Computational Approach for the Firm's Cost Minimization Problem Using the Selective Infimal Convolution Operator

L. Bayón, P. Fortuny Ayuso, R. García-Rubio, J. M. Grau & M. M. Ruiz

Computational Economics

ISSN 0927-7099
Volume 54
Number 2

Comput Econ (2019) 54:535-549
DOI 10.1007/s10614-018-9841-6

Springer

Computational Economics (2019) 54:535–549
https://doi.org/10.1007/s10614-018-9841-6

Computational Approach for the Firm's Cost Minimization Problem Using the Selective Infimal Convolution Operator

L. Bayón[1] · P. Fortuny Ayuso[1] · R. García-Rubio[2] · J. M. Grau[1] · M. M. Ruiz[1]

Accepted: 7 August 2018 / Published online: 14 August 2018

Abstract

The Infimal Convolution operator is well known in the context of convex analysis. This operator admits a very precise micro-economic interpretation: if several production units produce the same output, the Infimal Convolution of their cost functions represents the joint cost function distributing the production among all of them in the most efficient possible way. The drawback of this operator is that it does not discriminate whether one of some of the production units is not profitable (in the sense that it would be preferable to do without it).This is the motivating idea for the present work, in which we introduce a new operator: the Selective Infimal Convolution. We give not just its definition and basic properties but also an algorithm for its exact computation. Using this, we avoid the combinatorial blowing-up of other classical methods used for solving similar problems. Even more, our approach solves a one-parameter family of problems, not just a single one. We provide an application to the Firm's Cost Minimization Problem, one of the most important problems in Microeconomics.

Keywords Infimal Convolution · Unit Commitment · Cost Minimization Problem

Mathematics Subject Classification 91B38 · 47S99

1 Introduction

In the context of Electrical Engineering, there arises a problem which needs to be solved frequently: The Unit Commitment (UC) problem, which consists in determining the schedule in which production units are to be used and how much each unit should produce in order to meet a power demand, while satisfying operational and technological constraints, over a time horizon. Due to its combinatorial nature and the

✉ L. Bayón
bayon@uniovi.es

1 Department of Mathematics, University of Oviedo, Oviedo, Spain

2 Department of Economy, University of Salamanca, Salamanca, Spain

nonlinearities presented, solving the UC problem (for real sized examples) is a hard computational and optimization task: it is a NP-hard problem.

The UC problem dates back to the 1940s and has since been extensively studied in the literature. Several review articles have been written, like Padhy (2004), where the author reviews more than 150 published articles. More recently, other interesting reviews have come out, like Tung et al. (2012), Samani et al. (2013), Dai et al. (2015) and Singh and Kumar (2016). In them, several optimization techniques, based both on exact and on approximate algorithms have been reported, as well as in economic environments (Santos and Vigo-Aguiar 1998; Vigo-Aguiar et al. 2017). These approaches can be classified in three types: classical, non-classical and hybrid methods. Some methods are focusing on speed and others on accuracy.

In the First type, several approaches based on exact methods have been used, such as: Exhaustive Enumeration, Priority List, Branch and Bound, Dynamic Programming, Mixed-Integer Programming or Lagrangian Relaxation. The main drawbacks of all these techniques come of the dimensionality problem, not only in computational time, but also in storage requirements. For instance, the branch-and-bound method has a exponential growth in the computational time with problem dimension. Also, in Lagrangian Relaxation, as the number of units increases, there some difficulties arise for obtaining feasible solutions. We refer the reader to the review papers for the details of each method.

More recently, several meta-heuristic methods and hybrids of them have been proposed. These approaches have, in general, better performance than the traditional heuristics. The most commonly used meta-heuristic methods are simulated annealing, evolutionary programming, memetic algorithms, particle swarm optimization, tabu search, and genetic algorithms. These UC solution techniques use approximations of the problem and the approximation may result in inaccurate solutions, which are undesirable.

Obviously, from an Economics point of view, this problem has a great relevance for companies, and needs to be efficiently solved. In this paper we present the problem in a more general economic framework: we shall consider one of the most important issues for firms in the field of Microeconomics (Varian 2005): the Firm's Cost Minimization Problem, which can be stated as follows: needing to produce a given output ξ, choose the optimal inputs x_i, $(i = 1, \ldots, N)$ which minimize the cost. In this paper we consider a firm that operates under perfect competition, i.e. its prices are independent of the firm's input and output decisions. The production function [see Luenberger (1995), Jehle and Reny (2001)] expresses how inputs are transformed into outputs. The most widely used production functions are the Leontief production function, Cobb-Douglas' model and the one that we consider in this paper: the Linear production function. We shall also generalize the problem by adding box constraints for the inputs.

Let $A = \{1, \ldots, N\}$ and $\{F_i\}_{i \in A}$ be a family of strictly convex functions. We denote by $\Pr^A(\xi)$ the problem consisting in:

$$\begin{array}{ll} \text{minimizing:} & \sum\limits_{i \in B} F_i(x_i) \\ \text{subject to:} & \sum\limits_{i \in B} x_i = \xi \\ & m_i \le x_i \le M_i,\ \forall i = 1, \ldots, N \\ & B \subset A \end{array} \tag{1}$$

and having to decide which of the N inputs are committed or uncommitted.

