\max}] = [0, 500]$, measured in ($10^6$ kg yr^{-1}), i.e. open-access fishery, which means that the fishermen do as they choose, without any regulations being applied.

As noted in the previous section, the steady-state solution can be calculated a priori using Eq. (78). Solving this equation in x_s, we obtain very similar values to those obtained by means of our numerical solution:

$$x_s = 862.096\ 10^6\ (\text{kg}) \tag{89}$$

$$h_s = 207.105\ 10^6\ (\text{kg yr}^{-1}). \tag{90}$$

However, it should be recalled that what is most important is to determine both this value and the dynamics of the process towards said steady state. Using a discretization of 100 subintervals for each year, the algorithm ran very quickly. Greater discretization was not needed to obtain the desired accuracy. Starting out from two values for the coordination constant, K: $K_{\min}$ and $K_{\max}$ and using a conventional method such as the secant method, we achieve the prescribed tolerance in (53): $tol = 1 \cdot 10^{-5}$ in only 10 iterations (see Fig. 3(a)), the CPU time required by the program being 7.92 s on a personal computer (Intel Core 2/2.66 GHz). The optimal value of K is 7.715135951256791. The variation in $\lambda(t)$ and its asymptotic behaviour towards zero, can be seen in Fig. 3(b).

Finally, we present the value of the obtained optimum profits. In this case, it is more illustrative to consider a finite optimization interval so that the differences between the different cases considered can thus be more appreciable. For that reason, we show, for example, the optimum profits obtained during the 12 first years: $\max J_{12} = 22\,783.2$ (10^6 NOK).

4.2. Example 2: dynamics towards the steady state

In this example, we undertake some dynamics analysis for the model. We shall work once again with Denmark cod for the sake of comparison with the previous example. As already noted, once the steady state has been reached, the solution

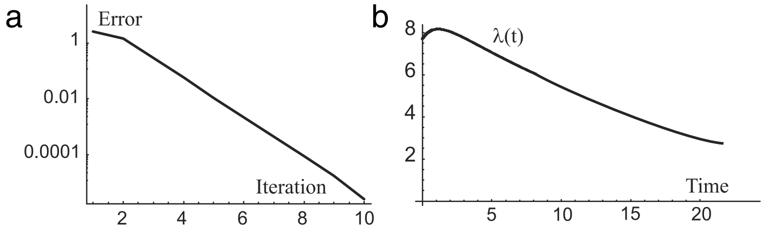

Fig. 3. Convergence of the algorithm.

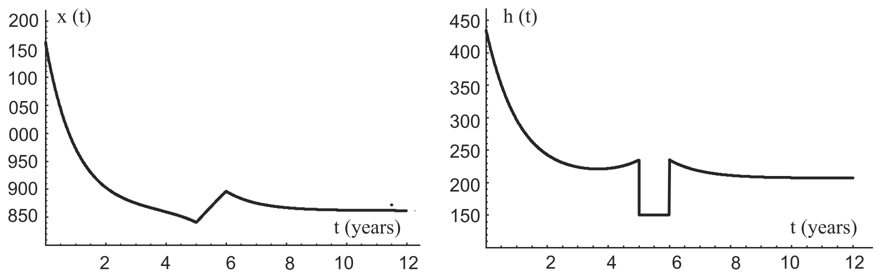

Fig. 4. Dynamic solution towards the steady state.

will remain constant as long as disturbances do not occur in the problem. That is precisely the situation that we shall now analyse, one that we consider of major interest. Suppose that for biological reasons, for instance, the harvesting rate needs to be reduced during a year. How does this affect the solution? Fig. 4 shows the optimal profile in the case that the harvesting of fish has a maximum limit of 150 (10^6 kg). That is, let us assume that $h(t) \in H(t) = [0, 150]$. This maximum limit, $h_{max} = 150$ (10^6 kg yr^{-1}), is notably lower than the value obtained for the steady state, $h^* = 207.10$ (10^6 kg yr^{-1}).

