

Mid-term bio-economic optimization of multi-species fisheries

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ABSTRACT

In this paper, we analyze the dynamics of a multi-species fisheries system in the presence of harvesting. We solve the problem of finding the optimal harvesting strategy for a mid-term horizon with a fixed final stock of each species, while maximizing the expected present value of total revenues. The problem is formulated as an optimal control problem. For its solution, we combine techniques derived from Pontryagin's Maximum Principle, cyclic coordinate descent and the shooting method. The algorithm we develop can solve problems both with inter-species competition and with predator–prey behaviors. Several numerical examples are presented to illustrate the different possibilities of the method and a study of the dependence of the behavior on some parameters is performed.

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1. Introduction

A large number of problems in applied mathematics is concerned with the analysis of dynamical processes in renewable resources. A classical introductory reference to the topic is the book [1], whose chapters contain abundant bibliography related to each of the traditionally considered stocks: fish, forests, or freshwater. More recently, the excellent handbooks [2] or [3] have appeared. Works centered on each of the stocks are [4] or [5] for the dynamics of fish populations; [6] –using Dynamic Programming– for the problem of forest management is studied and, quite recently, Mays [7] uses mathematical programming and differential dynamic programming techniques to study the optimization of groundwater management systems, or freshwater inflows among other topics.

From the economic point of view, the renewable resources problem is stated as the maximization of profit over a certain time horizon, where the problem is subject to the biological dynamic of the resource, an initial stock size of fish, and other technological or political constraints [8]. The economic value includes, by its very nature, a discount factor. The models related to fishery and marine economics and its relation to the economic foundation are discussed in [9]. In [10], the authors present a new approach to solve dynamic decision models in economics. Their is based on an iterative solution of optimal control problems in finite time horizons.

In this paper, we deal with the particular case of harvesting of fish [11]. Its optimization in different ways is a complex problem with both ecological and economical implications. The interaction between these two aspects is nowadays of great interest, and a large corpus has been written. In [12], the so-called social-ecological systems, characterized by strong links between both aspects, are analyzed. An analysis of how the technical tools should be used as a basis for policy making and management within the European context is presented in [13]. Also, [14] describes how the global economy can be

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restructured to make it compatible with the Earth's ecosystem. The study presented in [15] maximizes the net present value of a multi-species fishery where species interact both ecologically (in the ecosystem) and economically (in the output markets). The interactions taking place in an integrated ecological–economical model are also analyzed in [16].

From the biological point of view, most papers have focused on the optimal harvesting policies considering only one fish species, and an infinite-time horizon $[0, \infty)$ (see, for example, [17–19]). These works deal mostly with long-term stability and the existence of a steady-state solution. A method developed by Tsur and Zemel [20] calculates the optimal steady-state, and does not require the solution of the dynamic optimization problem.

Tsur's method was applied in [21] for a single-state, infinite-horizon, autonomous (except for the discount factor) model. The same problem was studied by some of the authors in a previous paper [22] but including, besides the classical infinite-time horizon problem, an example with long-term horizon. In the latter, we considered a fixed and finite optimization interval $[0, T]$, and we assumed it to be long enough for the steady state to be reached in its duration. We observed in those works that the solution leaves the steady state when approaching the end of the interval and seeks a final stock value. This optimal value is not part of the statement: it is freely sought by the system.

In recent years, the development of a proper model incorporating multi-species fisheries has gained interest. The biological interactions between species play an important role in optimal fisheries management, and it will be one of the main aspects studied in this paper. This multi-species modelization leads to complex dynamical problems. The most classical standpoint can be found, for instance, in [23] where the authors assume that the per capita growth rate of each species is a linear function of population abundance. A more sophisticated multi-species bioeconomic model is analyzed in [24]. This model (of an archetypal Caribbean coral reef community) includes two focal prey species (parrotfish and snapper) and a generalist predator (grouper). Again, the predation mortality of the two preys is modeled as linear on the predator population.

That type of linear model, with a food chain consisting of only two trophic levels and a prey–predator system is frequently used in the literature (see [25]). The same authors present, in [26], a system of n cooperating populations without harvesting. In this system of logistic populations, each one, by its mere presence, increases the carrying capacity of the other ones: one species may be a commensal to any number of species and likewise, one species may receive help from any number of species.

To cope with these complexities, authors sometimes analyze mainly the stability of the biological system in lack of fishing [27]. Other times, they use specific models which are only valid for particular groups of species (see [28–30]), or they develop more general methods but always under the infinite-horizon assumption and searching the optimal steady-state [31].

There is, then, a large variety of models. In the review paper [32], an interesting classification of three classes of ecological models used for an ecosystem approach to fisheries is presented: Single-species assessments, Models of Intermediate Complexity for Ecosystem assessments and Whole-of-ecosystem models. In the same reference, a subset of realistic models is shown (with the appropriate references), with the Region and the Species modeled. Sixteen other examples of population models with explicitly accounted-for predation mortality are also given in [33].

We end this brief survey citing [34], where the principal ways of modeling interactions between multiple species are presented. These include (among others): Competition, Predator–Prey, Host–Parasitoid, Infectious Diseases, Plant Competition, Facilitation. These models aim to incorporate multispecies interactions at large spatial extents using interaction matrices. Our approach differs radically: we propose first of all, a rather general modelization for studying multi-species interactions, with which we can deal both with pure competence between species and with predator–prey systems (and mixed competence–predator–prey ones).

Another important issue which can be taken into account in complex decision problems is the influence of uncertainty over fisheries. We shall deal, in this work, with a deterministic environment but we cannot omit the citation of several studies which include stochasticity. The book [2], includes a part dealing with fisheries management from the deterministic point of view and a comparison with the stochastic one. An analogue comparative study (but on forestry management) can be found in the same reference. The effects of uncertainty over fisheries are also covered in [5]. The uncertainties and climate change impacts on forests has been recently analyzed in [35], and [36] is an excellent review on forest management. Finally, Kvamsdal et al. [37] analyzes a continuous, nonlinear bioeconomic model to demonstrate how stochasticity in the growth of fish stocks affects the optimal exploitation policy when prices are stochastic.

Once the model, under a deterministic environment, is presented, we focus on an original setting: a multi-species biological system (say with n species) and a mid-term finite horizon $[0, T]$. In this context, we understand mid-term as a relatively short time in comparison with the time taken by the system to reach, by itself, a steady state, and at the same time, long enough that days or even a single fishing campaign are small in comparison. Following our computations, this gives a value for T of around 10 years. In this horizon, a single manager has a target for the end of the optimization interval: the final stock of each species $x_i(T)$ ($i = 1, \dots, n$) is fixed *a priori*. This problem of considering a specific final stock, which has barely appeared in the literature, is essential in today's society: rather than seeking long-term biological equilibrium by leaving the system to itself, it is important for authorities to have a tool enabling them to lead the system to a desired state, respecting its biological laws, while at the same time maximizing the profits of fishing during that time interval.

To solve the problem, we adapt an algorithm that takes advantage of the main result of Optimal Control: Pontryagin's Maximum Principle [38], combined with the shooting method [39] and an adapted version for functionals of the cyclic coordinate descent [40]. This resolution method can deal with arbitrarily complex multidimensional problems. The idea

lies in not trying to solve the problem globally but to deal with it as an iterative sequence of one-dimensional problems in which, at each step, each species considers the others as known functions. This property, together with other specifics which we shall show in detail, is what allows the method not to be gravely influenced by the dimensionality of the problem or by the complexity of the model.

The paper is organized as follows: Section 2 presents the different modelings needed for the problem being addressed; in Section 3, the optimization algorithm is shown; Section 4 shows a numerical example to illustrate the algorithm's performance and the dependence of the model on several variables is illustrated; finally, Section 5 includes the conclusions and perspectives for further research.

2. Statement of the problem

2.1. Biological model

Denoting by t the time variable (in units of years), we let $x(t)$ represent the stock of a single species at time t . We work in the continuous setting because we are modeling species which have large stocks (cod, capelin, herring, etc.) and because the fishing events, which are daily, happen at a very small scale relatively to our time interval (and the influence of a single event on the stock is very small). In this context, $x(t)$ will be a continuous function with piecewise continuous derivative (i.e. the plot of $x(t)$ may have corner points but no discontinuities).

