CrossMark

The Best-or-Worst and the Postdoc problems with random number of candidates

L. Bayón[1] · P. Fortuny[1] · J. Grau[1] · A. M. Oller-Marcén[2] · M. M. Ruiz[1]

Abstract
In this paper we consider two variants of the Secretary problem: The Best-or-Worst and the Postdoc problems. We extend previous work by considering that the number of objects is not known and follows either a discrete Uniform distribution $\mathcal{U}[1, n]$ or a Poisson distribution $\mathcal{P}(\lambda)$. We show that in any case the optimal strategy is a threshold strategy, we provide the optimal cutoff values and the asymptotic probabilities of success. We also put our results in relation with closely related work.

Keywords Secretary problem · Best-or-Worst problem · Postdoc problem · Combinatorial optimization

Mathematics Subject Classification 60G40 · 62L15

1 Introduction

The classical *Secretary problem* has been extensively studied in the fields of applied probability, statistics or decision theory and has been considered by many authors (see

✉ A. M. Oller-Marcén
oller@unizar.es

L. Bayón
bayon@uniovi.es

P. Fortuny
fortunypedro@uniovi.es

J. Grau
grau@uniovi.es

M. M. Ruiz
mruiz@uniovi.es

[1] Departamento de Matemáticas, Universidad de Oviedo, Avda. Calvo Sotelo s/n, 33007 Oviedo, Spain

[2] Centro Universitario de la Defensa de Zaragoza - IUMA, Ctra. Huesca s/n, 50090 Saragossa, Spain

Ferguson 1989; Ferguson et al. 1992; Szajowski 2009 for an extensive bibliography).
It can also be posed as a decision making problem in a game with the following rules:

(1) We have to choose one object from a set.
(2) The total number of objects in the set is known.
(3) The objects are rankable from best to worst.
(4) The objects appear sequentially and in random order.
(5) Each object must be accepted or rejected before the next one appears.
(6) The decision depends only on the relative ranks of the objects examined so far.
(7) Rejected objects cannot be called back.
(8) We want to maximize the probability of selecting the best object.

Dynkin (1963) and Lindley (1961) independently proved that, in the previous
setting, the best strategy consists in observing roughly n/e of the objects and then
choosing the first one that is better than all those observed so far. This strategy returns
the best object with a probability of at least $1/e$, this being its approximate value for
large values of n. This well-known solution was later refined by Gilbert and Mosteller
(1966), showing that $\lfloor (n - \frac{1}{2})e^{-1} + \frac{1}{2} \rfloor$ is a better approximation than $\lfloor n/e \rfloor$, although
the difference is never greater than 1.

We mention here that the classical Secretary problem is just a special case of
the problem of stopping without recall on the very last *interesting* event, since it
suffices to define interesting as *better than the previous ones*. The solution is therefore
a corollary of the odds-theorem of optimal stopping Bruss (2000). Moreover, Bruss
(2003) shows that the lower bound of $1/e$ for the success probability holds, remarkably,
in all generality for whatever law of interesting events. For further developments see
also (Dendievel 2013, 2015; Louchard 2017).

If we modify rule (8) above, we can get variants of the secretary problem. Some of
them also have simple, elegant solutions. For example, if we consider

(8′) We want to maximize the probability of selecting the second best object.

we obtain the so-called *Postdoc problem* in Vanderbei (1983). In this setting the prob-
ability of success for an even number of applicants is exactly $\frac{n}{4(n-1)}$. This probability
tends to 1/4 as n tends to infinity, illustrating the fact that it is easier to pick the best
than the second best. This variant was also considered in Bayón et al. (2018), Rose
(1982) and Szajowski (1982).

On the other hand, if we consider

(8″) We want to maximize the probability of selecting either the best or the worst
object.

we get the so-called *Best-or-Worst problem*. This variant can be found on Ferguson
(1992) as a multicriteria problem in the perfect negative dependence case. In Bayón
et al. (2018) we considered the Best-or-Worst and the Postdoc problems proving that
both of them share the same threshold strategy as optimal stopping rule and that the
probability of success in the Best-or-Worst problem is twice the probability of success
in the Postdoc problem.

Besides these, many other variants of the classical Secretary problem have been
recently studied, specially in the framework of partially ordered objects Freij and

Wastlund (2010), Georgiou et al. (2008) and Garrod and Morris (2013) or matroids Babaioff et al. (2007), Gharan and Vondrák (2013) and Soto (2013).

Interesting lines of work also arise if we also modify rule (2) above. If the number of objects is unknown, the decision maker faces an additional risk because if he rejects an object, he may then discover that it was the last one, in which case he fails. In Presman and Sonin (1972) the case in which the number of objects follows a discrete Uniform distribution $\mathcal{U}[1, n]$ was studied for the classical secretary problem. In this setting, the cutoff value for large N is approximately Ne^{-2} and the probability of success is $2e^{-2}$. This same paper also dealt with the case in which the number of candidates follows a Poisson distribution $\mathcal{P}(\lambda)$ showing that the optimal stopping limit relation is $r^*(\lambda)/\lambda \to e^{-1}$ and that this is also the asymptotic value of the probability of success. This was first studied in a continuous time setting by Cowan and Zabczyk (1978) for a Poisson process of candidates with known arrival rate and then generalized by Bruss (1987) for an unknown arrival rate. See also Szajowski's work (2007) for a corresponding game version.

In the present paper, we want to extend the work done in Bayón et al. (2018) by considering the Best-or-Worst and the Postdoc problems when the number of objects follows a Uniform distribution $\mathcal{U}[1, n]$ or a Poisson distribution $\mathcal{P}(\lambda)$.

The paper is organized as follows: In Sect. 2, we recall the relation between the Best-or-Worst and the Postdoc problems for a known number of objects and extend it to our setting. In Sect. 3 we show that in the considered situations; i.e., if the random number of candidates follows either a discrete Uniform distribution $\mathcal{U}[1, n]$ or a Poisson distribution $\mathcal{P}(\lambda)$ the optimal strategy is still a threshold strategy. After that, Sects. 4 and 5 deal with the Uniform case and with the Poisson case, respectively. Finally, Sect. 6 presents a comparative table of the results and some concluding remarks.

2 The relation between the Best-or-Worst problem and the Postdoc problem

The following theorem establishes the relation between the optimal strategies in the Best-or-Worst problem and the Postdoc problem when the number of objects in known.

Theorem 1 (See Sect. 4 in Bayón et al. (2018)) *Let us define a nice candidate as an object which is either better or worse than all the preceding ones (in the Best-or-Worst problem) or which is the second better than all the preceding ones (in the Postdoc problem). Then, if n is the total number of objects, the following strategy is optimal:*

(1) *Reject the $\lfloor \frac{n}{2} \rfloor$ first inspected objects regardless their rank.*
(2) *After that, accept the first nice candidate.*

Moreover, if $P_{BW}(n)$ and $P_{PD}(n)$ are the probabilities of success following this strategy in the Best-or-Worst and in the Postdoc problem respectively, then we have that

$$2P_{PD}(n) = P_{BW}(n) = \begin{cases} \frac{n}{2(n-1)}, & \text{if } n \text{ is even}; \\ \frac{n+1}{2n}, & \text{if } n \text{ is odd}. \end{cases}$$

In Bayón et al. (2018) it was also shown that, in the Postdoc problem, selecting a candidate that is better than all the previous ones has the same probability of success as waiting for the next nice candidate (second better than the previous ones). This means that the optimal strategy can neglect if a given candidate is better than all the preceding ones and focus only on whether the candidate is the second better than all the preceding ones.

Note that Theorem 1 implies that, when the number of objects is known, both problems share the same optimal threshold strategy. Moreover, under this strategy the probabilities of success in both problems are closely related (one is twice the other). We will now see that, as long as we follow a threshold strategy, this relationship still holds even if the number of objects is unknown. To do so, we first need two easy results.

Proposition 1 *If n is the total number of objects, let $A_n^{BW}(r)$ and $A_n^{PD}(r)$ denote the probability of success if we accept a nice candidate at the r-th step in the Best-or-Worst and in the Postdoc problem, respectively. Then,*

$$A_n^{BW}(r) = \frac{r}{n}$$
$$A_n^{PD}(r) = \frac{r(r-1)}{n(n-1)}.$$

Recall that a threshold strategy with cutoff value r consists of rejecting the first r candidates and then accepting the first nice candidate after that. The following result is a direct consequence of the previous proposition.

Proposition 2 *If $n > 1$ is the total number of objects, let $R_n^{BW}(r)$ and $R_n^{PD}(r)$ denote the probability of success following a threshold strategy with cutoff value r in the Best-or-Worst and in the Postdoc problem, respectively. Then,*

$$2R_n^{PD}(r) = R_n^{BW}(r) = \begin{cases} 2/n, & \text{if } r = 0; \\ \dfrac{2r(n-r)}{n(n-1)}, & \text{if } n \geq r; \\ 0, & \text{if } n < r. \end{cases}$$

Now, we can extend part of Theorem 1 to the case when the number of objects is unknown.