We shall consider two types of cost functions:

Case 1 Linear functions:

$$F_i(x_i) = a_i + b_i x_i \tag{2}$$

Case 2 Quadratic functions:

$$F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2 \tag{3}$$

where $\gamma_i > 0$. The compactness of the set defined by the constraints guarantees that $\Pr^A(\xi)$ has a solution $\forall \xi \in [\sum_{i \in A} m_i, \sum_{i \in A} M_i]$, and the strict convexity of each F_i, that this solution is unique.

In the context of optimization (and especially in Convex Analysis), the Infimal Convolution (IC) operator is widely known (Moreau 1970; Rockafellar 1970); we recall its definition:

Definition 1 Let $F, G : \mathbb{R} \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ be two functions. The Infimal Convolution of F and G is the following function:

$$F \odot G(x) := \inf_{y \in \mathbb{R}} \{F(x) + G(y - x)\} \tag{4}$$

For a survey of the properties of this operation, see Strömberg (1996) or Bauschke and Combettes (2011).

Remark 1 It is well known that $(\mathcal{F}(\mathbb{R}, \bar{\mathbb{R}}), \odot)$ is a commutative semigroup, where $\mathcal{F}(\mathbb{R}, \bar{\mathbb{R}})$ is the set of functions $f : \mathbb{R} \longrightarrow \bar{\mathbb{R}}$.

The following equality holds

$$\left(\bigodot_{i \in A} F_j \right)(K) = \inf_{\sum\limits_{i \in A} x_i = K} \left(\sum_{i \in A} F_i(x_i) \right) \tag{5}$$

for every finite set $A \subset \mathbb{N}$.

When the functions are restricted to a specific domain $Dom(F_i) = [m_i, M_i]$, the definition above remains valid, just by letting $F_i(x) = +\infty$ if $x \notin Dom(F_i)$. In this case, one has the following equivalent definition:

Definition 2 We shall call

$$(F_1 \odot F_2)(K) := \min_{\substack{x_1+x_2=K \\ m_i \le x_i \le M_i}} (F_1(x_1) + F_2(x_2)) = \min_{\substack{m_1 \le x \le M_1 \\ m_2 \le K - x_1 \le M_2}} ((F_1(x) + F_2(K - x)) \tag{6}$$

$$\Psi^A(K) := \bigodot_{i \in A} F_j(K) = \min_{\substack{\sum_{i \in A} F_i(x_i) = K \\ m_i \le x_i \le M_i}} \left(\sum_{i \in A} F_i(x_i) \right) \tag{7}$$

With this definition, if Ψ^A is the Infimal Convolution of several production cost production functions, then $\Psi^A(K)$ represents the joint cost for the production level K when this is distributed among the several units in the most efficient way. This operator has already been used in Mathematical Economics in Bayón et al. (2016).

This notion leads to a more realistic one which allows using only those production units which are profitable: that is, to disregard those whose use in the productive process would be costlier than their omission.

This is the motivating idea for the introduction of the operator we have called the **Selective Infimal Convolution (SIC).** Even though its formal definition is presented in this paper, the underlying idea has already been considered (see the Introduction) in the framework of Electrical Engineering (the UC problem), although in that setting it presents technical complications which prevent a rigorous and abstract statement as the one we propose. Even more, we do not limit ourselves to a specific problem but to a one-parameter family of problems, obtained by varying the value ξ of the output.

The rest of the paper is organized as follows. Section 2 presents the definition and main properties of the SIC operator. Section 3 outlines the Optimization Algorithm for the exact calculation of the SIC operator. Section 4 presents several numerical examples and analyzes the operational complexity of the algorithm. Section 5 concludes the paper and proposes some future work. In "Appendix", we include the proof of the formula for the SIC in the quadratic case, for the sake of completeness.

2 Definition and Properties of the SIC Operator

We give now the elementary properties of this new operator. Proofs are omitted, as they consist of elementary group calculations which provide no insight into the problem at hand.

Definition 3 Let $F, G : \mathbb{R} \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ be two functions. The Selective Infimal Convolution (SIC) of F and G is the following function:

$$(F \circledS G)(x) := \min\{F(x), G(x), (F \odot G)(x)\} \tag{8}$$

The first two results give the basic properties of the SIC operator and describe how one can compute its value for a family of functions from the IC operator.

Proposition 1 $(\mathcal{F}(\mathbb{R}, \bar{\mathbb{R}}), \circledS)$ *is a commutative semigroup.*

Proposition 2 *Let $A \subset \mathbb{N}$ be an initial segment and $G_i : \mathbb{R} \longrightarrow \bar{\mathbb{R}}$for $i \in A$. Then:*

$$\textcircled{s}_{i\in A} G_i(x) = \min_{B\in P(A)} \left(\bigodot_{i\in B} G_i(x) \right) = \min_{B\subseteq A} \left\{ \bigodot_{i\in B} G_i(x) \right\} \tag{9}$$

where $P(A)$ represents the set of non-empty subsets of A.

The SIC is the solution to a family of mixed-integer programming problems; this is the content of the following result.