As can be seen, the optimal paths change with respect to the previous case. The stock logically recovers during the period of limited harvesting, $t \in [5, 6]$. However, the most noteworthy effect is that, before this period begins, the optimal profile of h no longer follows the trend towards $h*$, and harvesting rises to a value of 235.1 10^6 (kg yr^{-1}). Just after this period $[5, 6]$, harvesting commences with this same value and the solution then tends towards the steady state (86) and (87). The profits now obtained during the first 12 years are slightly lower than before: $\max J_{12} = 22753.1$ (10^6 NOK). The optimal solution is now obtained with a value of $K = 7.710904992048142$.

Note that this is precisely another of the features of our approach: the possibility of imposing, in a simple way, constraints on the control of the following kind: $h(t) \in H(t)$. This situation may be straightforwardly generalized to the case of considering more periods with constraints.

4.3. Example 3: sensitivity analysis

In this example we consider questions such as how a system would reach a steady state if it were not already in one. Let us now assume the biological parameters to be fixed for each country and species. This assumption is fairly realistic, at least for the medium-term horizon. Accordingly, we shall analyse the influence of the initial stock, as this is the parameter that is most likely to vary as a function of different biological circumstances like, for example, environmental catastrophes, diseases, etc. As regards the economic parameters and values of the prices, we shall only analyse the influence of the discount rate and the rest of parameters will be fixed.

The results are shown in Fig. 5 and Table 3. First we fixed $\delta = 0.05$ and we considered different values for the initial stock, x_0, within its maximum range. These values can be seen in the first column of Table 3. We have included very few legends in Fig. 5 as each case can be easily recognized on the basis of the values of x_0 and $h(0)$. Then we fixed $x_0 = 1162$ (10^6 kg), and we considered different values for the discount rate, δ. In this example, we have not wanted to consider constraints on the harvest, except the natural constraint of $h_{min} = 0$ (10^6 kg yr^{-1}), in order to be able to compare all the results.

As can be seen, the further x_0 moves away from the steady-state value, $x^* = 862.06$ (10^6 kg), the greater the time needed to achieve that state, t^*, increases. It is also plain to see that the greater x_0, the greater the profits, $\max J_{12}$. The table also shows the optimal K obtained by our algorithm. The units are: x_0 in 10^6 (kg), δ in (%), t^* in (year), $\max J_{12}$ in 10^6 NOK, and $h(0)$ in 10^6 (kg yr^{-1}).

Finally, the behaviour of $h(t)$ is worthy of comment. We see the directly proportional relationship obtained between x_0 and $h(0)$ and that harvesting is negligible for a certain time interval from values of $x_0 \leq 400$ 10^6 (kg) upward. It should be noted that part of this interval corresponds to the constraint $h_{min} = 0$ imposed by the problem. However, another part corresponds to an interval when harvesting is almost negligible. In order to be able to quantify this phenomenon, we define two new variables: t^0, which represents the time during which the constraint $h_{min} = 0$ is active; and t^1, defined as the time

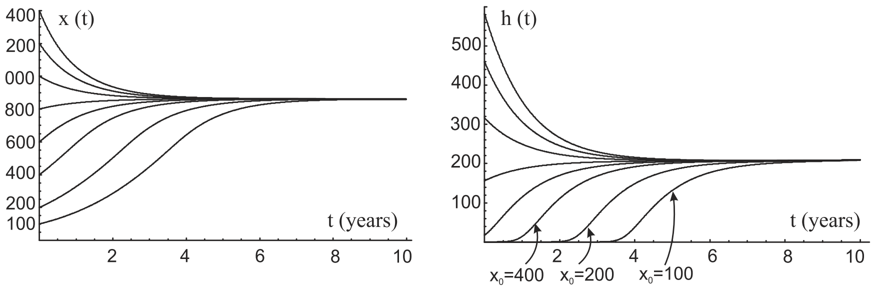

Fig. 5. Influence of the initial stock.