The model for the pattern of biological growth of one resource. The literature [1] presents several models in which the actual growth rate with respect to time $\dot{x}(t)$, depends on the stock size $x(t)$. A commonly used functional form is the simple logistic function or Verhulst equation (1), where $r > 0$ denotes the intrinsic growth rate and $k > 0$ the carrying capacity of the species. This model is a good approximation to the natural growth processes of many fish populations and will be the one we use henceforward. Thus, from now on $f_l(x)$ will denote the right hand side of (1):

$$\dot{x}(t) = f_l(x) = rx(t) \left(1 - \frac{x(t)}{k} \right) \quad (1)$$

In this model, the maximum amount of growth or maximum sustainable yield, x_{MSY} , happens when the stock size is equal to $k/2$. We consider upper and lower limits for the stock $x(t)$, expressed as $x \in [x_{\min}, k]$. These are imposed, respectively, by the biological minimum of the species, x_{\min} which allows its reproduction and by the value k of the model.

There are other generalizations of the logistic growth model such as the modified logistic model (2) or the Gompertz equation (3). Model (2) is used, for example, in [28] for cod in Norway, with $\gamma = 2$

$$\dot{x}(t) = rx^\gamma(t) \left(1 - \frac{x(t)}{k} \right); \quad \gamma > 1 \quad (2)$$

$$\dot{x}(t) = rx(t) \ln \frac{k}{x(t)} \quad (3)$$

There are also discrete-population models such as the Beverton–Holt model (which can be considered as a discrete version of the simple logistic function), the Hassell model or the Ricker model (see a review in [41]). We do not study any of these.

2.2. Harvesting model

Human harvesting is included in the model as follows: denote by $h(t)$ the rate of biomass harvest, which we assume is a piecewise continuous function. For the sake of simplicity, many authors (see, for example, [1,18,21]) assume that all the different dimensions of the harvesting activity (e.g. size of nets, number of trawlers, number of fishing days) can be aggregated into one single magnitude called effort, $E(t)$. Thus, a first model proposed by them is:

$$h(t) = qE(t)x(t) \quad (4)$$

where q is a constant, often called the catchability coefficient, $E(t)$ is the fishing effort, and $x(t)$ is the fish stock level at time t . The proportionality constant q encodes the easiness of fish harvesting. This approach has obvious advantages in terms of mathematical tractability but, in our view, it constitutes an unnecessary simplification.

In this paper, we model the dynamics in a more general form as:

$$\dot{x}(t) = f_l(x(t)) - h(t) \quad (5)$$

where $h(t)$ does not depend on any other quantities.

2.3. Economic model

The economic model for each species follows. Let $\pi(x(t), h(t))$ be the instantaneous net revenue from the harvest of the stock biomass, given as in [42]:

$$\pi(x(t), h(t)) = p(h(t))h(t) - c(x(t), h(t)) \quad (6)$$

where $p(h(t))$ is the price function and $c(x(t), h(t))$ the cost function associated with the harvest. We assume an economic model verifying the following three natural conditions:

$$\frac{\partial p(h)}{\partial h} < 0; \frac{\partial c(x, h)}{\partial h} > 0; \frac{\partial c(x, h)}{\partial x} < 0 \tag{7}$$

With these assumptions, we adopt the very general models:

$$p(h(t)) = p_0 - p_1 h \tag{8}$$

$$c(x(t), h(t)) = \frac{ch(t)^\alpha}{x(t)} \tag{9}$$

where p_0 is the stock price, p_1 is the strength of demand, c is the cost of exploitation and α is the harvest cost parameter. These models correspond to real-world fisheries (see [28] where it is used for Norway, Iceland and Denmark, and [30] where it is used for the Barents Sea), where the price of the harvest depends on the amount harvested and the cost of harvesting depends on the stock biomass. Substituting (8) and (9) in (6), the profit function for each species is:

$$\pi(x(t), h(t)) = p_0 h(t) - p_1 h(t)^2 - \frac{ch(t)^\alpha}{x(t)} \tag{10}$$

We remark that other authors use simpler models, in which either the stock price is constant [31], or the cost is linear in the harvesting ($\alpha = 1$) or even constant [27]. In [30], some species have constant price and the price of others depend on the harvesting. As regards the cost of harvesting, in those works it does not depend on the stock for some species. Our model, inspired by that of [17], is the most complete.

2.4. Multi-species biological model

Finally, we present the multi-species model for the biological system. As stated in the Introduction, we consider the classical modeling (see, for example [23]) as that in which the per capita growth rates of each species is a linear function of population abundances:

$$\frac{dn_i}{dt} = \left(\sum_{j=1}^s a_{ij} n_j \right) n_i \tag{11}$$

where a_{ij} is the effect of species i on the per capita growth rate of species j . The value of a_{ij} is constant in the model, negative if either species j exploits species i or species j and i compete directly with each other. Its value is positive if, on the contrary, species i exploits species j and $a_{ji} \leq -a_{ij}$. One can also define, following [23] the energy conversion rate from prey i to predator j as:

$$m_{ji} = -\frac{a_{ji}}{a_{ij}} \tag{12}$$

The pattern above is the most usual. See, for example [27] where two mutually competing fish species are considered, both subject to harvesting, and both of them have a common unharvested predator. A modified version is shown in the same reference, where a predator–prey system subject to harvesting is considered using a modified version of the Leslie–Gower scheme: it is assumed that the reduction in a predator population has a reciprocal relation with per capita availability of its preferred food.

See [28] for other more particular models in which linear functions for the relation between two species is also considered but with power models for each of them separately. In [30], three fish species (capelin, cod and herring) are studied and the pairwise interactions are (at times) considered as square roots. The problem with this models is that they are too specific and unfit for generalization.

In this work, we are going to adopt a model which is even more general than the one used in [31], where three species interact in a linear way but only pairwise: we referred to this type of relation schema in the Introduction, when citing [34]. In it, three kinds of biological interactions are distinguished: (1) simple qualitative linkages between species; (2) quantitative interaction coefficients that reflect the power of these interactions; and (3) parameters which measure –by way of a proxy environmental variable– how the triple interactions affect each species. All the approaches above can be described using interaction matrices. However, there is no study including all the possible parameters we are going to include.

We propose a model with the following parameters:

- The effect c_{ij} of species j on the per-capita growth rate of species i .
- The simultaneous effect c_{ijk} of both species j and k on the per-capita growth rate of species i .
- The convexity parameter γ_{ij} of the influence of species j on species i . It needs not be true that effects are linear on the population: this parameter measures this (possible) non-linearity.
- The convexity parameter γ_{ijk} of the pairwise influences of species j and k on species i , measured in species j .

- The catalysis level β_{ij} of the species i when interacted by species j : it might happen that the rate at which i increases is lower than the rate at which the influence of j does.
- The catalysis level β_{ijk} of species i when interacted by both species j, k .

One should take into account that a linear model assumes explicitly that interactions between individuals are straight-forward (one more individual means exactly the same change in the quantity of the interaction). This assumption, although highly successful for basic modeling is a very elementary description of real-world systems. Introducing non-linearity allows us to provide a richer structure. A very simple case might be that, after some point, the increase in a prey species is irrelevant for the increase in a predator one, at least when the size of the predator species is small (or vice versa). This is, for instance, what β_{ij} and/or γ_{ij} can model, taking their values as > 1 or < 1 . This is just for pairwise interactions. For triads (or larger) of interactions, our model copes perfectly well.