Proposition 3 *Let $\{F_i\}_{i\in A} \subset \mathcal{F}(\mathbb{R}, \bar{\mathbb{R}})$. The following holds:*

$$\textcircled{s}_{i\in A} F_i(\xi) = \inf_D \sum_{i\in A} z_i \cdot F_i(x_i) \tag{10}$$

with:

$$D = \{(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^n \times \{0, 1\}^n : \big| : \sum_{i\in A} z_i \cdot x_i = \xi \tag{11}$$

Finally, for the sake of completeness, we present two Propositions which give the explicit expression of the Infimal Convolution of two linear or quadratic functions. These are required to compute the SIC in a symbolic way.

Let F_i, for $i = 1, 2$ be two linear functions as:

$$F_i(x_i) = a_i + b_i x_i \tag{12}$$

Proposition 4 *Let $F_i(x_i) = a_i + b_i x_i$, $(i = 1, 2)$ with domains $[m_i, M_i]$. Let us assume that $b_1 \le b_2$. The following equality holds:*

$$(F_1 \odot F_2)(\xi) := \begin{cases} F_1(\xi - m_2) + F_2(m_2) & \text{if } \xi \in [m_1 + m_2, M_1 + m_2] \\ F_1(M_1) + F_2(\xi - M_1) & \text{if } \xi \in [M_1 + m_2, M_1 + M_2] \end{cases} \tag{13}$$

For two strictly convex quadratic functions

$$F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2 \tag{14}$$

(convex means $\gamma_i > 0$) with $i = 1, 2$, we have the following result.

Proposition 5 *Let $F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2$ $(i = 1, 2)$ with domains $[m_i, M_i]$. Let us assume that $F_1'(m_1) \le F_2'(m_2)$. Define*

$$l_1 = \frac{(-\beta_1 + \beta_2 + 2\gamma_2 m_2)}{2\gamma_1};\ l_2 = \frac{(\beta_1 - \beta_2 + 2\gamma_1 M_1)}{2\gamma_2};\ l_3 = \frac{(-\beta_1 + \beta_2 + 2\gamma_2 M_2)}{2\gamma_1} \tag{15}$$

and

$$F_{12}(\xi) = \alpha_1 + \alpha_2 - \frac{(\beta_1 - \beta_2)^2}{4(\gamma_1 + \gamma_2)} + \frac{\gamma_2\beta_1 + \gamma_1\beta_2}{\gamma_1 + \gamma_2}\xi + \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}\xi^2. \tag{16}$$

Then

(A) If $F_1'(m_1) \leq F_2'(m_2) \leq F_1'(M_1) \leq F_2'(M_2)$, then:

$$(F_1 \odot F_2)(\xi) := \begin{cases} F_1(\xi - m_2) + F_2(m_2) & \textit{if } \xi \in [m_1 + m_2, m_2 + l_1] \\ F_{12}(\xi) & \textit{if } \xi \in [m_2 + l_1, M_1 + l_2] \\ F_2(\xi - M_1) + F_1(M_1) & \textit{if } \xi \in [M_1 + l_2, M_1 + M_2] \end{cases} \tag{17}$$

(B) If $F_1'(m_1) \leq F_2'(m_2) \leq F_2'(M_2) \leq F_1'(M_1)$, then:

$$(F_1 \odot F_2)(\xi) := \begin{cases} F_1(\xi - m_2) + F_2(m_2) & \textit{if } \xi \in [m_1 + m_2, m_2 + l_1] \\ F_{12}(\xi) & \textit{if } \xi \in [m_2 + l_1, M_2 + l_3] \\ F_1(\xi - M_2) + F_2(M_2) & \textit{if } \xi \in [M_2 + l_3, M_1 + M_2] \end{cases} \tag{18}$$

(C) If $F_1'(m_1) \leq F_1'(M_1) \leq F_2'(m_2) \leq F_2'(M_2)$, then:

$$(F_1 \odot F_2)(\xi) := \begin{cases} F_1(\xi - m_2) + F_2(m_2) & \textit{if } \xi \in [m_1 + m_2, M_1 + m_2] \\ F_1(M_1) + F_2(\xi - M_1) & \textit{if } \xi \in [M_1 + m_2, M_1 + M_2] \end{cases} \tag{19}$$

Details on these propositions are given in Bayón et al. (2011, 2014).

3 An Exact Algorithm for Computing the SIC

We present in this section an exact method for computing the SIC in an exact way. It makes use of Propositions 4 and 5 in the previous sections. Using them, we can compute the SIC of a family of functions $F_1, \ldots, F_n$ in a recursive way. To this end, we implement the following collection of modules.

(Module 1) IC of 2 cost functions

To implement this we only need to apply Propositions 4 and 5 to a pair of functions F_1 and F_2 to obtain

$$F_1 \odot F_2 \tag{20}$$

(Module 2) Minimum function of several functions

This module computes the minimum of several functions (i.e. for their graphs, the enveloping curve which is lowest).

$$R(x) = \min\{F_1(x), F_2(x), \ldots, F_n(x)\} \tag{21}$$

This is, in general, a piece-wise defined function.