Table 3
Influence of initial stock and discount rate.

x_0	K	t^*	$\max J_{12}$	$h(0)$	δ	K	t^*	$\max J_{12}$	$h(0)$
1400	6.6603393	4.93	24573.0	582.81	3	7.8764491	4.33	25240.0	422.25
1200	7.5413941	4.51	23154.0	459.76	4	7.7951229	4.37	24007.0	428.24
1000	8.4606491	3.65	21554.0	316.69	5	7.7151359	4.41	22783.0	434.14
800	9.3205921	2.79	19772.0	156.12	6	7.6364540	4.44	21803.0	439.95
600	10.0185747	4.41	17836.0	17.13	7	7.5590614	4.48	20817.0	445.66
400	11.5816150	5.36	15710.0	0.0					
200	18.0849817	6.47	12904.0	0.0					
100	31.4387574	7.74	10580.0	0.0					

Table 4
Restriction on harvest.

x_0	t^0	$x(t^0)$	t^1	$x(t^1)$
400	–	–	0.59	509.8
300	0.06	308.6	1.22	509.6
200	0.87	308.0	2.03	508.7
100	2.15	307.8	3.31	508.4

required for harvesting to be $h(t) > 1\ 10^6$ (kg yr^{-1}). The results are given in Table 4. The units are: x_0, $x(t^0)$ and $x(t^1)$ in 10^6 (kg); t^0 and t^1 in (year).

As can be seen, t^0 and t^1 vary notably as a function of x_0. Nevertheless, although they are not strictly constant, the stock values corresponding to these values can in fact be considered to approach a threshold value. This shows that, although they are not an exclusive function of the model parameters, these two stock values can thus be considered an approximation. Furthermore, bear in mind the somewhat arbitrary nature of the definition of t^1. In addition, the behaviour of the control, $h(t)$, so close to 0 during the interval $\left[t^0, t^1\right]$, is not surprising, considering that this is a solution that tends towards a bang-singular path.

The influence of discount rate, δ, is only shown in Table 3, because the plots of $x(t)$ and $h(t)$ do not give relevant information. As can be seen, the greater δ, the greater the time needed to achieve the stationary state, t^*, and also $h(0)$ increases. Logically, the benefit decreases.

4.4. Example 4: model of the logistic function

Let us now see the influence of the model of the logistic function. To do so, we compare the case of Denmark cod with that of Norway cod, which, as already stated, was modelled by [5] by means of a modified logistic function of the form:

$$f_{ml}(x) = rx^2(t)\left(1 - \frac{x(t)}{k}\right). \tag{91}$$

The results are shown in Fig. 6. We have assumed an initial stock value $x_0 = 2400$ (10^6 kg), $x_0 \in [0, k]$, with $k = 2473$ (10^6 kg), and a 5% discount rate, δ. The steady state in this model is characterized by:

$$x^* = 2172.31\ 10^6\ \text{(kg)} \tag{92}$$

$$h^* = 381.55\ 10^6\ (\text{kg yr}^{-1}) \tag{93}$$

$$t^* = 2.56\ \text{(yr)}. \tag{94}$$

The optimal value of K is 0.6736459964308549 and the optimum profits obtained during the 12 first years: 15927.1 (10^6 NOK). As can be seen, the behaviour is qualitatively similar to that obtained for the simple logistic function. The only

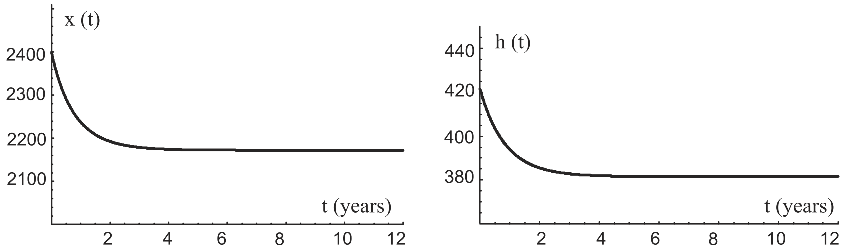

Fig. 6. Influence of logistic function.

Table 5
Parameters of the quasi-linear model.

Case	p_1	t^*	x^*	h^*	Case	α	t^*	x^*	h^*
0	p_1^0	4.4	862.0	207.1	0	α^0	4.4	862.0	207.1
1	$\frac{p_1^0}{5}$	2.4	838.6	209.9	1	$1+\frac{\alpha^0-1}{5}$	4.2	808.0	217.0
2	$\frac{p_1^0}{10}$	2.1	836.1	210.1	2	$1+\frac{\alpha^0-1}{10}$	4.2	802.4	217.4

Case	p_1	α	t^*	x^*	h^*
0	p_1^0	α^0	4.4	862.0	207.1
1	$\frac{p_1^0}{5}$	$1+\frac{\alpha^0-1}{5}$	2.2	790.4	213.7
2	$\frac{p_1^0}{10}$	$1+\frac{\alpha^0-1}{10}$	1.8	783.6	214.1

observed difference is in the speed at which the steady state is reached. In this case, the parameter t^* is only 2.56 (yr). The explanation lies in the steeper slope of the logistics curve.