Hence, in the general case of dimension n , we model the dynamics of the i th species as:

$$\dot{x}_i(t) = f_{i,i}(x_i) - g_i(t, x_1, \dots, x_i, \dots, x_n) - h_i(t) \tag{13}$$

where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ and $g_i(\mathbf{x}(t))$ is the function representing the competence or “coupling” among the different species, which we assume is of the form:

$$g_i(t, x_1, \dots, x_i, \dots, x_n) = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} c_{ij} x_i(t)^{\beta_{ij}} x_j(t)^{\gamma_{ij}} + \sum_{\substack{1 \leq j < k \leq n \\ j \neq i \neq k}} c_{ijk} x_i(t)^{\beta_{ijk}} x_j(t)^{\gamma_{ijk}} x_k(t)^{\gamma_{ik}} \tag{14}$$

This function we are proposing expresses the relations between those species which compete with the i th one. It is the most general one we have found in the literature. It allows modeling with a high degree of detail, including, for instance, simultaneous interactions of up to 3 species. Notice that this value might be increased without a substantial modification of the complexity of the underlying problem. Each species x_i is influenced by its exponent β_{ij} when affected by species x_j .

If $\beta_{ij} = \gamma_{ij} = 1$, $\beta_{ijk} = \gamma_{ijk} = 0$ one obtains the classical models [27,28,31]. In [30] one also finds simultaneous interactions between 3 species, as in our model (but his is a specific case, with the constants already fixed). Notice that (14) can deal both with problems of pure competence between species and with predator–prey models. In the former ones, the coefficients c_{ij} and c_{ijk} are positive, whereas in the latter ones, predators have negative c_{ij} and c_{ijk} and preys positive.

In order to clarify the notation used in (14), we show now, by way of example, the ground problem we shall analyze later on and which corresponds to the three-species case, $n = 3$ which, as seen in [27], already constitutes a highly dynamic system. Using the simple logistic function for $f_{i,i}$, we get:

$$\dot{x}_1(t) = r_1 x_1(t) \left(1 - \frac{x_1(t)}{k_1} \right) - [c_{12} x_1(t)^{\beta_{12}} x_2(t)^{\gamma_{12}} + c_{13} x_1(t)^{\beta_{13}} x_3(t)^{\gamma_{13}} + c_{123} x_1(t)^{\beta_{123}} x_2(t)^{\gamma_{123}} x_3(t)^{\gamma_{132}}] - h_1(t) \tag{15}$$

$$\dot{x}_2(t) = r_2 x_2(t) \left(1 - \frac{x_2(t)}{k_2} \right) - [c_{21} x_2(t)^{\beta_{21}} x_1(t)^{\gamma_{21}} + c_{23} x_2(t)^{\beta_{23}} x_3(t)^{\gamma_{23}} + c_{213} x_2(t)^{\beta_{213}} x_1(t)^{\gamma_{213}} x_3(t)^{\gamma_{231}}] - h_2(t) \tag{16}$$

$$\dot{x}_3(t) = r_3 x_3(t) \left(1 - \frac{x_3(t)}{k_3} \right) - [c_{31} x_3(t)^{\beta_{31}} x_1(t)^{\gamma_{31}} + c_{32} x_3(t)^{\beta_{32}} x_2(t)^{\gamma_{32}} + c_{312} x_3(t)^{\beta_{312}} x_1(t)^{\gamma_{312}} x_2(t)^{\gamma_{321}}] - h_3(t) \tag{17}$$

Once this complex modelization is adopted, we remark that the choice of the method we are going to develop in the next section is not just casual. Our resolution method is not intrinsically affected by the complexity of the model: as we use a cyclic coordinate descent method, what we do is to solve a sequence of one-dimensional problems. In each of these, each species “considers” the other ones as fixed, so that the modeling of the coupling term is, in this case, of the simple form $g_i(t, x_i)$.

2.5. Objective functional

Our model considers an open-access fishery model, in which a single manager takes the market price of fish as given. The manager’s objective is to maximize profits from the harvest schedule of the multi-species over a finite time horizon $[0, T]$, fixing a target for the end of the optimization interval: the final stock of each species, $x_i(T)$. Moreover, the solution is subject to the dynamic constraint equations (13) and other natural and policy restrictions involving limits for the harvest and the stock. Hence, letting $\pi_i(x(t), h(t))$ as in Eq. (10) be the profit function for each species i , our problem is:

$$\begin{aligned} & \max_{h_i(t)} \int_0^T \sum_{i=1}^n \pi_i(x_i(t), h_i(t)) e^{-\delta t} dt \\ \text{s.t. } & \dot{x}_i(t) = f_{i,i}(x_i(t)) - g_i(x_1(t), \dots, x_i(t), \dots, x_n(t)) - h_i(t) \\ & x_i(0) = x_{i0}; \quad x_i(T) = x_{iT} \\ & h_i(t) \in H_i = [h_{i\min}, h_{i\max}] \\ & i = 1, \dots, n \end{aligned} \tag{18}$$

where $\delta > 0$ is the discount rate, x_{i0} is the initial stock level and x_{iT} the final desired stock level.

3. Optimization algorithm

Notice that Problem (18) can be stated as a Multi-dimensional Optimal Control Problem. The numerical algorithm we propose for solving it uses a particular strategy related to the cyclic coordinate descent (CCD) method [40]. This method minimizes a function of n variables cyclically with respect to the coordinates. With our method, the Multi-dimensional problem with n species can be solved as a sequence of problems with one single species assuming the others fixed (what we shall call the Unidimensional Problem). Thus, starting from an admissible solution of the multi-dimensional problem, \mathbf{s}^0 , we compute a sequence (\mathbf{s}^k) and the algorithm will calculate:

$$\lim_{k \rightarrow \infty} \mathbf{s}^k \tag{19}$$

We shall also give a necessary condition of maximum for the Unidimensional Problem from which we shall compute the solution with the following two steps: the construction of x_i^K for $i = 1, \dots, n$ and the calculation of the optimal K .

With this method, the problem can be solved like a sequence of problems each of whose error functional converges to zero.

3.1. Step 1: The multi-dimensional problem

As explained above, Problem (18) can be stated as a Multi-dimensional Optimal Control Problem simply setting the stock as the state variable, $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ and the harvesting $\mathbf{u}(t) = (h_1(t), \dots, h_n(t))$ as the control variable. Taking into account that $\dot{x}_i(t) = f_{i,i}(x_i) - g_i(\mathbf{x}) - h_i(t)$, we can write

$$h_i(t) = f_{i,i}(x_i) - g_i(\mathbf{x}(t)) - \dot{x}_i(t) = G_i(t, \mathbf{x}(t), \dot{x}_i(t)) \tag{20}$$

and impose the specified constraints on both variables.

In what follows, a function $F(t, \mathbf{x}, \mathbf{u})$ (the integrand) will be given. Whenever a path $\mathbf{x}(t) = \mathbf{s}(t)$ and a coordinate i are fixed, we shall denote F_i^s the corresponding function $F_i^s(t) = F(t, \mathbf{s}(t), \mathbf{u}(t))$. There is also a vector function $\mathbf{f}(t) = (f_1(t), \dots, f_n(t))$.

Remark 1. The following hypotheses are assumed: (i) F_i^s and f_i are continuous; (ii) F_i^s and f_i have continuous partial first derivatives with respect to t and x_i ; (iii) the control $u_i(t)$ is piecewise continuous; (iv) the state variable $x_i(t)$ is continuous and its derivative is piecewise continuous (i.e. $x_i(t)$ admits corner points). The set of admissible controls is compact and convex. These hypothesis arise from the natural assumptions of our continuous model. As is customary, the set of differentiable functions of a real variable in $[0, T]$ with continuous derivative is denoted $\hat{C}^1[0, T]$.