(Module 3) IC of N cost functions

We also need a way to compute the IC of a family of cost functions,

$$F_1(x), \ldots, F_N(x). \tag{22}$$

It is calculated by computing all the ICs of all the pairs of functions $F_i \odot F_j$ and then computing the minimum:

$$F_1 \odot F_2 = \min_{(i,j)}(F_{1i} \odot F_{2j});\ i = 1, \ldots, k(1);\ j = 1, \ldots, N \tag{23}$$

(Module 4) SIC of 2 cost functions

Once the IC of 2 functions is computed, the SIC of 2 functions can be computed using the minimum:

$$(F_1 \circledS F_2)(x) = \min\{F_1(x), F_2(x), (F_1 \odot F_2)(x)\} \tag{24}$$

Which is, in general, another piece-wise defined function.

(Module 5) SIC of N cost functions

Bearing in mind the associative nature of the SIC operation, the SIC of N cost functions can now be calculated by means of a recursive process, carrying out N SIC operations using the recurrence:

$$F_1 \circledS F_2 \circledS \cdots \circledS F_N = (F_1 \circledS F_2 \circledS \cdots \circledS F_{N-1}) \circledS F_N \tag{25}$$

That is: once we have obtained the SIC of the first two units, we calculate the SIC of the obtained result $F_1 \circledS F_2$ with the third F_3 and so on, sequentially.

We might also consider the divide-and-conquer method:

$$F_1 \circledS F_2 \circledS \cdots \circledS F_N = (F_1 \circledS F_2 \circledS \cdots \circledS F_{\frac{n}{2}}) \circledS (F_{\frac{n}{2}+1} \circledS \cdots \circledS F_n) \tag{26}$$

The analytic expression of the SIC of the N cost functions yields the total cost of the optimal solution for any ξ.

4 Numerical Examples

Based on the above results, we are now ready to present two examples: the linear case and the quadratic case. For this purpose, we implemented the aforementioned algorithms in Mathematica®.

4.1 Linear Case

We first consider a case test with 5 inputs, where the parameters of the linear cost functions $F_i(x_i)$, $(i = 1, \ldots, 5)$ are presented in Table 1.

$$F_i(x_i) = a_i + b_i x_i \tag{27}$$

Table 1 Parameters of the linear cost functions $F_i(x_i)$

$F_i(x_i)$	a_i	b_i	m_i	M_i
F_1	2	1	0	7
F_2	1	3	2	9
F_3	3	2	1	10
F_4	4	5	0	8
F_5	5	4	3	6

Table 2 Optimal total cost and committed inputs

Output level ξ	Inputs					Total cost	
$[l_i, u_i]$	1	2	3	4	5	A_i	B_i
[0, 7]	1					2	1
[7, 8]	1		3			6	1
[8, 17]	1		3			−2	2
[17, 19]	1	2	3			1	2
[19, 26]	1	2	3			−18	3
[26, 28]	1	2	3	4		−66	5
[28, 29]	1	2	3		5	−10	3
[29, 32]	1	2	3		5	−39	4
[32, 40]	1	2	3	4	5	−67	5

Unlike the methods mentioned above, our algorithm provides the analytic solution for all values of the output level ξ. The SIC is a piece-wise-linear function (Z pieces) of the form:

$$F_1 \, ⓢ \, F_2 \, ⓢ \cdots ⓢ \, F_5 = c(\xi) = \{H_i(\xi)\} = \{A_i + B_i \cdot \xi\}\,;\ \xi \in [l_i, u_i]\,,\ i = 1, \ldots, Z \qquad (28)$$

The total cost $(A_i + B_i \cdot \xi)$ for each interval $(i = 1, \ldots, Z)$ of output level ξ is listed in Table 2. This table shows also the inputs which are *committed*.

As Fig. 1 shows, the SIC for this example has $Z = 9$ pieces and shows both continuous non-convex areas and discontinuities.

The computation of the SIC does not only provide the minimum value of the total cost but also, for any ξ, the production distribution among the N inputs. The procedure is as follows: first, given a certain ξ, choose the interval $[l_i, u_i]$, $i = 1, \ldots, Z$, for which $\xi \in [l_i, u_i]$. Then, order the F_i of the inputs $i \in \{k_1, \ldots k_r\}$ which are used in that interval in increasing order of their slopes, b_i, say $b_{i_1} < b_{i_2} < \ldots, b_{i_r}$. The distribution of inputs is then as follows: each input i, starting from i_1 is used up to its maximum capacity M_i up until the output level is reached, at which point, no more inputs are used.