Corollary 1 *If X is the random variable defining the number of objects, let $P_X^{BW}(r)$ and $P_X^{PD}(r)$ denote the probability of success following a threshold strategy with cutoff value $r > 1$ in the Best-or-Worst and in the Postdoc problem, respectively. Then,*

$$P_X^{BW}(r) = 2P_X^{PD}(r).$$

Proof Taking into account the previous proposition, it is enough to observe that

$$P_X^{BW}(r) = \sum_{i \geq r+1} R_i^{BW}(r) \cdot p(X=i) = \sum_{i \geq r+1} 2R_i^{PD}(r) \cdot p(X=i) = 2P_X^{PD}(r).$$

□

This corollary will be important in the sequel because, once we show that the optimal strategy is a threshold strategy and regardless the distribution followed by the unknown number of objects, both problems will share the same optimal cutoff value and the probability of success in the Best-or-Worst problem will be twice the probability of success in the Postdoc problem. Consequently, we will be able to focus on just one of them, namely the Best-or-Worst problem.

3 Threshold strategies for a random number of objects

As we have already mentioned, when the number of objects is known, the optimal strategy is a threshold strategy. Unfortunately, this is not necessarily the case if the number of objects is random. For example, let us assume that in the classical secretary problem the number of objects is a discrete random variable X such that $p(X = 100) = 0.99$ and $p(X = 1000) = 0.01$. Clearly, in such a situation the optimal strategy is not a threshold strategy. In fact, if at the 100-th step we inspect an object which is better than all the preceding ones, it must be accepted and the probability of success is greater than 0.99. However, at the 101-th step we should reject an object even if it is better than all the preceding ones because accepting it would be equivalent to accepting it if the number of objects was equal to 1000 and, as we know, that would not be optimal.

In this section we will prove that if the random number of objects X satisfies certain properties, then the optimal strategy is still a threshold strategy. Moreover, we will see that the required properties are fulfilled in the case of a Uniform distribution $\mathcal{U}[1, n]$ as well as in the case of a Poisson distribution $\mathcal{P}(\lambda)$. We will address the Best-or-Worst and the Postdoc problems separately but all the hard work will be done in the former case.

3.1 The Best-or-Worst problem

Recall that in the Best-or-Worst problem, a *nice candidate* is an object which is either better or worse than all the preceding ones.

Definition 1 If the random number of objects is a discrete random variable X, let us define the following probabilities.

- $P_A^X(r)$ is the probability of success if we accept a nice candidate at the r-th step. Note that if $X = k$ and we accept a nice candidate at the r-th step, the probability of success is r/k. Thus,

$$P_A^X(r) = \mathbb{E}\left(\frac{r}{X}\bigg| X \geq r\right) = \frac{\sum_{k=r}^{\infty} \frac{r}{k} p(X = k)}{\sum_{k=r}^{\infty} p(X = k)}.$$

- $P_R^X(r)$ is the probability of success if we reject an object (regardless it is nice or not) at the r-th step in order to accept the next nice candidate to be found. If $X = k$ and we reject an object at the r-th step in order to accept the next nice candidate to be found, the probability of success is $\dfrac{2r(k-r)}{k(k-1)}$ (see Proposition 2). Thus,

$$P_R^X(r) = \mathbb{E}\left(\frac{2r(X-r)}{X(X-1)}\bigg| X \geq r\right) = \frac{\sum_{k=r}^{\infty} \frac{2r(k-r)}{k(k-1)} p(X=k)}{\sum_{k=r}^{\infty} p(X=k)}.$$

- $\widetilde{P}_R^X(r)$ is the probability of success if we reject an object at the r-th step in order to adopt the optimal strategy later on. If we consider $q_r = p(X > r | X \geq r)$ it is easy to see that

$$\widetilde{P}_R^X(r) = q_r \frac{2}{r+1} \max\left\{P_A^X(r+1), \widetilde{P}_R^X(r+1)\right\} + q_r \frac{r-1}{r+1} \widetilde{P}_R^X(r+1).$$

In this setting and in terms of dynamic programming, the following strategy is obviously optimal at the r-th step:

- If the r-th object is not a nice candidate, then reject it.
- If the r-th object is a nice candidate but $P_A^X(r) < \widetilde{P}_R^X(r)$, then reject it.
- If the r-th object is a nice candidate and $P_A^X(r) \geq \widetilde{P}_R^X(r)$, then accept it.

Observe that, from the very definition it is clear that $P_R^X(r) \leq \widetilde{P}_R^X(r)$ for every r. It is also clear that, if the range of X is infinite, then the probability of success rejecting an object at the r-th step is strictly positive for every r; i.e., $\widetilde{P}_R^X(r) > 0$. The following result will allow us to work with $P_R^X(r)$ rather than with the more complex $\widetilde{P}_R^X(r)$.

Lemma 1 *Let X be a non-negative discrete random variable such that, either its range is finite or $\liminf P_A^X(r) > 1/2$. Assume that there exists r_0 such that $P_A^X(r) \geq P_R^X(r)$ for every $r > r_0$. Then, $P_A^X(r) \geq \widetilde{P}_R^X(r)$ for every $r > r_0$.*

Proof Given r_0, let us consider the set $S = \{r > r_0 : P_A^X(r) < \widetilde{P}_R^X(r)\}$. We claim that S is bounded. If the range of X is finite, this is trivially the case. If, on the other hand, the range of X is infinite, $\liminf P_A^X(r) > 1/2$ implies that there exists $\widetilde{r}$ such that $P_A^X(r) > 1/2$ for every $r > \widetilde{r}$. Since $P_A^X(r) + \widetilde{P}_R^X(r) \leq 1$, this implies that $P_A^X(r) > \widetilde{P}_R^X(r)$ for every $r > \widetilde{r}$ and hence S is bounded (by $\widetilde{r}$).

If $S = \emptyset$ the result follows so let us assume that S is nonempty and let r' be its maximum. This means that $\widetilde{P}_R^X(r') > P_R^X(r')$ while $P_A^X(r'+1) \geq \widetilde{P}_R^X(r'+1)$ but this is a contradiction.

This is because, if the probability of success rejecting an object at the r'-th step is bigger than the probability of success rejecting it in order to accept the next nice candidate; i.e., if $\widetilde{P}_R^X(r') > P_R^X(r')$, then accepting a nice candidate at the next step cannot be optimal; i.e., it is not possible that $P_A^X(r'+1) \geq \widetilde{P}_R^X(r'+1)$. □

Now, we can prove the following general result which shows that, under certain conditions, the optimal strategy is a threshold strategy.

Theorem 2 *In the Best-or-Worst problem, let the number of objects X be a non-negative discrete random variable such that, either its range is finite or $\liminf P_A^X(r) > 1/2$. Furthermore, assume that*

$$P_A^X(r) \geq P_R^X(r) \Rightarrow P_A^X(r+1) \geq P_R^X(r+1).$$

Then, there exists r_0 such that the following strategy is optimal:

(1) *Reject the r_0 first inspected objects.*

(2) *After that, accept the first nice candidate which is inspected.*

Proof Just consider $r_0 = \max\{r : P_A^X(r) < P_R^X(r)\}$ and apply the previous lemma. $\square$

The remaining of the section will be devoted to see that we can apply Theorem 2 either if the random number of objects follows a Uniform distribution $X \sim \mathcal{U}[1, n]$ or a Poisson distribution $X \sim \mathcal{P}(\lambda)$. In particular, we will see that in both situations the conditions from Theorem 2 holds.

The following lemma is devoted to explicitly compute $P_A^{\mathcal{U}[1,n]}(r)$ and $P_R^{\mathcal{U}[1,n]}(r)$, which are defined as in Definition 1 but for the particular case of $X \sim \mathcal{U}[1, n]$.

Lemma 2 *Let ψ denote the digamma function. Then,*

(i) $P_A^{\mathcal{U}[1,n]}(r) = \dfrac{r\,(\psi(n+1) - \psi(r))}{n+1-r},$

(ii) $P_R^{\mathcal{U}[1,n]}(r) = \dfrac{2r\,(r - n + n\psi(n) - n\psi(r))}{n(n+1-r)}.$

Proof Let $X \sim \mathcal{U}[1, n]$ be the random variable defining the number of objects. If $X = k$ (i.e., if there are k objects) and we accept a nice candidate at the r-th step, the probability of success is r/k. Thus, taking into account that

$$p(X = k | X \geq r) = 1/(n+1-r),$$

we have that

$$P_A^{\mathcal{U}[1,n]}(r) = \sum_{k=r}^{n} \frac{r}{k(n+1-r)} = \frac{r\,(\psi(n+1) - \psi(r))}{n+1-r}$$

because, for any positive integer m it holds that $\psi(m) = \sum_{k=1}^{m-1} \frac{1}{k} - \gamma$, ($\gamma$ being the Euler-Mascheroni constant).

On the other hand, if $X = k$ and we reject a nice candidate at the r-th step in order to accept the next nice candidate to be found, the probability of success is $P_k(r) = \dfrac{2r(k-r)}{k(k-1)}$. Hence,

$$P_R^{\mathcal{U}[1,n]}(r) := \sum_{k=r+1}^{n} P_k(r)\frac{1}{n+1-r} = \frac{2r\,(r - n + n\psi(n) - n\psi(r))}{n(n+1-r)}$$

using again the definition of the digamma function. $\square$

Once we have computed the values of $P_A^{\mathcal{U}[1,n]}(r)$ and $P_R^{\mathcal{U}[1,n]}(r)$ we can prove the following result that guarantees that we can apply Theorem 2 in the Uniform case.