The objective is to maximize the discounted profits given the state dynamics and fixing a final value for the stock:

$$\max_{\mathbf{u}(t)} J = \int_0^T F(t, \mathbf{x}(t), \mathbf{u}(t)) dt \tag{21}$$

where $\mathbf{u}(t) = (h_1(t), \dots, h_n(t))$, subject to satisfying, for each $t \in [0, T]$:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(0) = \mathbf{x}_0; \quad \mathbf{x}(T) = \mathbf{x}_T \tag{22}$$

$$h_{1 \min} \leq h_1(t) \leq h_{1 \max}, \dots, h_{n \min} \leq h_n(t) \leq h_{n \max} \tag{23}$$

Hence, we say that $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ is an admissible solution if it belongs to the set

$$\Theta := \{ \mathbf{z} \in (\hat{C}^1[0, T])^n / \mathbf{z}(0) = \mathbf{x}_0, \mathbf{z}(T) = \mathbf{x}_T, h_{i \min} \leq G_i(t, \mathbf{z}(t), \dot{z}_i(t)) \leq h_{i \max}, i = 1, \dots, n \} \tag{24}$$

This problem presents the following remarkable features: First, it is a multidimensional problem. Second, the optimization interval is finite and the final stock is fixed. Third, the time t is not explicitly present in the problem (it is a time-autonomous problem), except in the discount factor and finally –fourth– constraints on the control are imposed. Due to the nature of problem, Optimal Control Theory, and more specifically Pontryagin’s Maximum Principle [38] applies straightforwardly.

As we explained in the Introduction, we are not interested, in this work, in the equilibrium or steady-state solution, at which the resource stock size is unchanging over time (a biological equilibrium) and the harvesting is constant. We aim to find the dynamic solution, i.e. the adjustment path towards the final desired state starting from a given initial value.

From the Multi-dimensional problem with n species we construct a sequence of problems with one single species, assuming the others fixed.

Let $\mathbf{s} = (s_1, \dots, s_n) \in \Theta$, with $u_i(t) = h_i(t) = G_i(t, \mathbf{s}(t), \dot{s}_i(t))$ and set:

$$F_i^s(t, x_i, u_i) := F(s_1(t), \dots, s_{i-1}(t), x_i(t), s_{i+1}(t), \dots, s_n(t), h_1(t), \dots, h_{i-1}(t), u_i(t), h_{i+1}(t), \dots, h_n(t)) \tag{25}$$

The problem which consists in finding:

$$\max_{u_i(t)} J_i^s(x_i) := \int_0^T F_i^s(t, x_i, u_i) dt \tag{26}$$

in the set

$$\Theta_i^s := \{z_i \in \hat{C}^1[0, T] / z_i(0) = x_{i0}, z_i(T) = x_{iT}, h_{i\min} \leq G_i(s_1(t), \dots, s_{i-1}(t), x_i(t), s_{i+1}(t), \dots, s_n(t), \dot{x}_i(t)) \leq h_{i\max}\} \tag{27}$$

is (certainly) a unidimensional problem.

We advanced in the Introduction that the solution to the Multi-dimensional problem will be obtained by applying the following algorithm:

Remark 2. Notice that Problem (26) with the conditions (27) is a unidimensional optimization problem which admits a single solution $s_i^*(t)$.

Definition 1. We define the i th maximizing map φ_i in Θ as the map which substitutes the i th state function by the corresponding solution to problem (26), leaving the others unmodified: let $(s_1, \dots, s_n) \in \Theta$ be an admissible element. Let s_i^* be the solution of Problem (26) subject to (27). Then

$$\varphi_i(s_1, \dots, s_i, \dots, s_n) = (s_1, \dots, s_i^*, \dots, s_n), \tag{28}$$

so that $J_i^s(s^*) \geq J_i^s(x_i)$ for all $x_i \in \Theta_i^s$ (i.e. s_i^* maximizes J_i^s).

We shall denote by φ the map associated with the ascent algorithm, which will be the composition of the i th maximizing map:

$$\varphi := \varphi_n \circ \dots \circ \varphi_1 \tag{29}$$

In every k th iteration of the algorithm, “the n components of \mathbf{s} will have been maximized” by means of the i th maximizing applications for each $i = 1, \dots, n$ in this order, thus obtaining the new admissible element \mathbf{s}^k :

$$\mathbf{s}^k = \varphi(\mathbf{s}^{k-1}) = (\varphi_n \circ \varphi_{n-1} \circ \dots \circ \varphi_2 \circ \varphi_1)(\mathbf{s}^{k-1}) \tag{30}$$

Starting with some admissible \mathbf{s}^0 , we construct the sequence (\mathbf{s}^k) .

The convergence of the algorithm, taking into account Zangwill’s global convergence Theorem [43], is justified in the same way as the authors did in [44]. The limit of this ascending succession

$$\lim_{k \rightarrow \infty} \mathbf{s}^k \tag{31}$$

provides the maximum.

3.2. Step 2: The unidimensional problem

Problem (26), (27) is a unidimensional optimal control problem with fixed end-time T , fixed initial state $x_i(0)$ and fixed end state $x_i(T)$, which can be expressed as

$$\begin{aligned} \max_{u_i(t)} J_i^s &= \max_{u_i(t)} \int_0^T F_i^s(t, x_i(t), u_i(t)) dt \\ \text{subject to: } \dot{x}_i(t) &= f_i(t, x_i(t), u_i(t)) \\ x_i(0) &= x_{i0}; \quad x_i(T) = x_{iT} \\ u_i(t) &\in [h_{i\min}, h_{i\max}] \text{ for each } t \in [0, T] \end{aligned} \tag{32}$$

where $f_i(t, x_i(t), u_i(t)) = f_{1,i}(x_i) - g_i(s_1, \dots, s_{i-1}, x_i, s_{i+1}, \dots, s_n) - u_i(t)$.

Based on PMP [38], we establish the necessary conditions of optimality for our unidimensional problem (32). We require the following definition.

Definition 2. Let $x_i \in \Theta_i^s$. The coordination function of x_i , $\mathbb{F}_{x_i}(t)$ is the function in $[0, T]$, defined as follows:

$$\mathbb{F}_{x_i}(t) = (F_i^s(t, x_i(t), u_i(t)))_{u_i} \cdot e^{\int_0^t (f_i(t, x_i(v), u_i(v)))_{x_i} dv} + \int_0^t \left((F_i^s(t, x_i(v), u_i(v)))_{x_i} e^{\int_0^v (f_i(t, x_i(z), u_i(z)))_{x_i} dz} \right) dv \tag{33}$$

where sub-indices $(\dots)_{u_i}$ or $(\dots)_{x_i}$ mean partial differentiation with respect to the corresponding coordinate.

Theorem 1 (A necessary maximum condition). *Let u_i^* be the optimal control; let $x_i^* \in \hat{C}^1$ be a solution of the above problem. Then there exists a constant $K \in \mathbb{R}$ such that:*

$$\begin{aligned} \text{If } h_{i\min} < u_i^* < h_{i\max} &\implies \mathbb{F}_{x_i^*}(t) = K \\ \text{If } u_i^* = h_{i\min} &\implies \mathbb{F}_{x_i^*}(t) \leq K \\ \text{If } u_i^* = h_{i\max} &\implies \mathbb{F}_{x_i^*}(t) \geq K \end{aligned} \tag{34}$$

Proof. Let H be the Hamiltonian function associated with the problem

$$H(t, x_i, u_i, \lambda) = F_i^s(t, x_i, u_i) + \lambda \cdot f_i(t, x_i, u_i) \tag{35}$$

where $\lambda(t)$ is the co-state variable. Pontryagin's Maximum Principle (PMP) asserts that in order for $u_i^* \in [h_{i\min}, h_{i\max}]$ to be optimal, a nontrivial function λ must exist satisfying the following conditions:

$$\dot{x}_i^* = H_\lambda = f_i(t, x_i^*, u_i^*); \quad x_i^*(0) = x_{i0}^*, \quad x_i^*(T) = x_{iT}^* \tag{36}$$

$$\dot{\lambda} = -\left(H(t, x_i^*, u_i^*, \lambda)\right)_{x_i} \tag{37}$$

$$H(t, x_i^*, u_i^*, \lambda) = \max_{u_i(t) \in [h_{i\min}, h_{i\max}]} H(t, x_i^*, u_i, \lambda) \tag{38}$$

Eqs. (36) and (37) are a Boundary Value Problem for x_i^* and λ , which has solution as piecewise C^1 functions. Specifically, $\lambda(t)$ satisfies:

$$\dot{\lambda}(t) = -\left(H(t, x_i^*, u_i^*, \lambda)\right)_{x_i} = -\left(F_i^s\right)_{x_i} - \lambda(t) \cdot (f_i)_{x_i} \tag{39}$$

and hence:

$$\lambda(t) = \left[K - \int_0^t \left(F_i^s\right)_{x_i} e^{\int_0^v (f_i)_{x_i} dz} dv \right] e^{-\int_0^t (f_i)_{x_i} dv} \tag{40}$$

where $K = \lambda(0)$.