For instance, for $\xi = 27$ we need to consider the interval $[l_i, u_i] = [26, 28]$ where the used inputs are: 1, 2, 3, 4. The optimal cost is given by:

$$A_i + B_i \cdot \xi = -66 + 5 \cdot 27 = 69 \qquad (29)$$

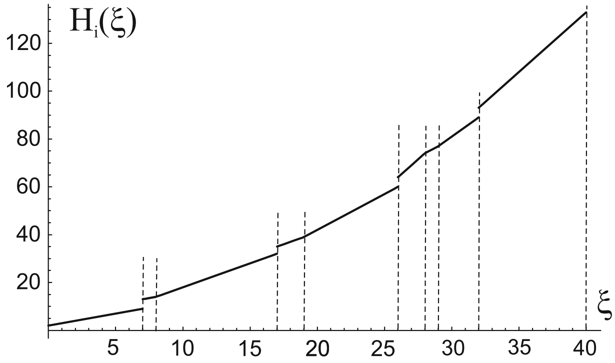


Fig. 1 SIC of the linear cost functions for Table 1

Table 3 Parameters of the quadratic cost functions $F_i(x_i)$

$F_i(x_i)$	α_i	β_i	γ_i	m_i	M_i
F_1	26.97	0.3975	0.002176	0	100
F_2	21.13	0.3059	0.001861	50	200
F_3	21.13	0.5500	0.001861	40	90

Considering each b_i, the order of the inputs is: 1, 3, 2, 4. Hence, just taking into account the M_i, we get:

$$x_1 = 7;\ \ x_3 = 10;\ \ x_2 = 9;\ \ x_4 = 1 \tag{30}$$

because of the condition

$$\sum x_i = \xi. \tag{31}$$

4.2 Quadratic Case

Secondly, we consider a case test with 3 inputs, where the the cost functions $F_i(x_i)$, $(i = 1, \ldots, 3)$ follow a quadratic model:

$$F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2 \tag{32}$$

The parameters are listed in Table 3.

The SIC is now a piece-wise quadratic function of the form:

$$\begin{aligned} F_1 \circledS F_2 \circledS F_3 = c(\xi) &= \{H_i(\xi)\} \\ &= \left\{A_i + B_i \cdot \xi + C_i \cdot \xi^2\right\};\ \xi \in [l_i, u_i],\ i = 1, \ldots, Z \end{aligned} \tag{33}$$

The SIC obtained for any output level ξ is presented in Table 4.

As Fig. 2 shows, the SIC in this example has also $Z = 9$ pieces.

Table 4 Optimal total cost and inputs *commitment*

Output level ξ	Inputs			Total cost		
$[l_i, u_i]$	1	2	3	A_i	B_i	C_i
[0.00, 40.00]	1			26.97	0.3975	0.002176
[40.00, 41.92]			3	21.13	0.5500	0.001861
[41.92, 50.00]	1			26.97	0.3975	0.002176
[50.00, 200.00]		2		21.13	0.3059	0.001861
[200.00, 241.53]	1	2		47.58	0.3481	0.001003
[241.53, 300.00]	1	2		97.63	-0.0663	0.001861
[300.00, 317.49]	1	2	3	65.15	0.4188	0.000651
[317.49, 345.58]	1	2	3	93.24	0.2418	0.000930
[345.58, 390.00]	1	2	3	204.37	-0.4012	0.001861

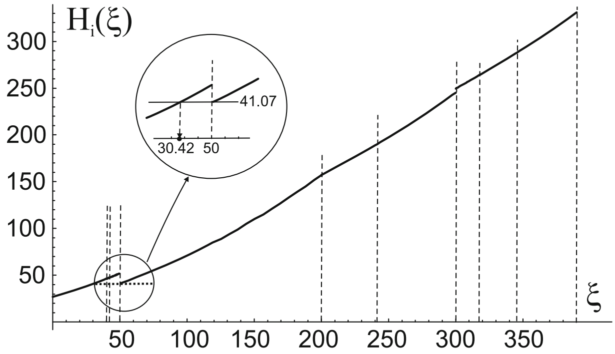


Fig. 2 SIC of the quadratic cost functions

A remarkable behaviour is noticeable in this example. We see how, for $\xi = 50$, the cost function $c(\xi)$ is discontinuous. The left and right limits are, respectively:

$$A_i + B_i \cdot \xi + C_i \cdot \xi^2 = 26.97 + 0.3975 \cdot 50 + 0.002176 \cdot 50^2 = 52.28$$
$$A_i + B_i \cdot \xi + C_i \cdot \xi^2 = 21.13 + 0.3059 \cdot 50 + 0.001861 \cdot 50^2 = 41.07 \quad (34)$$

The point given by that cost on the left is

$$41.07 = 26.97 + 0.3975 \cdot \xi + 0.002176 \cdot \xi^2 \Rightarrow \xi = 30.42 \quad (35)$$

and we conclude that, in this example, producing 50 units of output is cheaper than producing any other quantity between 30.42 and 50.

4.3 Profit Maximization Problem

In the previous sections, we solved what is traditionally known as the *Firm's Cost Minimization (FCM) Problem.* Using the computed SIC, we can also solve an associated

Fig. 3 FPM problem, linear case

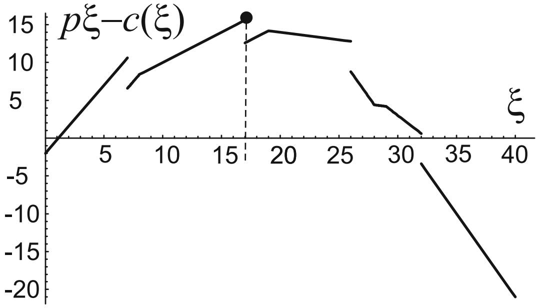
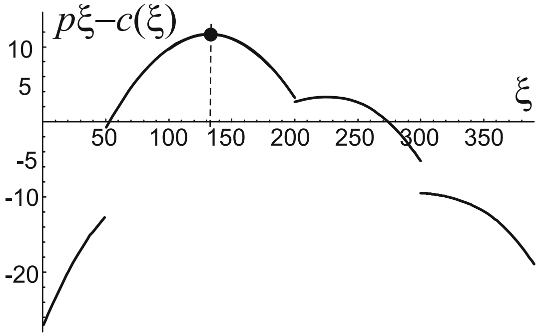