4.5. Example 5: quasi-linear model

As already stated, two of the representative parameters of the model behave distinctively in the real case being addressed in this paper (Denmark cod). The strength of demand, p_1, is thus close to 0 and the harvest cost parameter, α, is close to 1. Let us see what happens if this effect is accentuated and we force the model to adopt to a quasi-linear equation of the form:

$$\pi(x,h) = p_0 h - p_1 h^2 - \frac{ch^\alpha}{x} \simeq p_0 h - \frac{ch}{x}. \tag{95}$$

This assumption is not farfetched. It suffices to observe in Table 2 what happens to p_1 when capelin and herring species are considered in the three countries. We now compare the solution with that obtained in Example 1 and shall therefore once again consider the same general conditions and the same constraints on the harvest $h(t)$, with $H(t) = [h_{\min}, h_{\max}] = [0, 500]$, measured in $(10^6\ \text{kg yr}^{-1})$. While they did not exert an influence in Example 1, we see that they now do so decisively. The results are shown in Table 5 and Fig. 7. The units are: t^* (year), x^* 10^6 (kg) and $h^* 10^6$ (kg yr^{-1}).

We start from the baseline case, with $p_1^0 = 0.006344$ and $\alpha^0 = 1.069$, and then progressively reduce their value. We study the influence of each coefficient separately and also jointly. Fig. 7(a) shows the behaviour when reducing p_1, Fig. 7(b) when reducing α and, finally, Fig. 7(c) presents the joint study.

As can be seen, the coefficient p_1 has the most pronounced effect on t^*. The coefficient α barely reduces t^*, although it does modify the values of x^* and h^*. As can be appreciated, the optimal solution for $h(t)$, when reducing both coefficients, tends progressively towards the bang-singular form, as befits a problem of a quasi-linear nature. It can thus be seen that the steady state is reached much faster. In this problem, the unique solution would be that corresponding to the steady state.

4.6. Example 6: comparison with a fixed end-time

Finally, we present an example in which we consider the end time to be fixed. Though up until now we have considered a range of infinite optimization, let us now analyse what happens if the interval is considered finite, $[0, T]$, considering for example, a $T = 10$, 15, and 20 (yr) planning period. Once again, we compare the results with the baseline case presented in Example 1: Denmark cod. The optimal results obtained can be seen in Fig. 8. A very striking fact can be observed on the optimal path. Yet again, the steady state (86) and (87) is reached at time $t^* = 4.4$ (yr). However, by setting an end-time for the harvesting of the biomass, the solution leaves the steady state when approaching the end of the interval and seeks a final stock value of approximately $x(T) \simeq 534.2\ 10^6$ (kg). It should be stressed here that this optimal value is not fixed in the approach, but is freely sought by the problem. Furthermore, it is also the same value regardless of the initial stock, x_0.

This end state value cannot be predicted a priori when the system has a discount, unlike what occurred with the steady-state value. However, it can be calculated when the discount is zero and therefore the system is autonomous. As