Proposition 3 *Let $n \in \mathbb{N}$ and $r \in [1, n)$. Then,*

$$P_A^{\mathcal{U}[1,n]}(r) > P_R^{\mathcal{U}[1,n]}(r) \Rightarrow P_A^{\mathcal{U}[1,n]}(r+1) > P_R^{\mathcal{U}[1,n]}(r+1).$$

Proof It is easy to see that, for every $n/2 < r < n$ it holds that $P_A^{\mathcal{U}[1,n]}(r) > P_R^{\mathcal{U}[1,n]}(r)$. Hence, we can restrict ourselves to the case $1 < r \leq n/2$.

Let us assume that $P_A^{\mathcal{U}[1,n]}(r) > P_R^{\mathcal{U}[1,n]}(r)$ and that $P_A^{\mathcal{U}[1,n]}(r+1) \leq P_R^{\mathcal{U}[1,n]}(r+1)$. Using Lemma 2, we have that

$$P_R^{\mathcal{U}[1,n]}(r+1) - P_A^{\mathcal{U}[1,n]}(r+1) = \frac{(r+1)\big((r-n)(2r+1)+nr(\psi(n)-\psi(r))\big)}{n(n-r)r} \geq 0$$

Consequently, we have that

$$A := \frac{(r-n)(2r+1)}{nr} + \psi(n) - \psi(r) \geq 0.$$

On the other hand, using Lemma 2 again we have that

$$P_A^{\mathcal{U}[1,n]}(r) - P_R^{\mathcal{U}[1,n]}(r) = \frac{r\big(1+2n-2r-n\psi(n)+n\psi(r)\big)}{n(n+1-r)} > 0$$

and, consequently, that

$$B := \frac{1+2n-2r}{n} - \psi(n) + \psi(r) > 0.$$

Now, since $A, B > 0$, it follows that

$$0 < A + B = \frac{1+2n-2r}{n} + \frac{(r-n)(2r+1)}{nr} = \frac{2r-n}{nr}.$$

Since this implies that $r > n/2$ we have reached a contradiction and the result follows.
□

Now, we turn to the Poisson case. The following lemma is devoted to explicitly compute $P_A^{\mathcal{P}(\lambda)}(r)$ and $P_R^{\mathcal{P}(\lambda)}(r)$, which are defined as in Definition 1 but for the particular case of $X \sim \mathcal{P}(\lambda)$.

Lemma 3 *For any $\lambda > 0$ let us define*

$$\Psi(r,\lambda) := \sum_{k=r}^{\infty} \frac{\lambda^k}{e^\lambda k!}.$$

Then,

(i) $P_A^{\mathcal{P}(\lambda)}(r) = \dfrac{1}{\Psi(r,\lambda)} \displaystyle\sum_{k=r}^{\infty} \dfrac{r}{k} \dfrac{\lambda^k}{e^\lambda k!}$,

(ii) $P_R^{\mathcal{P}(\lambda)}(r) = \dfrac{1}{\Psi(r,\lambda)} \displaystyle\sum_{k=r}^{\infty} \dfrac{2(k-r)r}{(k-1)k} \dfrac{\lambda^k}{e^\lambda k!}$.

Proof Let $X \sim \mathcal{P}(\lambda)$ be the random variable defining the number of objects. Then,

$$p(X = k | X \geq r) = \frac{p(X = k)}{p(X \geq r)} = \frac{\frac{\lambda^k}{e^\lambda k!}}{\Psi(r, \lambda)}$$

and the result follows.   □

Now that we have explicit expressions for $P_A^{\mathcal{P}(\lambda)}(r)$ and $P_R^{\mathcal{P}(\lambda)}(r)$, the following results show that the conditions of Theorem 2 also hold in the Poisson case under consideration.

Proposition 4 *For any $\lambda > 0$ it holds that*

$$\lim_{r \to \infty} \frac{1}{\Psi(r, \lambda)} \sum_{k=r}^{\infty} \frac{r}{k} \frac{\lambda^k}{e^\lambda k!} = 1.$$

Proof Let us denote $S(r, \lambda) = \sum_{k=r}^{\infty} \frac{r}{k} \frac{\lambda^k}{e^\lambda k!}$. Then it is enough to apply the Stolz-Cesàro theorem taking into account that:

$$\Psi(r + 1, \lambda) - \Psi(r, \lambda) = -\frac{\lambda^r}{e^\lambda r!},$$

$$S(r + 1, \lambda) - S(r, \lambda) = \sum_{k=r+1}^{\infty} \frac{\lambda^k}{k e^\lambda k!} - \frac{\lambda^r}{e^\lambda r!},$$

□

Proposition 5 *Let $\lambda > 0$. Then*

$$P_A^{\mathcal{P}(\lambda)}(r) > P_R^{\mathcal{P}(\lambda)}(r) \Rightarrow P_A^{\mathcal{P}(\lambda)}(r + 1) > P_R^{\mathcal{P}(\lambda)}(r + 1).$$

Proof As in Proposition 4, using the Stolz-Cesàro theorem, it can be easily proved that $\lim_{r \to \infty} P_A^{\mathcal{P}(\lambda)}(r) = 1$ and that $\lim_{r \to \infty} P_R^{\mathcal{P}(\lambda)}(r) = 0$. In this situation, the statement is equivalent to prove that there exists at most one integer $r_0 \geq 1$ such that $P_A^{\mathcal{P}(\lambda)}(r_0 - 1) \leq P_R^{\mathcal{P}(\lambda)}(r_0 - 1)$ and $P_A^{\mathcal{P}(\lambda)}(r_0) > P_R^{\mathcal{P}(\lambda)}(r_0)$. In other words, either $P_A^{\mathcal{P}(\lambda)}(r)$ is always greater than $P_R^{\mathcal{P}(\lambda)}(r)$ or they "intersect" just once.

If, for $r \geq 1$, we define

$$f(r) := \sum_{k=r}^{2r-1} \frac{2r - k - 1}{k(k - 1)} \frac{\lambda^k}{k!}, \quad g(r) := \sum_{k=2r}^{\infty} \frac{k - 2r + 1}{k(k - 1)} \frac{\lambda^k}{k!}$$

it is straightforward to see that $P_A^{\mathcal{P}(\lambda)}(r) - P_R^{\mathcal{P}(\lambda)}(r) = \dfrac{re^{-\lambda}}{\Psi(r,\lambda)} (f(r) - g(r))$. Thus, we will see that $f(r)$ and $g(r)$ "intersect" at most once.

Note that, since $P_A^{\mathcal{P}(\lambda)}(r)$ is ultimately bigger than $P_R^{\mathcal{P}(\lambda)}(r)$, then $f(r)$ is ultimately bigger than $g(r)$.

Now,

$$G_1(r) := g(r+1) - g(r) = \sum_{k=2r+1}^{\infty} \frac{-2}{k(k-1)} \frac{\lambda^k}{k!} - \frac{1}{2r(2r-1)} \frac{\lambda^{2r}}{(2r)!} < 0,$$

$$G_2(r) := G_1(r+1) - G_1(r)$$
$$= \frac{1}{2r(2r-1)} \frac{\lambda^{2r}}{(2r)!} + \frac{2}{2r(2r+1)} \frac{\lambda^{2r+1}}{(2r+1)!} + \frac{1}{(2r+1)(2r+2)} \frac{\lambda^{2r+2}}{(2r+2)!} > 0.$$

This means that $g(r)$ strictly decreases (to 0) and that it is "convex".

On the other hand,

$$F_1(r) := f(r+1) - f(r) = \sum_{k=r+1}^{2r-1} \frac{2}{k(k-1)} \frac{\lambda^k}{k!} + \frac{1}{2r(2r-1)} \frac{\lambda^{2r}}{(2r)!} - \frac{1}{r} \frac{\lambda^r}{r!}.$$

This implies that F_1 changes sign at most once. Since $F_1(r)$ is clearly negative for big values of r, this means that $f(r)$ is either strictly decreasing or it first increases and then decreases with only one change in its monotony. Furthermore,

$$F_2(r) := F_1(r+1) - F_1(r) = G_2(r) + \frac{\lambda^r}{r(r+1)^2 r!} \phi(r),$$

with $\phi(r) = (r^2 + (2-\lambda)r + (1-2\lambda))$. This implies that $F_2(r) > G_2(r)$ for every $r > \tilde{r}$, with $\tilde{r}$ the biggest root of $\phi(r) = 0$. In other words, for $r > \tilde{r}$, $f(r)$ is "more convex" than $g(r)$.