Fig. 4 FPM problem, quadratic case



problem: the *Firm's Profit Maximization (FPM) Problem.* The idea is that, in order to solve the FPM problem, we first compute analytically the minimum cost function:

$$c(\xi) = \min_{x_i} \sum F_i(x_i) \tag{36}$$

and then, maximize over the output quantity:

$$\max_{\xi}(p\xi - c(\xi)) \tag{37}$$

where x_i are the inputs, ξ is the output and p is the price of the output.

In the simplest case, when $c(\xi)$ is of class C^1, it is necessary to determine the optimum level of output ξ for which the marginal cost $c'(\xi)$ coincides with the price p. In our case, the problem is more complicated, as $c(\xi)$ is not even continuous. In Fig. 3 we give the graphical representation of the benefit $p\xi - c(\xi)$, for the previous linear example, with $p = 2.8$. As we see, the maximum of the benefit coincides with the maximum of

$$2.8\xi - H_3(\xi) \tag{38}$$

which happens for $\xi = 17$, at which point inputs 1 and 3 are used. In the linear case, as the $H_i(\xi)$ are also linear, the maximum is always achieved at the one of the endpoints of the intervals $[l_i, u_i]$.

Table 5 Optimal total cost for the IC

Output level ξ	Inputs			Total Cost		
$[l_i, u_i]$	1	2	3	A_i	B_i	C_i
[90.00, 111.71]	1	2	3	96.01	0.0058	0.002176
[111.71, 214.83]	1	2	3	81.36	0.2678	0.001003
[214.83, 317.49]	1	2	3	65.15	0.4188	0.000651
[317.49, 345.58]	1	2	3	93.24	0.2418	0.000930
[345.58, 390.00]	1	2	3	204.37	-0.4012	0.001861

Table 6 Comparison between SIC and IC

Method	Cost	Inputs
SIC	$21.13 + 0.3059 \cdot 150 + 0.001861 \cdot 150^2 = 108.88$	{2}
IC	$81.36 + 0.2678 \cdot 150 + 0.001003 \cdot 150^2 = 144.09$	{1, 2, 3}

On the other hand, in Fig. 4 we plot the profit for the quadratic case computed above, for the specific value $p = 0.8$. In the quadratic case, as the $H_i(\xi)$ are also quadratic, the maximum can be either at the endpoints or in the interior of one of the sub-intervals $[l_i, u_i]$. In our example, it comes from maximizing

$$p\xi - H_4(\xi) = 0.8 \cdot \xi - (21.13 + 0.3059 \cdot \xi + 0.001861 \cdot \xi^2) \Rightarrow \xi = 132.75 \quad (39)$$

and in this case, only input 2 is used.

4.4 Comparison with the Infimal Convolution

In order to provide some insight, we show how the SIC in the quadratic example above (Table 3) compares against the IC (Bayón et al. 2016). As the IC requires the use of all the inputs at all times, the optimal solution obviously different, as Table 5 shows.

Notice first of all, that the IC provides values only starting at $\xi = 90$, which is the minimum output level when using the three inputs. Also, the IC only divides the whole ξ interval into 5 pieces, in contrast with the 9 into which the SIC divides it. The cost functions coincide for the last three sub-intervals (as must be, as in both cases all the inputs are used).

Comparing the costs obtained for the SIC and the IC for $\xi = 150$, for example, we get:

Table 6 shows, as expected, that the cost is much less for the SIC, as one is allowed to use not all the inputs but only the most efficient ones.

5 Conclusions

The Unit Commitment (UC) problem is a well-known combinatorial optimization problem arising in operations planning of power systems. Numerous algorithms have been formulated in the past six decades for optimization of the UC problem. But researchers coincide in considering it an open problem, in which novel algorithms are required. This paper addresses this issue for the solution of the unit commitment problem.

We have presented its definition and basic properties and a new algorithm for computing an exact solution. This algorithm does not show the combinatorial blowing-up of other classical methods like Exhaustive Enumeration or Branch and Bound. Our application to the Firm's Cost Minimization Problem and to the Firm's Profit Maximization Problem shows the potential of our method in several problems of Economy. The most relevant point is that we solve not just a specific case but a family of problems which arise when varying the output value. This way, one can obtain qualitative properties of the solution, as we show in our examples.