From (38) follows that, for each t the value u_i^* maximizes

$$H(u_i) := H(t, x_i^*, u_i, \lambda) = F_i^s(t, x_i^*, u_i, u) + \lambda \cdot f_i(t, x_i^*, u_i), \quad \forall u_i(t) \in [h_{i\min}, h_{i\max}] \tag{41}$$

Bearing in mind that

$$\frac{\partial H(u_i)}{\partial u_i} = \left(F_i^s\right)_{u_i} + \lambda(t)(f_i)_{u_i} \tag{42}$$

there are three possibilities:

1. $h_{i\min} < u_i^* < h_{i\max}$. In this case, $\frac{\partial H(u_i)}{\partial u_i} = 0$. From (40) and (42) and noticing that $(f_i)_{u_i} = -1$, we obtain:

$$0 = \mathbb{F}_{x_i^*}(t) - K \Rightarrow \mathbb{F}_{x_i^*}(t) = K \tag{43}$$

2. $h_{i\min} = u_i^*$. In this case, $\frac{\partial H(u_i)}{\partial u_i} \leq 0$. an analogous reasoning gives:

$$\mathbb{F}_{x_i^*}(t) \leq K \tag{44}$$

3. $u_i^* = h_{i\max}$. In this case, $\frac{\partial H(u_i)}{\partial u_i} \geq 0$ and the same argument provides:

$$\mathbb{F}_{x_i^*}(t) \geq K \tag{45}$$

This ends the proof. \square

From the computational point of view, the construction of the solution consists of two main steps which we proceed to explain.

3.3. Step 3: The construction of x_i^K

In this section, we describe how, using the i th maximizing map, we compute an approximation to the function x_i^K for a given value K , using the fact that for each i , x_i^K maximizes Problem (32). This approximate construction of x_i^K can be performed using a discretized version of the following equation, which we shall call the *coordination equation*:

$$K = \left(F_i^s\right)_{u_i} \cdot e^{\int_0^t (f_i)_{x_i} dz} + \int_0^t \left(F_i^s\right)_{x_i} e^{\int_0^v (f_i)_{x_i} dz} dv \tag{46}$$

For given K , we compute x_i^K , by means of (46). When the values obtained do not obey the control constraints, we force the solution to belong to the boundary until the moment established by conditions (44) and (45). This computation is carried out using polygons (an adaptation of Euler's method).

In summary, we divide the interval $[0, T]$ using N nodes:

$$t_0 < t_1 < \dots < t_{N-1} \tag{47}$$

with $n = 0, \dots, N - 1$ so that $t_0 = 0$ and $t_{N-1} = T$.

Next, we solve the coordination Eq. (46) at each node. Starting at $t_0 = 0$, where x_{i0} is known, we compute, from the discretized version of the coordination Eq. (46), the corresponding control u_{i0} . Using this value for the control, we calculate the derivative of the state equation

$$\dot{x}_i(t) = f_i(t, x_i(t), u_i(t)) = f_{i,i}(x_i) - g_i(s_1, \dots, s_{i-1}, x_i, s_{i+1}, \dots, s_n) - u_i(t) \tag{48}$$

In this case:

$$\dot{x}_{i0} = f_i(t, x_{i0}, u_{i0}) \quad (49)$$

And using this value we jump to the next node, using Euler's method, imposing the condition:

$$x_{i1} = x_{i0} + f_i(t, x_{i0}, u_{i0})d \quad (50)$$

And, in general:

$$x_{ik} = x_{i(k-1)} + f_i(t, x_{i(k-1)}, u_{i(k-1)})d \quad (51)$$

where the step d is:

$$d = \frac{T}{N} \quad (52)$$

That way, for $k = 0, \dots, N - 1$, we go through all the nodes until the whole interval $[0, T]$ is covered, linking the arcs continuously.

We must also take into account the restrictions on the control:

$$h_{i\min} \leq u_i(t) \leq h_{i\max}, \quad 0 \leq t \leq T. \quad (53)$$

When these are not verified by the value obtained from solving the coordination Eq. (46), we force the solution to remain on the border until the moment established by conditions (44) and (45):

$$\mathbb{F}_{x_i}(t) \leq K \quad (54)$$

$$\mathbb{F}_{x_i}(t) \geq K \quad (55)$$

3.4. Step 4: Computation of the optimal K

The computation of the optimal K is achieved by means of an adaptation of the shooting method. We search for the extremal x_i^K fulfilling the second boundary condition by varying the coordination constant K (57). The procedure is similar to the shooting method used to solve second-order differential equations with boundary conditions, which may be performed approximately using elemental procedures. Starting from two values for the coordination constant, $K: K_{\min}$ and K_{\max} and using a conventional method such as the secant method, our algorithm converges satisfactorily, as we shall see in the examples.

In summary, the problem consists in finding, for each K , the function x_i^K satisfying the initial condition

$$x_i^K(0) = x_{i0} \quad (56)$$

the conditions of Theorem 1 and, among these, the one that satisfies the second boundary condition:

$$x_i^K(T) = x_{iT} \quad (57)$$

Consider the shooting function:

$$\varphi(K) := x_i^K(T) \quad (58)$$

Solving our final condition problem is equivalent to solving the (finite dimensional) equation

$$\varphi(K) - x_{iT} = 0 \quad (59)$$

We solve this equation by means of the secant method, imposing a stopping criterion based on a pre-established tolerance:

$$\text{error} = |\varphi(K) - x_{iT}| \leq \text{tol}_K \quad (60)$$

The secant method gives:

$$K_{j+1} = K_j - \frac{K_j - K_{j-1}}{\varphi(K_j) - \varphi(K_{j-1})} \varphi(K_j) \quad (61)$$

Thus, for each i th component, we construct a sequence $\{K_j\}_{j \in \mathbb{N}}$ such that $x_i^{K_j}(T)$ converges to x_{iT} . The optimal K is the one verifying

$$\lim_{j \rightarrow \infty} K_j = K \quad (62)$$

with $x_i^K(T) = x_{iT}$.

The stopping criterion for the algorithm is based on the desired tolerance. We shall see this fact in greater detail in Example 1 in Section 4.

Fig. 1 shows a schematic picture of our algorithm.

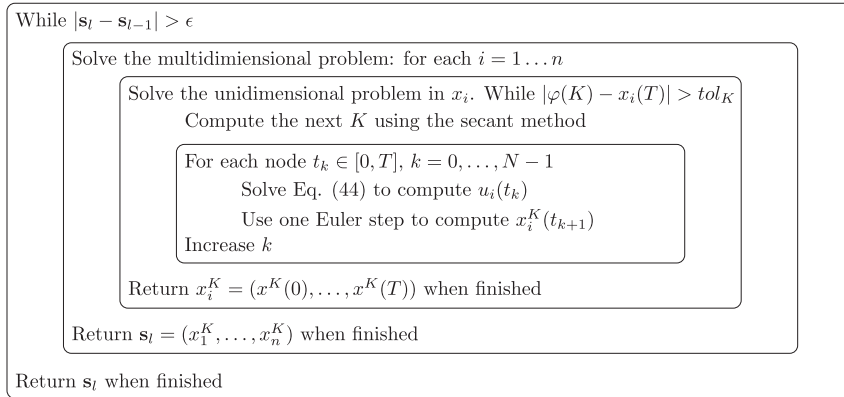


Fig. 1. Schematic flow of the whole algorithm.

Table 1
Parameters of the cost model.

Specie	p_0	p_1	c	α
1	0.9	0.01	75	1.1
2	1.9	0.02	85	1.2
3	2.8	0.03	60	1.4

Table 2
Parameters of the biological models.