Finally, assume that $f(r)$ and $g(r)$ intersect at some point in which both of them are decreasing. Since $f(r)$ must be ultimately bigger that $g(r)$, this contradicts either the fact that $f(r)$ only has at most one change in monotony or the fact that $F_2(r) > G_2(r)$ for some moment on. This means that either $f(r)$ and $g(r)$ do not intersect at all or that they do so only once, which is what we wanted to prove. □

Remark We have just seen that $P_A^{\mathcal{P}(\lambda)}(r)$ and $P_R^{\mathcal{P}(\lambda)}(r)$ intersect at most once. Since $P_A^{\mathcal{P}(\lambda)}(r)$ increases monotonically to 1 while $P_R^{\mathcal{P}(\lambda)}(r)$ tends to 0, the intersection of both functions depends only on the relationship between $P_A^{\mathcal{P}(\lambda)}(1)$ and $P_R^{\mathcal{P}(\lambda)}(1)$. Namely, $P_A^{\mathcal{P}(\lambda)}(r) > P_R^{\mathcal{P}(\lambda)}(r)$ for every r if and only if $P_A^{\mathcal{P}(\lambda)}(1) > P_R^{\mathcal{P}(\lambda)}(1)$. Solving the equation $P_A^{\mathcal{P}(\lambda)}(1) = P_R^{\mathcal{P}(\lambda)}(1)$ leads to the approximate value of $\lambda_0 = 2.2197719\ldots$ which means that, for every $\lambda < \lambda_0$ it holds that the functions $P_A^{\mathcal{P}(\lambda)}(r)$ and $P_R^{\mathcal{P}(\lambda)}(r)$ never intersect and for every $\lambda \geq \lambda_0$ they intersect exactly once.

Finally, we can give the main result of this section to establish that the optimal strategy in both considered situations is a threshold strategy.

Theorem 3 *In the Best-or-Worst problem let the number of objects follow a Uniform distribution $\mathcal{U}[1, n]$ (resp. a Poisson distribution $\mathcal{P}(\lambda)$). Then, there exists $r(n)$ (resp. $r(\lambda)$) such that the following strategy is optimal:*

(1) *Reject the $r(n)$ (resp. $r(\lambda)$) first inspected objects.*
(2) *After that, accept the first nice candidate which is inspected.*

We will refer to the value $r(n)$ (resp. $r(\lambda)$) as the optimal cutoff value.

Proof It follows from Theorem 2. If $X \sim \mathcal{U}[1, n]$ it is enough to apply Proposition 3 because the range of X is finite. On the other hand, if $X \sim \mathcal{P}(\lambda)$, we have to apply both Propositions 4 and 5. □

Remark The proof of Theorem 3 can also be approached in terms of Markov chains as it was done in Presman and Sonin (1972) for the classical Secretary problem. In fact, in order to prove that the optimal strategy is a threshold strategy, the only relevant factor is the function that determines the probability of success if we accept a nice candidate at the r-th step with a known number of objects k. As it turns out, this function is the same in the classical problem as in the Best-or-Worst problem. Thus, the proof would go just as in the aforementioned paper Presman and Sonin (1972). However, we decide to provide full explicit proofs, avoiding Markov chains, to keep the paper self-contained and elementary in nature.

3.2 The Postdoc problem

Now we turn to the Postdoc problem. In this setting, a *nice candidate* is an object which is the second better than all the preceding ones. First of all, we have the following analogue to Definition 1.

Definition 2 If the random number of objects is a discrete random variable X, let us define the following probabilities.

- $\mathbf{P}_A^X(r)$ is the probability of success if we accept a nice candidate at the r-th step. Note that if $X = k$ and we accept a nice candidate at the r-th step, the probability of success is r/k. Thus,

$$\mathbf{P}_A^X(r) = \mathbb{E}\left(\frac{r(r-1)}{X(X-1)} \middle| X \geq r \right) = \frac{\sum_{k=r}^{\infty} \frac{r(r-1)}{k(k-1)} p(X = k)}{\sum_{k=r}^{\infty} p(X = k)}.$$

- $\mathbf{P}_R^X(r)$ is the probability of success if we reject an object (regardless it is nice or not) at the r-th step in order to accept the next nice candidate to be found. If $X = k$ and we reject an object at the r-th step in order to accept the next nice candidate to be found, the probability of success is $\frac{r(k-r)}{k(k-1)}$ (see Proposition 2). Thus,

$$\mathbf{P}_R^X(r) = \mathbb{E}\left(\frac{r(X-r)}{X(X-1)} \middle| X \geq r \right) = \frac{\sum_{k=r}^{\infty} \frac{r(k-r)}{k(k-1)} p(X = k)}{\sum_{k=r}^{\infty} p(X = k)}.$$

- $\widetilde{\mathbf{P}}_R^X(r)$ is the probability of success if we reject an object at the r-th step in order to adopt the optimal strategy later on. If we consider $q_r = p(X > r | X \geq r)$ it is easy to see that

$$\widetilde{\mathbf{P}}_R^X(r) = \frac{q_r}{r+1} \max\left\{\mathbf{P}_A^X(r+1), \widetilde{\mathbf{P}}_R^X(r+1)\right\} + \frac{rq_r}{r+1}\widetilde{\mathbf{P}}_R^X(r+1).$$

After these definitions, we could state and prove the direct analogues to Lemma 1 and to Theorem 2. On the other hand, note that it is straightforward to check that

$$\mathbf{P}_A^X(r) - \mathbf{P}_R^X(r) = P_A^X(r) - P_R^X(r).$$

Consequently, the corresponding analogues to Propositions 3 and 5 also hold in this setting. Finally, as in Proposition 4, it can be easily proved using the Stolz-Cesàro theorem that

$$\lim_{r \to \infty} \mathbf{P}_A^{\mathcal{P}(\lambda)}(r) = 1.$$

This being said, it follows that the following analogue to Theorem 3 also holds, showing that in the Postdoc problem the optimal strategy is still a threshold strategy.

Theorem 4 *In the Postdoc problem let the number of objects follow a Uniform distribution $\mathcal{U}[1, n]$ (resp. a Poisson distribution $\mathcal{P}(\lambda)$). Then, there exists $r(n)$ (resp. $r(\lambda)$) such that the following strategy is optimal:*

(1) *Reject the $r(n)$ (resp. $r(\lambda)$) first inspected objects.*
(2) *After that, accept the first nice candidate which is inspected.*

We will refer to the value $r(n)$ (resp. $r(\lambda)$) as the optimal cutoff value.

Now that we have seen than both in the Best-or-Worst problem and in the Postdoc problem the optimal strategies are threshold strategies, we are in the conditions to apply Corollary 1. Thus, we can focus just on one of the problems (we choose the Best-or-Worst problem). The forthcoming sections will be devoted to study the optimal cutoff values as well as the associated probabilities of success for each of the considered distributions.

4 The Best-or-Worst problem when the number of objects follows a uniform distribution $\mathcal{U}[1, n]$

Taking into account Proposition 2, if there were $k > 1$ objects, the probability of success using a threshold strategy with cutoff value $1 \leq r < k$ would be $P_k(r) = \frac{2r(k-r)}{k(k-1)}$. On the other hand, if the random variable defining number of objects follows a discrete Uniform distribution $X \sim \mathcal{U}[1, n]$, we have that $p(X = k) = 1/n$ for every k. Hence, the probability of success is given in this situation by the function

$$F^{\mathcal{U}}(r, n) = \sum_{k=r+1}^{n} P_k(r)p(X = k) = \sum_{k=r+1}^{n} \frac{P_k(r)}{n} = \sum_{k=r+1}^{n} \frac{2r(k-r)}{k(k-1)n}.$$

Now, let us denote by $\mathcal{M}(n) \in [1, n]$ the optimal cutoff value; i.e., the value for which the function $F^{\mathcal{U}}(\cdot, n)$ reaches its maximum. Also, let us denote by $P(n) := F^{\mathcal{U}}(\mathcal{M}(n), n)$. Thus, $P(n)$ denotes the probability of success in the Best-or-Worst problem when the number of objects follows a discrete Uniform distribution $\mathcal{U}[1, n]$ using the optimal threshold strategy.

Remark With the previous notation, it is straightforward to see that $\mathcal{M}(1) = \mathcal{M}(2) = 0$ and also that $P(1) = P(2) = 1$. This corresponds to the fact that, in the Best-or-Worst problem, if there is only one or two objects, we will always succeed if we accept the first one.

In order to study the behavior of $\mathcal{M}(n)$ and $P(n)$ we shall first prove that, with the only exception of the previous remark, P is strictly decreasing. To do so we first prove an adaptation of the strategy-stealing argument in the following lemma.