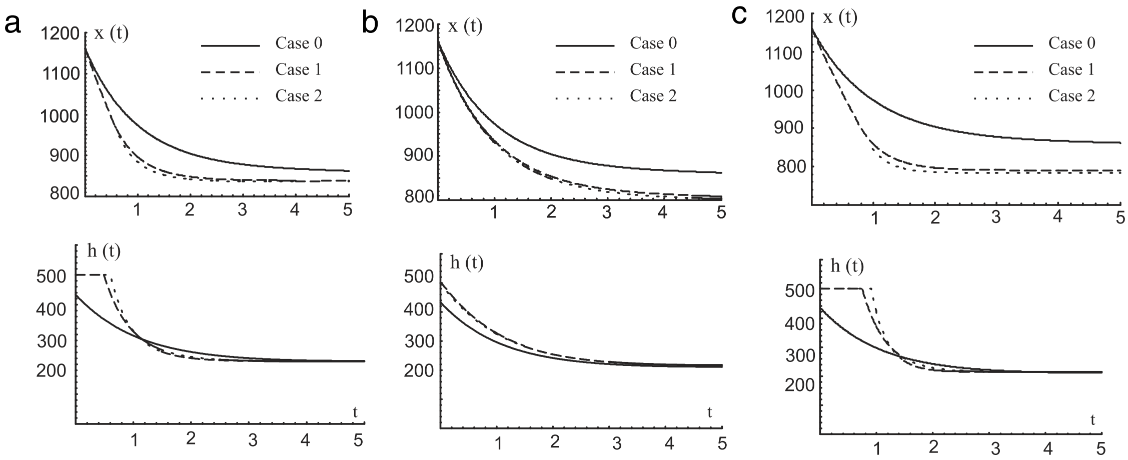

Fig. 7. Solution for the quasi-linear model.

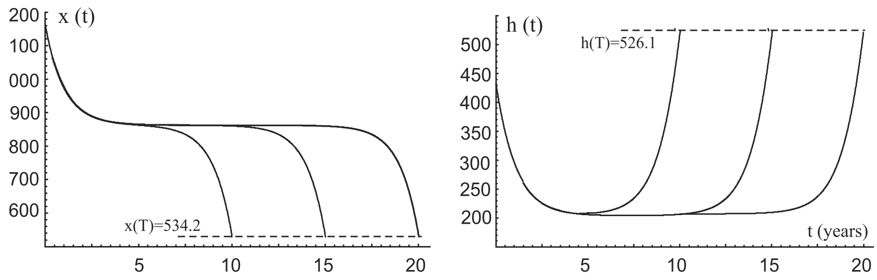

Fig. 8. Optimal solution with a fixed end-time.

Table 6
Influence of T on the end state.

T	1	2	3	4	5	6	7	8	9
$x(T)$	678.49	576.29	548.77	539.60	536.25	534.98	534.49	534.30	534.23

we saw earlier, in order to calculate this value, it suffices to solve the system (83), i.e., once we have previously calculated the value of the constant corresponding to the steady-state value (82). The optimization interval must be long enough for the steady state to be reached. In our example with $t^* = 4.4$, we have verified that it suffices to consider $T > 10$.

The following are the results for the same model as in Example 1 (Denmark cod), but assuming $\delta = 0$ instead of 5%. Solving Eq. (78) in x_s, we now have that:

$$x_s = 895.530\ 10^6\ (\text{kg}) \tag{96}$$

$$h_s = 202.538\ 10^6\ (\text{kg yr}^{-1}). \tag{97}$$

With these values, the value of the constant (82) becomes $cte = 2251.09$. Moreover, solving the system (83), we obtain the end state, $x(T)$, $h(T)$:

$$x(T) = 542.14\ 10^6\ (\text{kg}) \tag{98}$$

$$h(T) = 538.548\ 10^6\ (\text{kg yr}^{-1}). \tag{99}$$

Numerical simulations using our algorithm have fully confirmed these predicted values. These values, $x(T)$ and $h(T)$, are therefore an exclusive function of the model parameters.

Returning to the case with a discount ($\delta = 0.05$), the value of $x(T)$ is different to that obtained with $\delta = 0$, but the latter can be taken as a good approximation. As for the influence of the length of the interval, T, it should be noted that the value of $x(T)$ is practically constant from $T > 10$ upward, thus showing the negligible influence of the discount term, $e^{-\delta t}$, for long t. For values of $T < 10$, marked variations in the value of the end state can be observed for the lowest values of T. The results are shown in Table 6.

The reader might intuitively think that, in this case with a fixed T, the optimal solution would seek to reduce the stock much more at the end of the interval. As can be seen, however, this is not the case. This is related to the special features that problems of renewable resources present. To achieve the optimal end state, $x(T)$, the harvest, $h(t)$, undergoes a very sharp final increase of $h(T) \simeq 526.1$ (10^6 kg yr^{-1}). This value is also very similar for all the cases with $T > 10$. To obtain $x(T)$ without any constraints, we have assumed that $h_{\max} = 600$ (10^6 kg yr^{-1}).

Note that in this case the solution was obtained using the same algorithm as before, though now the transversality condition (53) has been changed to that presented in the PMP (24), i.e. $\lambda(T) = 0$. The reader should likewise note that there are other possible extensions of the problem, which we discuss in the final section of the paper.