Appendix: Proof of the Formula for the SIC (Quadratic Case)

Proposition 5 *Let* $F_i(x_i) = \alpha_i + \beta_i x_i + \gamma_i x_i^2$ *with domains* $[m_i, M_i]$ *and* $\gamma_i > 0$ $(i = 1, 2)$. *Let us assume that* $F_1'(m_1) \leq F_2'(m_2)$. *Define*

$$l_1 = \frac{(-\beta_1 + \beta_2 + 2\gamma_2 m_2)}{2\gamma_1};\ l_2 = \frac{(\beta_1 - \beta_2 + 2\gamma_1 M_1)}{2\gamma_2};\ l_3 = \frac{(-\beta_1 + \beta_2 + 2\gamma_2 M_2)}{2\gamma_1} \tag{40}$$

and

$$F_{12}(\xi) = \alpha_1 + \alpha_2 - \frac{(\beta_1 - \beta_2)^2}{4(\gamma_1 + \gamma_2)} + \frac{\gamma_2\beta_1 + \gamma_1\beta_2}{\gamma_1 + \gamma_2}\xi + \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}\xi^2. \tag{41}$$

Then

(A) If $F_1'(m_1) \leq F_2'(m_2) \leq F_1'(M_1) \leq F_2'(M_2)$, *then:*

$$(F_1 \odot F_2)(\xi) := \begin{cases} F_1(\xi - m_2) + F_2(m_2) & \text{if } \xi \in [m_1 + m_2, m_2 + l_1] \\ F_{12}(\xi) & \text{if } \xi \in [m_2 + l_1, M_1 + l_2] \\ F_2(\xi - M_1) + F_1(M_1) & \text{if } \xi \in [M_1 + l_2, M_1 + M_2] \end{cases} \tag{42}$$

(B) If $F_1'(m_1) \leq F_2'(m_2) \leq F_2'(M_2) \leq F_1'(M_1)$, *then:*

$$(F_1 \odot F_2)(\xi) := \begin{cases} F_1(\xi - m_2) + F_2(m_2) & \text{if } \xi \in [m_1 + m_2, m_2 + l_1] \\ F_{12}(\xi) & \text{if } \xi \in [m_2 + l_1, M_2 + l_3] \\ F_1(\xi - M_2) + F_2(M_2) & \text{if } \xi \in [M_2 + l_3, M_1 + M_2] \end{cases} \tag{43}$$

(C) If $F_1'(m_1) \leq F_1'(M_1) \leq F_2'(m_2) \leq F_2'(M_2)$, *then:*

$$(F_1 \bigodot F_2)(\xi) := \begin{cases} F_1(\xi - m_2) + F_2(m_2) & \text{if } \xi \in [m_1 + m_2, M_1 + m_2] \\ F_1(M_1) + F_2(\xi - M_1) & \text{if } \xi \in [M_1 + m_2, M_1 + M_2] \end{cases} \tag{44}$$

Proof The case for n quadratic functions has been studied in Bayón et al. (2010). In this paper we only deal with the case $n = 2$.

(A) Let (x_ξ, y_ξ) be the minimum of $F_1(x) + F_2(y)$ subject to $x + y = \xi$, with $m_1 \leq x_\xi \leq M_1$ and $m_2 \leq y_\xi \leq M_2$.

We first show that the following holds:

(i) If $F_1'(m_1) < F_2'(m_2)$ or $(F_1'(m_1) = F_2'(m_2))$ then $y_\xi > m_2 \Rightarrow x_\xi > m_1$ (or $x_\xi = m_1 \Rightarrow y_\xi = m_2$).
(ii) If $F_2'(m_2) < F_1'(M_1)$ or $(F_2'(m_2) = F_1'(M_1))$ then $x_\xi = M_1 \Rightarrow y_\xi > m_2$ (or $y_\xi = m_2 \Rightarrow x_\xi < M_1$).
(iii) If $F_1'(M_1) < F_2'(M_2)$ or $(F_1'(M_1) = F_2'(M_2))$ then $x_\xi < M_1 \Rightarrow y_\xi < M_2$ (or $y_\xi = M_2 \Rightarrow x_\xi = M_1$).

We prove just the case (i), the other two follow from a similar reasoning.

(i) Let$F_1'(m_1) \leq F_2'(m_2)$. Assuming that $x_\xi = m_1$ and $y_\xi > m_2$ leads to a contradiction. Consider the function:

$$\Phi(\varepsilon) = F_1(x_\xi + \varepsilon) + F_2(y_\xi - \varepsilon) \tag{45}$$

Hence $\Phi'(0) = F_1'(m_1) - F_2'(y_\xi) < F_1'(m_1) - F_2'(m_2) \leq 0$, which contradicts the minimal nature of (x_ξ, y_ξ).

Notice that (i) guarantees that the minimum cannot be obtained for $x_\xi = m_1$ and $m_2 < y_\xi \leq M_2$; (ii) guarantees that the minimum cannot be obtained for $y_\xi = m_2$ and $x_\xi = M_1$, and finally (iii) guarantees that the minimum cannot be obtained for $y_\xi = M_2$ and $m_1 \leq x_\xi < M_1$.