Lemma 4 *For every pair of integers (r, n) with $1 < r < n$, one of the following identities holds:*

$$F^{\mathcal{U}}(r, n + 1) < F^{\mathcal{U}}(r - 1, n),$$
$$F^{\mathcal{U}}(r, n + 1) < F^{\mathcal{U}}(r, n).$$

Proof Let us denote $H(n, r) = \sum_{k=r}^{n-1} \frac{1}{i}$. Then,

$$F^{\mathcal{U}}(r, n) = \sum_{k=r+1}^{n} \frac{2k(k-r)}{k(k-1)n} = \frac{2r}{n} \sum_{k=r+1}^{n} \left(\frac{r}{k} + \frac{1}{k-1} + \frac{r}{k-1} \right)$$
$$= \frac{2r}{n} \left(\left(\frac{r}{n} - 1 \right) + \sum_{k=r+1}^{n} \frac{1}{k-1} \right) = \frac{2r}{n} \left(\left(\frac{r}{n} - 1 \right) + H(n, r) \right).$$

In the same way we have that

$$F^{\mathcal{U}}(r, n + 1) = \frac{2r}{n+1} \left(\left(\frac{r}{n+1} - 1 + \frac{1}{n} \right) + H(n, r) \right),$$
$$F^{\mathcal{U}}(r - 1, n) = \frac{2(r-1)}{n} \left(\left(\frac{r-1}{n} - 1 + \frac{1}{r-1} \right) + H(n, r) \right).$$

And, as a consequence, it follows that

$$F^{\mathcal{U}}(r, n+1) - F^{\mathcal{U}}(r-1, n) = \frac{2(1+n-r)}{n(n+1)} \left(-\frac{(1+2n)(1+n-r)}{n(n+1)} + H(n, r) \right),$$
$$F^{\mathcal{U}}(r, n+1) - F^{\mathcal{U}}(r, n) = \frac{2r}{n(n+1)} \left(\frac{2n(1+n-r) - r}{n(n+1)} - H(n, r) \right)$$

Now, if we assume that both $F^{\mathcal{U}}(r, n+1) - F^{\mathcal{U}}(r-1, n) \geq 0$ and $F^{\mathcal{U}}(r, n+1) - F^{\mathcal{U}}(r, n) \geq 0$, we get that

$$A := H(n, r)n(n+1) - (1+2n)(1+n-r) \geq 0,$$
$$B := -(H(n, r)n\ (n+1)) + 2n\ (1+n-r) - r \geq 0.$$

As a consequence, it follows that $0 \leq A + B = -1 - n < 0$. This is a contradiction and hence the result.   □

Using this lemma, we can now prove that P is decreasing.

Proposition 6 *Let $n > 1$ be any integer. Then, $P(n+1) < P(n)$.*

Proof If $n > 1$,

$$P(n+1) = F^{\mathcal{U}}(\mathcal{M}(n+1), n+1) < \max\{F^{\mathcal{U}}(\mathcal{M}(n+1), n),\ F^{\mathcal{U}}(\mathcal{M}(n+1)-1, n)\}$$
$$\leq F^{\mathcal{U}}(\mathcal{M}(n), n) = P(n),$$

where the first inequality follows from Lemma 4 and the second holds by the definition of $\mathcal{M}(n)$.   □

In order to provide further information about $\mathcal{M}(n)$ and $P(n)$ we first need two technical results. The first one was proved in Bayón (2018, Proposition 1), while the second is just an elementary Calculus exercise. Recall that for every $x > -1/e$, the principal branch of the Lambert W-function is the only real number $W(x) > -1$ such that $x = W(x)e^{W(x)}$.

Lemma 5 *Let $\{F_n\}$ be a sequence of real functions with $F_n \in \mathcal{C}[0, n]$ and let $\mathcal{M}(n)$ be the value for which the function F_n reaches its maximum. Assume that the sequence of functions $\{g_n\}_{n \in \mathbb{N}}$ given by $g_n(x) := F_n(nx)$ converges uniformly on $[0, 1]$ to a function g and that θ is the only global maximum of g in $[0, 1]$. Then,*

  (i) $\lim\limits_{n} \mathcal{M}(n)/n = \theta$.
 (ii) $\lim\limits_{n} F_n(\mathcal{M}(n)) = g(\theta)$.
(iii) *If $\mathfrak{M}(n) \sim \mathcal{M}(n)$ then* $\lim\limits_{n} F_n(\mathfrak{M}(n)) = g(\theta)$.

Lemma 6 *The function $g(x) = -2x \log x - 2x(1-x)$ reaches its absolute maximum in the interval $[0, 1]$ at the point*

$$\vartheta := -\frac{1}{2} W\left(-\frac{2}{e^2}\right) = 0.20318786\ldots,$$

where W denotes Lambert W-function. Moreover, the value of this maximum is:

$$g(\vartheta) = 2(\vartheta - \vartheta^2) = 0.32380511\ldots$$

Now, the following result provides estimations for the values $\mathcal{M}(n)$ and describes the asymptotic behavior of P.

Theorem 5 *With all the previous notation, the following hold:*

(i) *For every positive integer n,*

$$1 = P(1) = P(2) > P(3) > \cdots > P(n) > P(n+1) > \cdots > 2(\vartheta - \vartheta^2).$$

(ii) $\lim_n \dfrac{\mathcal{M}(n)}{n} = \vartheta$*; i.e.,* $\mathcal{M}(n) \sim \vartheta n$.

(iii) $\lim_n F^{\mathcal{U}}(\lfloor \vartheta \cdot n \rfloor, n) = \lim_{n \to \infty} P(n) = 2(\vartheta - \vartheta^2)$.

Proof As in the proof of Lemma 4, we have that

$$F^{\mathcal{U}}(r, n) = \frac{2r}{n}\left(\left(\frac{r}{n} - 1\right) + \sum_{k=r}^{n-1}\frac{1}{k}\right)$$

so, if we recall the definition of the digamma function $\psi(n) := \sum_{k=1}^{n-1}\frac{1}{k} - \gamma$, we get that

$$F^{\mathcal{U}}(r, n) = \frac{2r\,(r - n + n\psi(n) - n\psi(r))}{n^2}$$

Now, if we define $g_n(x) := F^{\mathcal{U}}(nx, n)$ we have that the sequence of functions $\{g_n\}$ converges uniformly to the function $g(x) := 2x\,(-1 + x - \log(x))$.

Since $P(n) = g_n\left(\frac{\mathcal{M}(n)}{n}\right)$ it is enough to apply Proposition 6 and Lemma 6 to obtain point (i). Points (ii) and (iii) readily follow from Lemmas 5 and 6 and the proof is complete. $\qquad\square$

Theorem 5 shows that $[\vartheta \cdot n]$ (the nearest integer to $\vartheta \cdot n$) constitutes a practical estimation of the optimal cutoff value in the optimal threshold strategy and that, following this strategy, the probability of success is greater than $2(\vartheta - \vartheta^2) = 0.3238\ldots$. The first few values of n for which the estimation $[n\vartheta]$ fails are

8, 13, 18, 23, 32, 37, 42, 47, 52, 57, 62, 67, 72, 77, 82, 96, 101, 106, 111, 116, 121, . . .

Even if the estimation fails about 20% of times, $[n\vartheta]$ differs at most 1 from the actual optimal cutoff value and the error is negligible if compared with the probability of success of the optimal strategy for large values of n.

We are now interested in finding a better estimate for $\mathcal{M}(n)$. As usual, we first need to introduce a technical result.

Lemma 7 *Let us consider the function* $f(r, n) := -\frac{2r}{n} + \frac{2r^2}{n^2} + \frac{2r}{n}\psi(n) - \frac{2r\log(r)}{n} + \frac{1}{n}$ *and let* $\alpha(n) \in [1, n]$ *be the value for which the function* $f(\cdot, n)$ *reaches its maximum. Then,* $\alpha(n) \approx \mathcal{M}(n)$.

Proof As we saw int he proof of Theorem 5,

$$F^{\mathcal{U}}(r,n) = -\frac{2r}{n} + \frac{2r^2}{n^2} + \frac{2r}{n}\psi(n) - \frac{2r}{n}\psi(r).$$

On the other hand, for any integer r it holds that $\psi(r) = \log r - \frac{1}{2r} + \epsilon(r)$, with $\epsilon(r) = O(1/2r)$ if $r \to \infty$. Thus,

$$F^{\mathcal{U}}(r,n) = -\frac{2r}{n} + \frac{2r^2}{n^2} + \frac{2r}{n}\psi(n) - \frac{2r\log(r)}{n} + \frac{1}{n} - \frac{2r\epsilon(r)}{n} = f(r,n) - \frac{2r\epsilon(r)}{n}.$$

Since $r \geq 1$, it follows that there exists k such that $|2r\epsilon(r)| \leq k$ for every r and hence:

$$g_1(r,n) = f(r,n) - \frac{k}{n} \leq P(r,n) \leq f(r,n) + \frac{k}{n} = g_2(r,n).$$

Obviously both functions $g_1(r,n)$ and $g_2(r,n)$ reach their maximum at $\alpha(n)$. In this situation, there exist points $\beta_1(n)$ and $\beta_2(n)$ such that $g_2(\beta_1(n),n) = g_2(\beta_2(n),n) = g_1(\alpha(n),n)$ and $\alpha(n) \in (\beta_1(n), \beta_2(n))$ and the inequality above implies that

$$\beta_1(n) \leq \mathcal{M}(n) \leq \beta_2(n).$$

Finally, for every r we have that $g_2(r,n) - g_1(r,n) = 2k/n$ so it follows that $|\beta_2(n) - \beta_1(n))| \xrightarrow[n \to \infty]{} 0$ and, consequently also $|\alpha(n) - \mathcal{M}(n)| \xrightarrow[n \to \infty]{} 0$ as we wanted to prove. $\qquad\square$

As a consequence of the previous result, if we compute the value of $\alpha(n)$ we can give another estimation for $\mathcal{M}(n)$. In fact, we have the following result.