Note. Similar results can be obtained using the values for Iceland cod, and for other varieties: capelin and herring. These results are not presented here for the sake of brevity. As the convergence of the algorithm was seen to behave in every case in a similar way to how it behaved in Example 1, we have therefore not shown this behaviour for the other examples.

5. Conclusions and future perspectives

In this paper, we have presented a completely general algorithm for solving optimal control problems within the framework of renewable resources. Although the case studied in detail in this paper corresponds to the model of harvesting marine resources, it can be straightforwardly extended to cover other problems of a similar nature (such as, for instance, forestry harvesting). The scope of the paper comprises all those problems in which mathematical modelling leads to models with variable parameters. Moreover, this variability often makes the type of solution to the optimal control problem vary. This means that many conventional algorithms become invalid, as they are not able to address conventional solutions, bang–bang solutions or solutions containing singular arcs simultaneously. The main advantages of our method worth highlighting are: rapid convergence, the possibility of easily imposing constraints on the control, versatility in tackling different logistic models, the ability to tackle quasi-linear models leading to solutions of the quasi-bang-singular type and models with different types of final conditions. The proposed algorithm, specially designed for this problem, shows good convergence properties.

We have also presented the adaptation to our model of an important result that allows us to obtain the value of the optimal solution of the steady state when the optimization interval is infinite. In the case of a finite optimization interval, though one with a sufficient value to reach the steady within said interval, we have presented a new result which, employing the simplification of an autonomous system, likewise allows us to obtain a priori the end value of the state and of the control.

Regarding future perspectives, we believe that the problem posed in this paper offers many possibilities. For example, a possible extension would be to consider the case of a free end-time, T, and fixed (or lower bounded) end state, $x(T)$. In this case, the optimum total time of harvesting of the resource would constitute the unknown variable in the problem. Another possible extension would be to consider several species simultaneously, so that there is some kind of biological interaction between them. In this case of multispecies, however, the OC problem becomes multidimensional and new mathematical tools must be designed. Finally, we cannot fail to mention another possible variant of the study, namely the addition of stochasticity to make the deterministic model more realistic.

Appendix. Formal construction of the dynamic solution

To formally construct the function x_K, we shall consider:

$$0 = t_0 < t_1 < \cdots < \infty \tag{100}$$

such that in each (t_{j-1}, t_j) the following is fulfilled:

$$u_{\min} < u < u_{\max} \quad \text{or} \quad u_{\min} = u \quad \text{or} \quad u = u_{\max}. \tag{101}$$

We shall carry out p steps, in each of which we shall construct $\omega_j \in C^1[t_{j-1}, t_j]$ such that $\omega_j(t_j) = \omega_{j+1}(t_j)$ and $f(t, \omega_j(t_j), u_i) = f(t, \omega_{j+1}(t_j), u_i)$ and that the function defined from these as:

$$x_K(t) := \omega_j(t) \quad \text{where } j \text{ is such that } t \in [t_{j-1}, t_j] \tag{102}$$

satisfies the maximality conditions expressed in Theorem 3.

Concatenation of the extremal arcs

Step [1] (the first arc)

(i) If $K \le -\frac{F_u(t,x_0,u_{\min})}{f_u(t,x_0,u_{\min})}$, we set $\omega_1(t)$ such that $\dot{\omega}_1(t) = f(t, \omega_1(t), u_{\min})$ in the maximal interval $[0, t_1]$, where:

$$K \le -\frac{F_u(t, \omega_1(t), u_{\min})}{f_u(t, \omega_1(t), u_{\min})} \exp\left(\int_0^t f_x(s, \omega_1(s), u_{\min}) ds\right) + \int_0^t \left(F_x(s, \omega_1(s), u_{\min}) \exp \int_0^s f_x(z, \omega_1(z), u_{\min}) dz\right) ds. \tag{103}$$