$f_{l,i}(x_i(t))$		$g_i(\mathbf{x}(t))$			
$r_1 = 0.5$	$c_{12} = 2.10^{-4}$	$c_{21} = 10^{-5}$	$c_{31} = 10^{-4}$	$\gamma_{12} = 1.1$	$\gamma_{123} = 1.1$
$r_2 = 0.3$	$c_{13} = 3.10^{-5}$	$c_{23} = 2.10^{-5}$	$c_{32} = 10^{-4}$	$\gamma_{13} = 1.0$	$\gamma_{132} = 1.2$
$r_3 = 0.2$	$c_{123} = 10^{-8}$	$c_{213} = 10^{-7}$	$c_{312} = 0$	$\gamma_{21} = 1.0$	$\gamma_{213} = 1.0$
$k_1 = 1000$	$\beta_{12} = 1.0$	$\beta_{21} = 1.0$	$\beta_{31} = 1.0$	$\gamma_{23} = 1.2$	$\gamma_{231} = 1.0$
$k_2 = 700$	$\beta_{13} = 1.2$	$\beta_{23} = 1.0$	$\beta_{32} = 1.0$	$\gamma_{31} = 1.1$	$\gamma_{312} = 0.$
$k_3 = 600$	$\beta_{123} = 0.9$	$\beta_{213} = 1.0$	$\beta_{312} = 0.$	$\gamma_{32} = 1.0$	$\gamma_{321} = 0.$

4. Numerical examples

We now show some numerical simulations, with a twofold aim: firstly, to illustrate how our algorithm works; secondly, to analyze the influence on the solution of some of the parameters of the problem and/or of the model.

4.1. Example 1: Base case

Let us begin with what we shall call the *base case* and which will allow us to compare the results obtained with different parameters. This is the 3-species model described in (15)–(17). The values of the cost model are summarized in Table 1, whereas Table 2 shows the parameters of the biological model, including the interactions among the species. Growth is measured in 10^6kg , and time in units of years. Prices of the economic parameters are in $\text{€}/\text{kg}$ and costs are in 10^6€ .

The values in Table 1 are not taken from any real species but they are inspired by Agnarsson et al. [28], where a detailed modeling was carried out for cod, capelin and herring in Norway, Iceland and Denmark. We have respected the ratio between income and cost in the revenue function which they checked against the real revenues. As for the parameters in Table 2 about species interaction, $g_i(\mathbf{x}(t))$ and the simple logistic function $f_{l,i}(x_i(t))$, the choice of parameters follows the ideas in [31]. All our results were obtained using a custom-written software implemented in Mathematica 10.0[®].

Fig. 2 shows the solution obtained in the absence of harvesting, that is, the long-term steady-state solution of the biological model (15)–(17), with no optimization of the functional. Starting from the initial stock values (in 10^6kg):

$$x_1(0) = 150; \quad x_2(0) = 250; \quad x_3(0) = 150 \tag{63}$$

the values obtained for the steady state of the stock (10^6kg) of each species after 100 years are:

$$x_1^* = 463.69; \quad x_2^* = 654.65; \quad x_3^* = 146.96 \tag{64}$$

However, our interest is not the steady-state: starting from the known initial stocks (63), we fix a time horizon of say $T = 10$ years and, for different reasons, (environmental, economical, ecological, legislative...), we state a final stock for each species

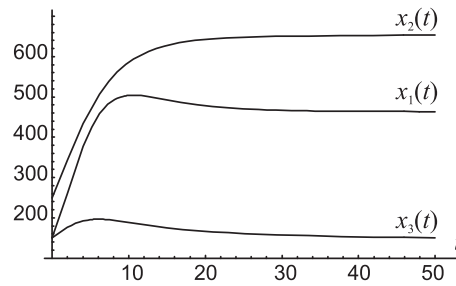


Fig. 2. Steady state of biological equilibrium, no harvesting.

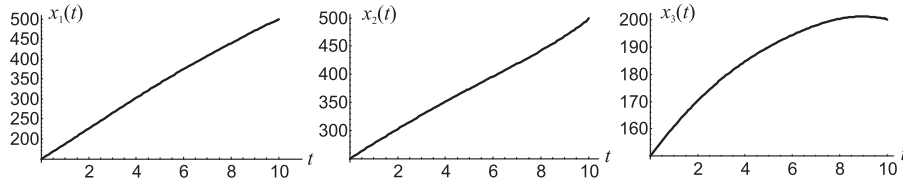


Fig. 3. Optimal solution for the stock profile.

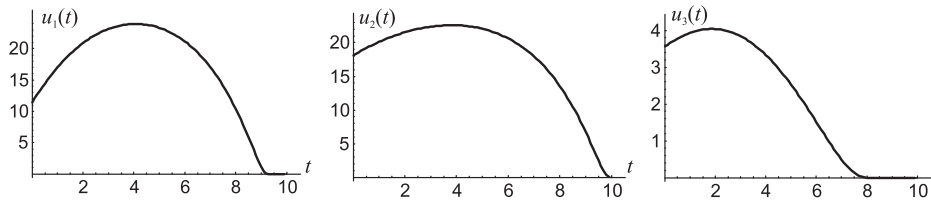


Fig. 4. Optimal solution for the harvest profile.

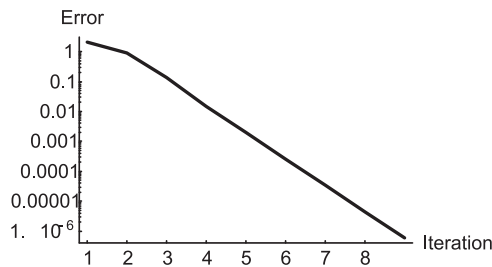


Fig. 5. Convergence of the multi-dimensional algorithm.

to reach in that time-span, different from the one set by the free biological evolution of the system (64). Let us set, in this example, the following target values:

$$x_1(T) = 500; \quad x_2(T) = 500; \quad x_3(T) = 200 \tag{65}$$

The solution will optimize the harvesting revenue during that time, by choosing the optimal controls, i.e. how each species should be fished. We set a $\delta = 5$ percent discount rate.

The results obtained for the optimal stock profile, $x(t)$ (in 10^6 kg), and the optimal harvest path, $h(t)$ (in 10^6 kg/year), of each species are shown in Figs. 3 and 4 respectively. In computing the solution, we used a discretization of 100 sub-intervals. For each species and for the secant method, we chose a tolerance (60) $tol_K = 5 \cdot 10^{-2}$. The stopping criterion for the multi-dimensional algorithm was that the difference between two consecutive iterations of the values of K was less than 10^{-6} . With these parameters, the algorithm converges in just 9 iterations to the final solution (see Fig. 5). The optimal values of K for each species are, respectively:

$$K_1^* = -0.0299251; \quad K_2^* = 0.4474817; \quad K_3^* = 1.6556858 \tag{66}$$

and the final revenue after 10 years is 235.381(in 10^6 €). In finding the solution, we have considered the control limited by $u_i(t) \in [0, 25]$ (in 10^6 kg). Fig. 4 shows how the maximum is not reached at any moment but the minimum is. For species 1, harvesting stops in the interval $[9.2, 10]$, and for species 3, in the interval $[8, 10]$. We consider these two facts quite remarkable, as they state roughly that the optimum is reached by fishing, mainly, in the middle part of the optimization interval

Table 3
Sweeping of the K -values in the shooting function $\varphi(K)$.