Theorem 6 *With all the previous notation, the following hold:*

(i) $\mathcal{M}(n) \approx -\dfrac{n}{2}W\left(-\dfrac{2e^{-2+\psi(n)}}{n}\right),$

(ii) $\mathcal{M}(n) \approx n\vartheta + \dfrac{1}{4 - 2e^{2-2\vartheta}}.$

Proof (i) Since $f(\cdot,n)$ (see Lemma 7) reaches its maximum at $\alpha(n)$, it follows that

$$0 = \frac{\partial f}{\partial r}(\alpha(n),n) = -\frac{4}{n} + \frac{4\alpha(n)}{n^2} + \frac{2}{n}\psi(n) - \frac{2\log(\alpha(n))}{n}.$$

From this, and taking into account the definition of Lambert W-function it follows that

$$\alpha(n) = -\frac{n}{2}W\left(-\frac{2e^{-2+\psi(n)}}{n}\right)$$

so it is enough to apply Lemma 7.

(ii) Using (i), we have that

$$\lim_n \left(\mathcal{M}(n) - n\vartheta\right) = \lim_n \left(-\frac{n}{2} W(-\frac{2e^{-2+\psi(n)}}{n}) - n\vartheta\right) = \frac{1}{4 - 2e^{2-2\vartheta}}.$$

Thus,

$$\mathcal{M}(n) \approx n\vartheta + \frac{1}{4 - 2e^{2-2\vartheta}}$$

as claimed.

$\square$

As far as our computing capabilities let us check, the estimation $\mathcal{M}(n) \approx \left(n\vartheta + \frac{1}{4-2e^{2-2\vartheta}}\right)$ fails for very few values of n (only four cases have been found: 2, 3, 23 and 2971). On the other hand, we have not found any value of $n > 4$ for which the estimation $\mathcal{M}(n) \approx \left[-\frac{n}{2} W\left(-\frac{2e^{-2+\psi(n)}}{n}\right)\right]$ fails.

5 The Best-or-Worst problem when the number of objects follows a Poisson distribution $\mathcal{P}(\lambda)$

Now, let us assume that the random variable defining number of objects follows a Poisson distribution $X \sim \mathcal{P}(\lambda)$. Hence, $p(X = k) = \frac{\lambda^k}{e^\lambda k!}$ and reasoning like at the beginning of the previous section, we conclude that the probability of success using a threshold strategy with cutoff value $1 \le r < k$ is given in this situation by the function

$$F^{\mathcal{P}}(r, \lambda) = \sum_{k=r+1}^{n} P_k(r) p(X = k) = \sum_{k=r+1}^{\infty} \frac{2r(k-r)}{k(k-1)} \frac{\lambda^k}{e^\lambda k!}.$$

and

$$F^{\mathcal{P}}(0, \lambda) = \frac{\lambda}{e^\lambda} + \frac{\lambda^2}{2e^\lambda} + \sum_{k=3}^{\infty} \frac{2}{k} \frac{\lambda^k}{e^\lambda k!}$$

Also, following the same notation as in the previous section, let us denote by $\mathcal{M}(\lambda)$ the value for which $F^{\mathcal{P}}(\cdot, \lambda)$ reaches its maximum value and let $P(\lambda) = F^{\mathcal{P}}(\mathcal{M}(\lambda), \lambda)$; i.e., $P(\lambda)$ denotes the probability of success in the Best-or-Worst problem when the number of objects follows a Poisson distribution $\mathcal{P}(\lambda)$ using the optimal threshold strategy.

Remark The value λ_m for which the probability of success $P(\lambda)$ reaches its maximum can be explicitly computed solving the equation

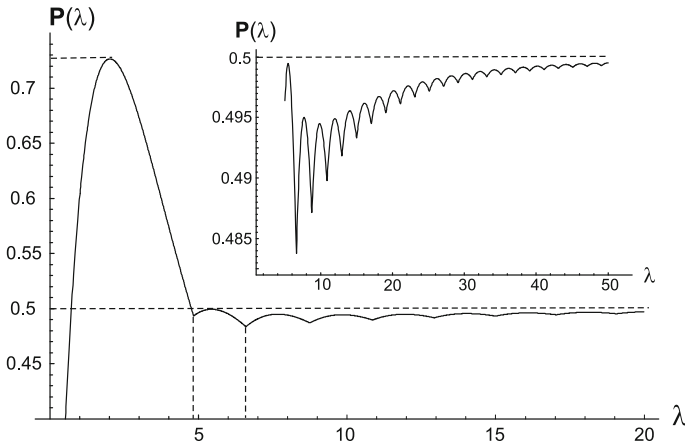$$-2 + 2e^x - x + 2\gamma x + x^2 + 2x\Gamma(-x) + 2x\log(-x) = 0,$$

Fig. 1 Graph of the function $P(\lambda)$

which leads to an approximate value of $\lambda_m = 2.01771\ldots$ and $P(\lambda_m) = 0.72647\ldots$ as can be seen in Fig. 1.

As we can see in Fig. 1, the graph of $P(\lambda)$ consists of a sequence of concave arcs. The rest of the section will be devoted to provide an estimation for $\mathcal{M}(\lambda)$ and to study the asymptotic behavior of $P(\lambda)$. In particular, we will see that, as suggested by Fig. 1, $\lim_{\lambda \to \infty} P(\lambda) = 1/2$. But first we need some technical results.

Lemma 8 *Let us consider the function*

$$f(r, \lambda) := \sum_{k=2}^{r} \frac{2r(k-r)}{k(k-1)} \frac{\lambda^k}{e^{\lambda} k!}.$$

Then, $\lim_{\lambda \to \infty} f(\lambda/2, \lambda) = 0.$

Proof First of all, note that

$$\left| \sum_{k=2}^{\lambda/2} \frac{2(\lambda/2)(k - \lambda/2)}{k(k-1)} \frac{\lambda^k}{e^{\lambda} k!} \right| < \lambda^2 e^{-\lambda} \sum_{k=2}^{\lambda/2} \frac{\lambda^k}{k!} < \frac{1}{\lambda}.$$

Thus, if we define

$$a_n := 2^2 n^2 e^{-2n} \sum_{k=2}^{n} \frac{2^k n^k}{k!},$$

we just need to prove that $\lim_n a_n = 0$.

Now,

$$a_{n+1} = 2^2(n+1)^2 e^{-2(n+1)} \sum_{k=2}^{n+1} \frac{2^k(n+1)^k}{k!}$$

and since

$$\sum_{k=2}^{n+1} \frac{2^k(n+1)^k}{k!} = \sum_{k=2}^{n} \frac{2^k n^k}{k!} + \frac{2^2}{2!}(2n+1) + \frac{2^3}{3!}(3n^2 + 3n + 1) + \cdots$$

$$+ \frac{2^n}{n!}\left(n^n + \frac{n}{1!}n^{n-1} + \frac{n(n-1)}{2!}n^{n-2} + \cdots + \frac{n(n-1)\cdots 2.1}{n!}\right)$$

$$+ \frac{2^{n+1}}{(n+1)!}(n+1)^{n+1},$$

we have that

$$a_{n+1} = 2^2(n+1)^2 e^{-2(n+1)} \sum_{k=2}^{n} \frac{2^k n^k}{k!} + 2^2(n+1)^2 e^{-2(n+1)}\frac{2^2}{2!}(2n+1) + \cdots$$

$$+ 2^2(n+1)^2 e^{-2(n+1)}\frac{2^n}{n!}\left(n^n + \frac{n}{1!}n^{n-1} + \frac{n(n-1)}{2!}n^{n-2} + \cdots + \frac{n(n-1)\cdots 2.1}{n!}\right)$$

$$+ 2^2(n+1)^2 e^{-2(n+1)}\frac{2^{n+1}}{(n+1)!}(n+1)^{n+1}.$$

On the other hand,

$$\sum_{k=2}^{n} \frac{2^k n^k}{k!} = \frac{a_n}{2^2 n^2 e^{-2n}}$$

so

$$a_{n+1} = \left(\frac{n+1}{n}\right)^2 \frac{a_n}{e^2} + 2^2(n+1)^2 e^{-2(n+1)}\frac{2^2}{2!}(2n+1) + \cdots$$

$$+ 2^2(n+1)^2 e^{-2(n+1)}\frac{2^n}{n!}\left(n^n + \frac{n}{1!}n^{n-1} + \frac{n(n-1)}{2!}n^{n-2} + \cdots + \frac{n(n-1)\cdots 2.1}{n!}\right)$$

$$+ 2^2(n+1)^2 e^{-2(n+1)}\frac{2^{n+1}}{(n+1)!}(n+1)^{n+1},$$

which clearly implies that

$$\lim_n \left(a_{n+1} - \left(\frac{n+1}{n}\right)^2 \frac{a_n}{e^2}\right) = 0. \tag{1}$$

Finally, since a_n is decreasing and non-negative, we have that $\lim_n a_{n+1} = \lim_n a_n = l$. Consequently, Eq. 1 implies that $l = l/e^2$; i.e., $l = 0$ as claimed. $\qquad\square$

Theorem 7 *Let us consider the function*

$$f^\star(r, \lambda) := \sum_{k=2}^{\infty} \frac{2r(k-r)}{k(k-1)} \frac{\lambda^k}{e^\lambda k!}.$$

For every $\lambda > 0$, let us denote by $\mathcal{M}^\star(\lambda)$ the value for which $f^\star(\cdot, \lambda)$ reaches its maximum and let $P^\star(\lambda) = f^\star(\mathcal{M}^\star(\lambda), \lambda)$. Then, the following hold:

(i) $\lim_{\lambda \to \infty} \mathcal{M}^\star(\lambda)/\lambda = 1/2$.