(ii) If $K \geq -\frac{F_u(t,x_0,u_{\max})}{f_u(t,x_0,u_{\max})}$, we set $\omega_1(t)$ such that $\dot{\omega}_1(t) = f(t, \omega_1(t), u_{\min})$ in the maximal interval $[0, t_1]$, where:

$$K \geq -\frac{F_u(t,\omega_1(t),u_{\max})}{f_u(t,\omega_1(t),u_{\max})} \exp\left(\int_0^t f_x(s,\omega_1(s),u_{\max})ds\right) + \int_0^t \left(F_x(s,\omega_1(s),u_{\max}) \exp \int_0^s f_x(z,\omega_1(z),u_{\max})dz\right) ds. \quad (104)$$

(iii) If $-\frac{F_u(t,x_0,u_{\max})}{f_u(t,x_0,u_{\max})} > K > -\frac{F_u(t,x_0,u_{\min})}{f_u(t,x_0,u_{\min})}$, then $\exists u^* \in (u_{\min}, u_{\max})$ such that $K = -\frac{F_u(t,x_0,u^*)}{f_u(t,x_0,u^*)}$, and we set $\omega_1(t)$ the arc of the extremal in its maximal domain $[0, t_1]$ (with $\omega_1(0) = x_0$, $\dot{\omega}_1(0) = f(t, x_0, u^*)$) which satisfies:

$$K = -\frac{F_u(t,\omega_1(t),u(t))}{f_u(t,\omega_1(t),u(t))} \exp\left(\int_0^t f_x(s,\omega_1(s),u(s))ds\right) + \int_0^t \left(F_x(s,\omega_1(s),u(s)) \exp \int_0^s f_x(z,\omega_1(z),u(z))dz\right) ds. \quad (105)$$

[j-th Step] (j-th arc)

(A) If ω_{j-1} has an interior extremal arc in $[t_{j-2}, t_{j-1}]$, there are two possibilities:

(I) If $\dot{\omega}_{j-1}(t_{j-1}) = f(t, \omega_{j-1}(t_{j-1}), u_{\min})$, we set $\omega_j(t)$ in the maximal interval $[t_{j-1}, t_j]$ which satisfies the differential equation $\dot{\omega}_j(t) = f(t, \omega_j(t), u_{\min})$ with the initial condition $\omega_j(t_{j-1}) = \omega_{j-1}(t_{j-1})$ and:

$$-\frac{F_u(t,\omega_{j-1}(t_{j-1}),u_{\min})}{f_u(t,\omega_{j-1}(t_{j-1}),u_{\min})} \leq -\frac{F_u(t,\omega_{j-1}(t_{j-1}),u_{\min})}{f_u(t,\omega_{j-1}(t_{j-1}),u_{\min})} \exp\left(\int_{t_{j-1}}^t f_x(s,\omega_j(s),u_{\min})ds\right) + \int_{t_{j-1}}^t \left(F_x(s,\omega_j(s),u_{\min}) \exp \int_{t_{j-1}}^s f_x(z,\omega_j(z),u_{\min})dz\right) ds. \quad (106)$$

(II) If $\dot{\omega}_{j-1}(t_{j-1}) = f_u(t, \omega_{j-1}(t_{j-1}), u_{\max})$, we set $\omega_j(t)$ in the maximal interval $[t_{j-1}, t_j]$ which satisfies the differential equation $\dot{\omega}_j(t) = f(t, \omega_j(t), u_{\max})$ with the initial condition $\omega_j(t_{j-1}) = \omega_{j-1}(t_{j-1})$ and:

$$-\frac{F_u(t,\omega_{j-1}(t_{j-1}),u_{\max})}{f_u(t,\omega_{j-1}(t_{j-1}),u_{\max})} \geq -\frac{F_u(t,\omega_{j-1}(t_{j-1}),u_{\max})}{f_u(t,\omega_{j-1}(t_{j-1}),u_{\max})} \exp\left(\int_{t_{j-1}}^t f_x(s,\omega_j(s),u_{\max})ds\right) + \int_{t_{j-1}}^t \left(F_x(s,\omega_j(s),u_{\max}) \exp \int_{t_{j-1}}^s f_x(z,\omega_j(z),u_{\max})dz\right) ds. \quad (107)$$

(B) If $[t_{j-2}, t_{j-1}]$ is the boundary interval, we set $\omega_j(t)$ the arc of the interior extremal (with $\omega_j(t_{j-1}) = \omega_{j-1}(t_{j-1})$) in its maximal domain $[t_{j-1}, t_j]$.

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