K	-2	-1.6	-1.2	-0.8	-0.4	0	0.4	0.8	1.2	1.6	2.0
$x_1^K(T)$	0.03	0.06	0.16	1.16	141.26	340.57	521.67	578.65	580.20	580.20	580.20

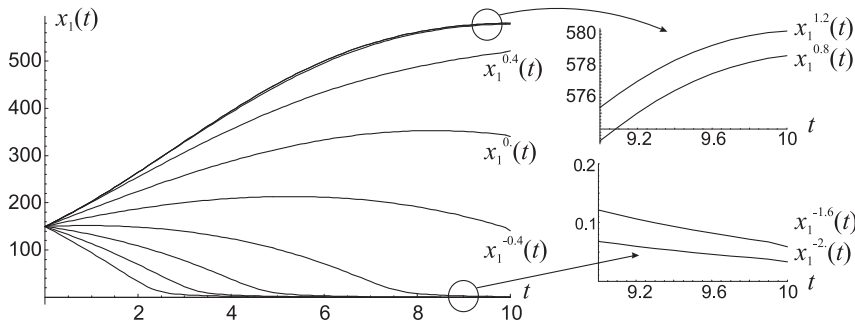


Fig. 6. Admissible interval for the values of the final stock.

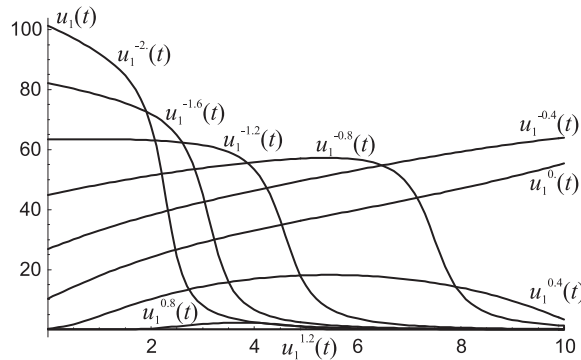


Fig. 7. Optimal controls for the sweeping of the K -values.

and practically stopping the harvest at its end. At this point, and almost without the intervention of the control, the system reaches the desired final state “by itself”.

4.2. Example 2: Influence of final stock and T

In this section we fix the biological parameters for each species, which is a fairly realistic assumption for our mid-term horizon. For the same reason, we fix the economic and price parameters. In this second example, we are going to study a problem we deem relevant: given some initial stocks, what are the possible achievable final values for them and what influence does the time-span have on them?

Varying the coordination constant K . Let us, for the sake of concreteness, assume that the final stocks of species 2 and 3 are imposed and are the same as in the previous example: $x_2(T) = 500$ and $x_3(T) = 200$ (in 10^6 kg). For species 1, starting from the initial stock $x_1(0) = 150$ (in 10^6 kg), what are the final stock values $x_1(T)$ that we can impose that are biologically possible after $T = 10$ years? For the sake of simplicity, we shall assume at this point that there are no upper limits for the controls.

In order to answer the question above, we are going to perform an adaptation of the algorithm previously presented. We are going to consider the one-dimensional problem on species 1, imposing, as known stock values, for each t , those obtained in the base example above, $x_2(t)$ and $x_3(t)$.

Then, instead of fixing the final state $x_1(T)$, we are going to carry out a sweeping of the admissible interval for the coordination constant K , using the shooting function:

$$\varphi(K) := x_1^K(T) \tag{67}$$

Table 3 shows the results obtained for the final stock of $x_1(T)$ in 10^6 kg when sweeping the K -interval $[-2, 2]$. Notice how the admissible values for $x_1^K(T)$ is the interval $[0.03, 580.20]$ (in 10^6 kg). Fig. 6 shows the optimal profiles of $x_1^K(t)$ for the values of K given in Table 3, whereas Fig. 7 shows the optimal controls obtained for those values.

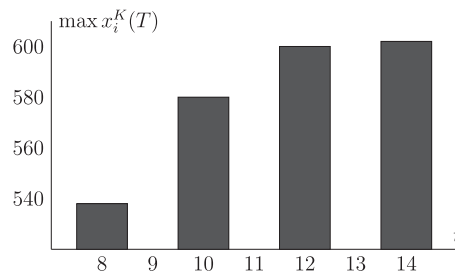


Fig. 8. Histogram corresponding to Table 4: influence of T on $\max x_1^K(T)$.

Table 4

Influence of T on the admissible interval.

T	8	10	12	14
$\max x_1^K(T)$	538.09	580.20	598.88	605.42

As one can see, we have constructed the extremal field for the variable $x_1(t)$: the zooms on the right side shows how the extremals do not meet.

The maximum value for the stock is quickly reached for values of K greater than 1.2 and it corresponds to those cases where the harvesting (the control) is 0 (see Fig. 7). On the other hand, the minimum value of the stock (essentially 0) is reached with the optimal harvesting profiles shown in Fig. 7. At the beginning of the interval, the fishing activity is remarkably large, so that when the stock of species 1 has already been greatly diminished, only the competency effects of species 2 and 3 and practically no harvesting, species 1 gets to its practical biological demise. Obviously, this is not the desired value at all: we show it as a validation of the theoretical soundness of our algorithm.

Influence of T on the admissible interval. This is the second question we are going to address. Table 4 summarizes the results for the values $T = 8, 10, 12$ and 14 years: the maximum achievable value of $x_1(T)$ for each T is shown (in 10^6 kg), corresponding to the maximum values of K . Notice that the minimum achievable value of $x_1(T)$ is the same in each case, essentially 0 as shown in the previous example. Notice how, the longer the time T , the larger the admissible variation rank for the final stock of species 1, $x_1(T)$ which is reachable. However, Fig. 8 shows clearly how these increase is not linear. Actually, the species has a biological limit and, notwithstanding the time-span, the maximum admissible stock does not grow indefinitely. We consider this result (which is reasonable from the biological point of view) provides an important contingency of which the harvesting managers have to be aware for the mid-term planning.

We finish this section by remarking that what we have obtained is just an *estimation* of the structure of the set of admissible solutions for $x_1(T)$, as the values of $x_2(t)$ and $x_3(t)$ were chosen from the base case. These values, even starting from the same initial stock value (each its own) and reaching the same final value (ibid.), might take different intermediate values and this would, certainly, influence the admissible interval of $x_1(T)$. However, several numerical simulations lead us to conjecture that this variation should be quite small and that our estimation is rather good.

5. Conclusions and future perspectives

Going further than the study of the long-term steady-state solution of the multi-species problem in the renewable resources problem, we have considered a joint finite mid-term horizon and a stock target at the end of the optimization interval. The solution of this problem involves a highly dynamic system which requires the development of a method allowing to solve it satisfactorily. This algorithm, an adaptation of the cyclic coordinate descent provides a procedure whose complexity is not essentially influenced by the dimensionality of the problem, as it reduces to an iteration of one-dimensional algorithms. Each of these is tackled by approximately solving what we call the *coordination equation*. This, together with a customization of the shooting method, eliminates the multi-dimensionality and thus, removes complexities due to the biological interactions from the resolution process.

In brief, we consider that the main novel characteristics of our work are:

- The setting of a fixed stock at a mid-term horizon as one of the constraints.
- The arbitrary dimensionality of the considered problem: we allow for any number of species, with arbitrary simultaneous interactions.
- The flexibility of the model, allowing for nonlinearity in the interactions, which greatly increases the modeling possibilities.
- The method used to solve the problem: despite the nonlinearity of the model, we have an algorithm whose complexity is not inherently increased by either the dimensionality or the interactions.
- Our model permits studying problems of very different nature: predator–prey models, competition, host–parasitoid, infectious disease, etc.

As for the solution we obtain, we remark that it is possible to achieve the desired final stock values while maximizing the revenues of harvesting during the optimization interval. As a matter of fact, we observe that the optimum is reached in such a way that the harvesting practically stops near the end of the interval and the biological system evolves by itself to the desired outcome. We have also analyzed the variation of the range of admissible values for the stock depending on the duration of the optimization interval: this gives the policy makers an a priori set of limit values of each species which they can reach without harm to the biological system. One remaining question which we do not know how to tackle yet is the structure of the joint set of admissible values for each species at the same time. This is an open question which we think requires a richer set of tools or a deeper insight than we have at present.