(ii) $\lim_{\lambda \to \infty} f^\star(\lambda/2, \lambda) = 1/2$.

(iii) $\mathcal{M}^\star(\lambda) \approx \lambda/2 - 1$.

Proof First of all, we are going to compute the value of $\mathcal{M}^\star(\lambda)$. To do so, let us consider the following functions:

$$S_1(x) = \sum_{k=2}^{\infty} \frac{x^k}{(k-1)k!},$$

$$S_2(x) = \sum_{k=2}^{\infty} \frac{x^k}{k(k-1)k!} = -\sum_{k=2}^{\infty} \frac{x^k}{kk!} + S_1(x),$$

$$S_3(x) = \sum_{k=2}^{\infty} \frac{x^k}{kk!}.$$

Then, we have that

$$S_1''(x) = \frac{e^x - 1}{x}, \quad S_3'(x) = \frac{e^x - 1}{x} - 1$$

and, by integrating these expressions we obtain that

$$S_1(x) = 1 - e^x + x - \gamma x + x \operatorname{E}(x) - x \log x,$$
$$S_3(x) = -\gamma + \operatorname{E}(x) - \log x - x,$$

where

$$\operatorname{E}(\lambda) = \gamma + \log \lambda + \int_0^\lambda \frac{e^x - 1}{x} dx.$$

Also, as a consequence we obtain that

$$S_2(x) = 1 - e^x + 2x + (\gamma - \operatorname{E}(x) + \log x)(1 - x).$$

Now, we observe that

$$f^\star(r, \lambda) = 2re^{-\lambda} \left(\sum_{k=2}^{\infty} \frac{\lambda^k}{(k-1)k!} - r \sum_{k=2}^{\infty} \frac{\lambda^k}{k(k-1)k!} \right)$$
$$= 2re^{-\lambda} S_1(\lambda) - 2r^2 e^{-\lambda} S_2 \lambda).$$

Thus,

$$\frac{\partial f^\star(r, \lambda)}{\partial r} = 2e^{-\lambda} S_1(\lambda) - 4re^{-\lambda} S_2(\lambda)$$

so, if we define

$$r_\lambda := \frac{1 - e^\lambda + \lambda - \gamma\lambda + \lambda \operatorname{E}(\lambda) - \lambda \log \lambda}{2 \left(1 - e^\lambda + 2\lambda + (\gamma - \operatorname{E}(\lambda) + \log \lambda)(1 - \lambda)\right)},$$

it is clear that $\frac{\partial f^\star(r, \lambda)}{\partial r}|_{r=r_\lambda} = 0$ and, consequently, we have just obtained that $\mathcal{M}^\star(\lambda) = r_\lambda$.

Once we have computed the value of $\mathcal{M}^\star(\lambda)$, we are in the condition to prove the three statements of the theorem.

(i) If we recall the definition of $\operatorname{E}(\lambda)$, we have that

$$\lim_{\lambda \to \infty} \frac{\mathcal{M}^\star(\lambda)}{\lambda} = \frac{1}{2} \lim_{\lambda \to \infty} \frac{1 - e^\lambda + \lambda - \gamma\lambda + \lambda \operatorname{E}(\lambda) - \lambda \log \lambda}{\lambda \left(1 - e^\lambda + 2\lambda + (\gamma - \operatorname{E}(\lambda) + \log \lambda)(1 - \lambda)\right)}$$
$$= \frac{1}{2} \lim_{\lambda \to \infty} \frac{1 - e^\lambda + \lambda + \lambda \int_0^\lambda \frac{e^x - 1}{x} dx}{\lambda \left(1 - e^\lambda + 2\lambda + (\lambda - 1) \int_0^\lambda \frac{e^x - 1}{x} dx\right)}$$

so, applying L'Hôpital's rule repeatedly we obtain:

$$\lim_{\lambda \to \infty} \frac{\mathcal{M}^\star(\lambda)}{\lambda} = \frac{1}{2} \lim_{\lambda \to \infty} \frac{1 + \lambda}{\lambda + 3} = \frac{1}{2},$$

as claimed.

(ii) We have that

$$\lim_{\lambda \to \infty} f^\star(\lambda/2, \lambda)$$
$$= \lim_{\lambda \to \infty} \frac{\lambda}{2e^\lambda} \left((-\lambda^2 + 3\lambda) \int_0^\lambda \frac{e^x - 1}{x} dx + (\lambda - 2)e^\lambda - 2\lambda^2 + \lambda + 1 \right)$$

so, using L'Hôpital's rule, we get that:

$$\lim_{\lambda \to \infty} f^\star(\lambda/2, \lambda) = \frac{1}{2} \lim_{\lambda \to \infty} \frac{6 + 2\lambda}{2\lambda + 5} = \frac{1}{2}.$$

(iii) Since

$$\lim_{\lambda\to\infty} (\mathcal{M}^\star(\lambda) - \lambda/2) = \frac{1}{2} \lim_{\lambda\to\infty} \frac{1 - e^\lambda + \lambda e^\lambda - 2\lambda^2 + (2\lambda - \lambda^2)\int_0^\lambda \frac{e^x-1}{x}dx}{1 - e^\lambda + 2\lambda + (\lambda - 1)\int_0^\lambda \frac{e^x-1}{x}dx},$$

L'Hôpital's rule leads to

$$\lim_{\lambda\to\infty} (\mathcal{M}^\star(\lambda) - \lambda/2) = \frac{1}{2} \lim_{\lambda\to\infty} \frac{-2\lambda - 4}{\lambda} = -1$$

and hence the result.   □

Note that the value $\mathcal{M}^\star(\lambda) \approx \lambda/2 - 1$ is not the optimal cutoff value that we are looking for. $\mathcal{M}^\star(\lambda)$ is the value for which the function $f^\star(\cdot, \lambda)$ reaches its maximum, while we are interested in finding $\mathcal{M}(\lambda)$ which is the value for which the function $F^{\mathcal{P}}(\cdot, \lambda)$ reaches its maximum. Of course, both functions are closely related and, as we will see, so are $\mathcal{M}^\star(\lambda)$ and $\mathcal{M}(\lambda)$.

Recall that, in the case when the number of objects is known, the probability of success following the optimal threshold strategy in the Best-or-Worst problem is given by

$$P_{BW}(n) = \begin{cases} \frac{n}{2(n-1)}, & \text{if } n \text{ is even;} \\ \frac{n+1}{2n}, & \text{if } n \text{ is odd.} \end{cases}$$

The next lemma holds.

Lemma 9

$$\lim_{\lambda\to\infty} \sum_{k=1}^\infty P_{BW}(k) \frac{\lambda^k}{e^\lambda k!} = 1/2.$$

Proof Taking into account the definition of P_{BW} we have the following decomposition

$$\sum_{k=1}^\infty P_{BW}(k) \frac{\lambda^k}{e^\lambda k!} = \sum_{k=1}^\infty P_{BW}(2k) \frac{\lambda^{2k}}{e^\lambda (2k)!} + \sum_{k=1}^\infty P_{BW}(2k-1) \frac{\lambda^{(2k-1)}}{e^\lambda (2k-1)!}$$

Now, it can be easily seen that

$$\sum_{k=1}^\infty P_{BW}(2k) \frac{\lambda^{2k}}{e^\lambda (2k)!} = \sum_{k=1}^\infty \frac{2k}{4k-2} \frac{\lambda^{2k}}{e^\lambda (2k)!} = \frac{\lambda}{2} e^{-\lambda} S(\lambda),$$

$$\sum_{k=1}^\infty P_{BW}(2k-1) \frac{\lambda^{(2k-1)}}{e^\lambda (2k-1)!} = \sum_{k=1}^\infty \frac{2k}{4k-2} \frac{e^{-\lambda}\lambda^{(2k-1)}}{(2k-1)!} = \frac{\sinh(\lambda)}{2 e^\lambda} + \frac{S(\lambda)}{2e^\lambda},$$

where

$$S(\lambda) = \int_0^\lambda \frac{\sinh(x)}{x}dx.$$

Consequently,

$$\lim_{\lambda \to \infty} \sum_{k=1}^{\infty} P_{BW}(k) \frac{\lambda^k}{e^\lambda k!} = \frac{\lambda}{2} e^{-\lambda} S(\lambda) + \frac{\sinh(\lambda)}{2e^\lambda} + \frac{S(\lambda)}{2e^\lambda}$$

and the result follows from L'Hôpital's rule.   □

Using this result we can compute the asymptotic probability of success using the optimal threshold strategy in the Poisson case.