Thus, we have the following possibilities:

– If $y_\xi = m_2$ and $m_1 \leq x_\xi < M_1$ then $F_1'(x_\xi) \leq F_2'(m_2)$. As F_1' is increasing, there must exist some $l_1 \geq x_\xi$ with $F_1'(l_1) = F_2'(m_2)$, that is,

$$l_1 = \frac{(-\beta_1 + \beta_2 + 2\gamma_2 m_2)}{2\gamma_1} \tag{46}$$

such that $y_\xi = m_2$ and $x_\xi = \xi - m_2 \in [m_1, l_1]$, from which $\xi \in [m_1 + m_2, l_1 + m_2]$ and certainly, in this interval, $(F_1 \bigodot F_2)(\xi) = F_1(\xi - m_2) + F_2(m_2)$.

– If $x_\xi = M_1$ and $m_2 < y_\xi \leq M_2$ then $F_1'(M_1) \leq F_2'(y_\xi)$. As F_2' is increasing, there must exist some $l_2 \leq y_\xi$ with $F_1'(M_1) = F_2'(l_2)$, that is,

$$l_2 = \frac{(\beta_1 - \beta_2 + 2\gamma_1 M_1)}{2\gamma_2} \tag{47}$$

such that $x_\xi = M_1$ and $y_\xi = \xi - M_1 \in [l_2, M_2]$, from which $\xi \in [M_1 + l_2, M_1 + M_2]$ and certainly, in this interval, $(F_1 \bigodot F_2)(\xi) = F_1(M_1) + F_2(\xi - M_1)$.

– If $m_1 < x_\xi < M_1$ and $m_2 < y_\xi < M_2$ then $F_1'(x_\xi) = F_2'(y_\xi)$. It is clear that, in this case, $\xi \in [l_1 + m_2, M_1 + l_2]$ and

$$\min_y\{F_1(\xi - y) + F_2(y)\} = \alpha_1 + \alpha_2 - \frac{(\beta_1 - \beta_2)^2}{4(\gamma_1 + \gamma_2)} + \frac{\gamma_2\beta_1 + \gamma_1\beta_2}{\gamma_1 + \gamma_2}\xi + \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}\xi^2 \tag{48}$$

which function we denote $F_{12}(\xi)$.

(B) and (C) are proved using a similar reasoning. □

References

Bauschke, H. H., & Combettes, P. L. (2011). *Convex analysis and monotone operator theory in Hilbert spaces* (pp. 167–180). Berlin: Springer.

Bayón, L., García-Nieto, P. J., García-Rubio, R., Grau, J. M., Ruiz, M. M., & Suárez, P. M. (2016). The operation of infimal/supremal convolution in mathematical economics. *International Journal of Computer Mathematics*, *93*(5), 735–748.

Bayón, L., García-Nieto, P. J., Grau, J. M., Ruiz, M. M., & Suárez, P. M. (2014). An economic dispatch algorithm of combined cycle units. *International Journal of Computer Mathematics*, *91*(2), 269–277.

Bayón, L., Grau, J. M., Ruiz, M. M., & Suárez, P. M. (2010). An analytic solution for some separable convex quadratic programming problems with equality and inequality constraints. *Journal of Mathematical Inequalities*, *4*(3), 453–465.

Bayón, L., Grau, J. M., Ruiz, M. M., & Suárez, P. M. (2011). Algorithm for calculating the analytic solution for economic dispatch with multiple fuel units. *Computers and Mathematics with Applications*, *62*(5), 2225–2234.

Dai, H., Zhang, N., & Su, W. (2015). A literature review of stochastic programming and unit commitment. *Journal of Power and Energy Engineering*, *3*(4), 206–214.

Jehle, G. A., & Reny, P. J. (2001). *Advanced Microeconomic Theory* (2nd ed.). Boston: Addison-Wesley.

Luenberger, D. G. (1995). *Microeconomic Theory*. New York: McGraw-Hill.

Moreau, J. J. (1970). Inf-convolution, sous-additivit e, convexit eriques. *Journal de Mathématiques Pures et Appliquées*, *49*, 109–154.

Padhy, N. P. (2004). Unit commitment—A bibliographical survey. *IEEE Transactions on Power Systems*, *19*(2), 1196–1205.

Rockafellar, R. T. (1970). *Convex Analysis*. Princeton: Princeton University Press.

Samani, H., Razmezani, M., & Naseh, M. R. (2013). Unit commitment and methods for solving; a review. *Journal of Basic and Applied Scientific Research*, *3*(2s), 358–364.

Santos, M. S., & Vigo-Aguiar, J. (1998). Analysis of a numerical dynamic programming algorithm applied to economic models. *Econometrica*, *66*(2), 409–426.

Singh, A., & Kumar, S. (2016). Solution to unit commitment scheduling problem—A proposed approach. *International Journal of Control Theory and Applications*, *9*(41), 499–507.

Strömberg, T. (1996). The operation of infimal convolution, Dissertationes Mathematicae, 352

Tung, N. S., Kaur, G., Kaur, G., & Bhardwaj, A. (2012). Optimization techniques in unit commitment a review. *International Journal of Engineering Science and Technology*, *4*(4), 1623–1627.

Varian, H. R. (2005). *Intermediate microeconomics: A modern approach* (7th ed.). New York: W.W. Norton & Company.

Vigo-Aguiar, J., Medina, J., & Garcia-Rubio, R. (2017). Current computational tools for science engineering and economics at CMMSE. *Journal of Computational and Applied Mathematics*, *318*, 1–2.