References

- [1] R. Perman, Y. Ma, J. McGilvray, M. Common, *Natural Resource & Environmental Economics*, Pearson Education, Harlow, England, 2003.
- [2] J. Kennedy, *Dynamic Programming: Applications to Agriculture and Natural Resources*, Springer Science & Business Media, 2012.
- [3] J.C. Cardenas, M. Janssen, F. Bousquet, Dynamics of rules and resources: three new field experiments on water, forests and fisheries., in: *Handbook on Experimental Economics and the Environment*, Edward Elgar Publishing, 2013.
- [4] R. Beverton, S.J. Holt, *On the Dynamics of Exploited Fish Populations*, 11, Springer Science & Business Media, 2012.
- [5] R. Hilborn, C.J. Walters, *Quantitative Fisheries Stock Assessment: Choice, Dynamics and Uncertainty*, Springer Science & Business Media, 2013.
- [6] P. Bettinger, et al., *Forest Management and Planning*, Academic Press, 2016.
- [7] L.W. Mays, *Optimal Control of Hydrosystems: 0*, CRC Press, 2018.
- [8] O. Sundström, D. Ambühl, L. Guzzella, On implementation of dynamic programming for optimal control problems with final state constraints, *Oil Gas Sci. Technol. Revue de l'Institut Français du Pétrole* 65 (1) (2010) 91–102.
- [9] L.G. Kronbak, D. Squires, N. Vestergaard, Recent developments in fisheries economics research, *Int. Rev. Environ. Res. Econ.* 7 (2014) 67–108.
- [10] L. Grüne, W. Semmler, M. Stieler, Using nonlinear model predictive control for dynamic decision problems in economics, *J. Econ. Dyn. Control* 60 (2015) 112–133.
- [11] P. Prouzet, *Value and Economy of Marine Resources*, John Wiley & Sons, New York, 2015.
- [12] M. Schlüter, et al., New horizons for managing the environment: a review of coupled social-ecological systems modeling, *Nat. Resour. Model.* 25 (2012) 219–272.
- [13] V.N. de Jonge, R. Pinto, R.K. Turner, Integrating ecological, economic and social aspects to generate useful management information under the EU directives' 'ecosystem approach', *Ocean Coast. Manag.* 68 (2012) 169–188.
- [14] L.R. Brown, *Eco-economy: Building an Economy for the Earth*, Routledge, 2013.
- [15] S. Kasperski, Optimal multi-species harvesting in ecologically and economically interdependent fisheries, *Environ. Resour. Econ.* 61 (4) (2015) 517–557.
- [16] A.E. Speers, E.Y. Besedin, J.E. Palardy, C. Moore, Impacts of climate change and ocean acidification on coral reef fisheries: an integrated ecological–economic model, *Ecol. Econ.* 128 (2016) 33–43.
- [17] D. Poudel, L.K. Sandal, S.F. Kvamsdal, Analyzing risk of stock collapse in a fishery under stochastic profit maximization, discussion paper, Norwegian School of Econ. Bus. Admin. 4 (2012) 1–23.
- [18] M.A. Shah, Optimal control theory and fishery model, *J. Dev. Agric. Econ.* 5 (12) (2013) 476–481.
- [19] H. Qiu, J. Lv, K. Wang, The optimal harvesting policy for non-autonomous populations with discount, *Appl. Math. Lett.* 26 (2013) 244–248.
- [20] Y. Tsur, A. Zemel, The infinite horizon dynamic optimization problem revisited: A simple method to determine equilibrium states, *Eur. J. Oper. Res.* 131 (3) (2001) 482–490.
- [21] R. Suri, *Optimal Harvesting Strategies for Fisheries: A Differential Equations Approach*, Massey University New Zealand (2008). thesis
- [22] L. Bayón, P.J. García-Nieto, R. García-Rubio, J.A. Otero, P.M. Suárez, C. Tasis, An algorithm for quasi-linear control problems in the economics of renewable resources: The steady state and end state for the infinite and long-term horizon, *J. Comput. Appl. Math.* 309 (2017) 456–472.
- [23] H. Matsuda, P.A. Abrams, Maximal yields from multispecies fisheries systems: rules for systems with multiple trophic levels, *Ecol. Appl.* 16 (2006) 225–237.
- [24] J.B. Kellner, J.N. Sanchirico, A. Hastings, P.J. Mumby, Optimizing for multiple species and multiple values: tradeoffs inherent in ecosystem-based fisheries management, *Conserv. Lett.* 4 (1) (2011) 21–30.
- [25] T. Legović, S. Klanjšček, S. Geček, Maximum sustainable yield and species extinction in ecosystems, *Ecol. Modell.* 221 (12) (2010) 1569–1574.
- [26] T. Legović, S. Geček, Impact of maximum sustainable yield on mutualistic communities, *Ecol. Modell.* 230 (2012) 63–72.
- [27] T.K. Kar, A. Ghorai, Dynamic behaviour of a delayed predator-prey model with harvesting, *Appl. Math. Comput.* 217 (22) (2011) 9085–9104.
- [28] S. Agnarsson, R. Arnason, K. Johannsdottir, L. Ravn-Jonsen, L.K. Sandal, S.I. Steinshamn, N. Vestergaard, in: *Comparative evaluation of the fisheries policies in denmark, Iceland and Norway: multispecies and stochastic issues*, Bergen, Norway, 2008. Report No 25/07.
- [29] S. Aanestad, L.K. Sandal, A. Eide, in: *Optimal fishing policy for two species in a three species predator-prey model. the case of capelin, cod and juvenile herring in the Barents sea*, Bergen Norway, 2009.
- [30] D. Poudel, L.K. Sandal, Stochastic optimization for multispecies fisheries in the barents sea, *Natur. Resour. Model.* 28 (2015) 219–243.
- [31] I. Agmour, M. Bentounsi, N. Achtaich, Y. Foutayeni, Optimization of the two fishermen's profits exploiting three competing species where prices depend on harvest, 2017, 2017, p. 3157294. 17 pages.
- [32] E.E. Plagányi, et al., Multispecies fisheries management and conservation: tactical applications using models of intermediate complexity, *Fish Fish.* 15 (1) (2014) 1–22.
- [33] M.C. Tyrrell, J.S. Link, H. Moustahfid, The importance of including predation in fish population models: implications for biological reference points, *Fish. Res.* 108 (1) (2011) 1–8.
- [34] W.D. Kissling, et al., Towards novel approaches to modelling biotic interactions in multispecies assemblages at large spatial extents, *J. Biogeogr.* 39 (12) (2012) 2163–2178.
- [35] M. Linder, et al., Climate change and european forests: what do we know, what are the uncertainties, and what are the implications for forest management? *J. Environ. Manag.* Vol. 146 (2014) 69–83.
- [36] R. Yousefpour, et al., A review of decision-making approaches to handle uncertainty and risk in adaptive forest management under climate change, *Ann. Forest Sci.* 69 (2012) 1–15.
- [37] S.F. Kvamsdal, D. Poudel, L.K. Sandal, Harvesting in a fishery with stochastic growth and a mean-reverting price, *Environ. Resour. Econ.* 63 (2016) 643–663.
- [38] A. Chiang, *Elements of Dynamic Optimization*, Waveland Press, 2000.
- [39] M.S. Aronna, J.F. Bonnans, P. Martinon, A shooting algorithm for optimal control problems with singular arcs, *J. Optim. Theory Appl.* 158 (2) (2013) 419–459.
- [40] Z.Q. Luo, P. Tseng, On the convergence of the coordinate descent method for convex differentiable minimization, *J. Optim. Theory Appl.* 72 (1) (1992) 7–35.
- [41] A. Brannstrom, D.J.T. Sumpter, The role of competition and clustering in population dynamics, *Proc. R. Soc. B* 272 (2005) 2065–2072.
- [42] L.K. Sandal, S.I. Steinshamn, A stochastic feedback model for optimal management of renewable resources, *Nat. Resour. Model.* 10 (1) (1997) 31–52.
- [43] W.L. Zangwill, *Nonlinear Programming: a Unified Approach*, Prentice Hall, New Jersey, 1969.
- [44] L. Bayón, J.M. Grau, M.M. Ruiz, P.M. Suárez, Cyclic coordinate descent in hydrothermal nonsmooth problems., *Numer. Algor.* 59 (2012) 227–247.