Theorem 8 *With all the previous notation, we have that*

$$\lim_{\lambda \to \infty} F^{\mathcal{P}}(\lambda/2, \lambda) = \lim_{\lambda \to \infty} F^{\mathcal{P}}(\mathcal{M}(\lambda), \lambda) = \lim_{\lambda \to \infty} P(\lambda) = 1/2.$$

Proof By definition, $P_{BW}(k)$ is the maximum probability of success in the Best-or-Worst problem with a known number of objects $k > 1$. Hence,

$$\frac{2\mathcal{M}(\lambda)(k - \mathcal{M}(\lambda))}{k(k-1)} = P_k(\mathcal{M}(\lambda)) \le P_{BW}(k).$$

Consequently,

$$P(\lambda) = F^{\mathcal{P}}(\mathcal{M}(\lambda), \lambda) = \sum_{k=\mathcal{M}(\ge)+1}^{\infty} \frac{2\mathcal{M}(\lambda)(n - \mathcal{M}(\lambda))}{k(k-1)} \frac{\lambda^k}{e^\lambda k!} \le \sum_{k=1}^{\infty} P_{BW}(k) \frac{\lambda^k}{e^\lambda k!}$$

so, if we take upper limits it is clear that $\varlimsup_{\lambda \to \infty} P(\lambda) \le 1/2$.

On the other hand, recalling Lemma 8 and Theorem 7 we have that

$$P(\lambda) = F^{\mathcal{P}}(\mathcal{M}(\lambda), \lambda) \ge F^{\mathcal{P}}(\lambda/2, \lambda) = f^\star(\lambda/2, \lambda) - f(\lambda/2, \lambda).$$

and also that

$$\lim_{\lambda \to \infty} f^\star(\lambda/2, \lambda) - f(\lambda/2, \lambda) = 1/2.$$

Thus, taking lower limits $\varliminf_{\lambda \to \infty} P(\lambda) \ge 1/2$.

In conclusion, $1/2 \le \varliminf_{\lambda \to \infty} P(\lambda) \le \varlimsup_{\lambda \to \infty} P(\lambda) \le 1/2$ and the proof is complete. □

Even if we did not explicitly find an estimation for the optimal cutoff value $\mathcal{M}(\lambda)$, this theorem satisfactorily solves the problem, since it implies that $\lambda/2$ is an acceptable estimation for the optimal cutoff value because it provides the same asymptotic probability of success as the exact value of $\mathcal{M}(\lambda)$ would do; i.e., $1/2$. Thus, Theorem 8 proves that, if the random number of objects follows a Poisson distribution $\mathcal{P}(\lambda)$, the optimal strategy consists in rejecting the first $\lambda/2$ objects and then accept the first nice candidate after them. With this strategy we will succeed approximately one half of the times.

In addition, as far as we were able to check, $\lfloor \lambda/2 - 1 \rfloor$ coincides with the exact value of $\mathcal{M}(\lambda)$ for every integer value of $\lambda > 1$. Hence, we propose the following conjecture.

Conjecture 1 $\mathcal{M}(\lambda) \approx \mathcal{M}^\star(\lambda) \approx \lambda/2 - 1$.

6 Concluding remarks

In the table below we compare the optimal cutoff value $\mathcal{M}$ and the asymptotic probability of success P in the classical, the Best-or-Worst and the Postdoc problems and in the different variants studied in this paper for the number of objects X. Recall the relationship between the Best-or-Worst problem and the Postdoc problem that was stated in Theorem 1. All the constants that appear in the table can be expressed in terms of $e = 2.71828\ldots$ and the rumor's constant $\vartheta := -\frac{1}{2}W(-2e^{-2}) = 0.20318\ldots$

	Classic		Best-or-Worst		Postdoc	
X	$\mathcal{M}$	P	$\mathcal{M}$	P	$\mathcal{M}$	P
$X = n$	ne^{-1}	e^{-1}	$n/2$	$1/2$	$n/2$	$1/4$
$X \sim \mathcal{U}[1,n]$	ne^{-2}	$2e^{-2}$	$n\vartheta$	$2(\vartheta - \vartheta^2)$	$n\vartheta$	$(\vartheta - \vartheta^2)$
$X \sim \mathcal{P}(\lambda)$	λ/e	e^{-1}	$\lambda/2$	$1/2$	$\lambda/2$	$1/4$

We close the paper with an intriguing remark pointed out by Havil (2003) relating the convergents for the continued fraction of e^{-1} and the optimal cutoff value $r(n)$ in the secretary problem with a known number of candidates n. In fact, the convergents for the continued fraction of e^{-1} (see sequences A007676 and A007677 in the OEIS) are given by

$$0, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}, \frac{7}{19}, \frac{32}{87}, \frac{39}{106}, \frac{71}{193}, \frac{465}{1264}, \frac{536}{1457}, \frac{1001}{2721}, \ldots$$

which exactly coincide with the fractions of the form $r(n)/n$.

Now, if we focus on the Best-or-Worst problem when the number of candidates follows a discrete Uniform distribution $\mathcal{U}[1, n]$ the same relation arises considering ϑ instead of e^{-1}. The convergents for the continued fraction of ϑ are

$$0, \frac{1}{4}, \frac{1}{5}, \frac{12}{59}, \frac{13}{64}, \frac{38}{187}, \frac{51}{251}, \frac{1262}{6211}, \frac{1313}{6462}, \frac{11766}{57907}, \frac{13079}{64369}, \frac{64082}{315383}, \ldots$$

which, as far as our computation capabilities allowed us to check, also coincide with fractions of the form $r(n)/n$, with $r(n)$ being the optimal cutoff value for the considered problem. However, as Havil's remark, this relation remains an open problem.

References

Babaioff M, Immorlica N, Kleinberg R (2007) Matroids, secretary problems, and online mechanisms. In: Proceedings of the eighteenth annual ACM-SIAM symposium on discrete algorithms. ACM, New York, pp 434–443

Bayón L, Fortuny Ayuso P, Grau JM, Oller-Marcén AM, Ruiz MM (2018) The Best-or-Worst and the Postdoc problems. J Comb Optim 35(3):703–723

Bruss FT (1987) On an optimal selection problem of Cowan and Zabczyk. J Appl Probab 24(4):918–928

Bruss FT (2000) Sum the odds to one and stop. Ann Probab 28(3):1384–1391

Bruss FT (2003) A note on bounds for the odds theorem of optimal stopping. Ann Probab 31(4):1859–1861

Cowan R, Zabczyk J (1978) An optimal selection problem associated with the Poisson process. Teor Veroyatnost i Primenen 23(3):606–614

Dendievel R (2013) New developments of the odds-theorem. Math Sci 38(2):111–123

Dendievel R (2015) Weber's optimal stopping problem and generalizations. Stat Probab Lett 97:176–184

Dynkin EB (1963) Optimal choice of the stopping moment of a Markov process. Dokl Akad Nauk SSSR 150:238–240

Ferguson TS (1989) Who solved the secretary problem? Stat Sci 4(3):282–296

Ferguson TS (1992) Best-choice problems with dependent criteria. Strategies for sequential search and selection in real time. In: Bruss FT, Ferguson TS, Samuels S (eds) Proceedings of the AMS-IMS-SIAM joint summer research conference (Amherst, MA, 1990), contemporary mathematics, vol 125. American Mathematical Society, Providence, RI, pp 135–151

Ferguson TS, Hardwick JP, Tamaki M (1992) Maximizing the duration of owning a relatively best object. Strategies for sequential search and selection in real time. In: Bruss FT, Ferguson TS, Samuels S (eds) Proceedings of the AMS-IMS-SIAM joint summer research conference (Amherst, MA, 1990), contemporary mathematics, vol 125. American Mathematical Society, Providence, RI, pp 37–58

Freij R, Wastlund J (2010) Partially ordered secretaries. Electron. Commun. Probab. 15:504–507

Garrod B, Morris R (2013) The secretary problem on an unknown poset. Random Struct Algorithms 43(4):429–451

Georgiou N, Kuchta M, Morayne M, Niemiec J (2008) On a universal best choice algorithm for partially ordered sets. Random Struct Algorithms 32:263–273

Gharan SO, Vondrák J (2013) On Variants of the Matroid Secretary Problem. J Algorithmica 67(4):472–497

Gilbert JP, Mosteller F (1966) Recognizing the maximum of a sequence. J Am Stat Assoc 61:35–73

Havil J (2003) Optimal choice. Gamma: exploring Euler's constant. Princeton University Press, Princeton, pp 34–138

Lindley DV (1961) Dynamic programming and decision theory. Appl Stat 10:39–51

Louchard G (2017) A refined and asymptotic analysis of optimal stopping problems of Bruss and Weber. Math Appl 45(2):95–118

Rose JS (1982) A problem of optimal choice and assignment. Oper Res 30(1):172–181

Presman ÈL, Sonin IM (1972) The problem of best choice in the case of a random number of objects. Teor Verojatnost i Primenen 17:695–706

Soto JA (2013) Matroid secretary problem in the random-assignment model. Siam J Comput 42(1):178–211

Szajowski K (1982) Optimal choice problem of a-th object. Matem Stos 19:51–65

Szajowski K (2007) A game version of the Cowan–Zabczyk–Bruss' problem. Stat Probab Lett 77(17):1683–1689

Szajowski K (2009) A rank-based selection with cardinal payoffs and a cost of choice. Sci Math Jpn 69(2):285–293

Vanderbei RJ (1983) The Postdoc variant of the secretary problem. http://www.princeton.edu/rvdb/tex/PostdocProblem/PostdocProb.pdf. Accessed 01 July 2018 (unpublished)