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The multi-returning secretary problem

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ABSTRACT

In this paper we consider the so-called Multi-returning secretary problem, a version of the Secretary problem in which each candidate has m identical copies. The case $m = 2$ has already been completely solved by several authors using different methods, but the case $m > 2$ had not been satisfactorily solved yet. Here, under the conjecture of certain (very likely true) uniform convergence results, we provide an efficient algorithm to compute the optimal threshold and the probability of success for every m. Moreover, we give a method to determine their asymptotic values based on the solution of a system of m ODEs.

1. Introduction

The so-called *Secretary Problem* is possibly one of the most famous problems in optimal stopping theory. This problem can be stated as follows: we want to select the best out of n ranked candidates. The candidates are inspected one by one in random order and we have to accept or reject the candidate immediately. At each step, we can rank the candidate among all the preceding ones, but we are unaware of the quality of yet unseen candidates. The goal is to determine the optimal strategy that maximizes the probability of selecting the best candidate.

Dynkin [5] and Lindley [13] independently proved that the best strategy consists in rejecting roughly the first n/e interviewed candidates and then selecting the first one that is better than all the preceding ones. Following this strategy, the probability of selecting the best candidate is at least $1/e$, this being its approximate value for large values of n. This well-known solution was later refined by Gilbert and Mosteller [10], showing that $\lfloor (n - \frac{1}{2})e^{-1} + \frac{1}{2} \rfloor$ is a better approximation than $\lfloor n/e \rfloor$, although the difference is never greater than 1. Furthermore, this problem can be addressed and solved in a rather straightforward manner using the so-called *odds-algorithm* devised by Bruss [4].

The Secretary problem has been addressed by many authors in different fields such as applied probability, statistics or decision theory. Extensive bibliographies on the topic can be found in [6,8,16] for instance. Among the several interesting modifications of the original problem we can mention, for example:

- The *Best or Worst problem*, in which the goal is to select either the best or the worst candidate [2,3].
- The *Postdoc problem*, in which the goal is to select the second best candidate [2,3,17] or, even more generally, the kth best candidate [15].
- The *Win, Lose or Draw marriage problem*, in which the payoff is 1 if the best candidate is selected, -1 if a non-best candidate is selected, and 0 if no candidate is selected; and the goal is to maximize the payoff [7].

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Table 1
Table from [9, p. 51]. Numbers in boldface are wrong in the original (correct values are in parentheses).

m	$\mathbf{k}_{100}^m$	$\mathbf{k}_{1000}^m$	$\lim_n \left(\frac{\mathbf{k}_n^m}{n}\right)$	$\mathbf{P}_{100}^m$	$\lim_n \mathbf{P}_n^m$
1	38	369	0.3679	**0.3708** (0.371042)	0.3678794
2	48	471	0.4709	0.76970661	0.7679742
3	50	493	?	**0.9354** (0.93518)	?
4	50	499	?	?	?
5	50	500	?	?	?
6	50	500	?	?	?
7	50	500	?	?	?
8	50	500	?	?	?
9	50	500	?	?	?
10	50	500	?	?	?

- The *Secretary problem with uncertain employment*, in which a candidate may refuse to be accepted with a given probability [14].
- The *One of the best two problem*, in which the goal is to select either the best or the second best candidate receiving different payoffs in each case [10,12].

Another interesting variant of the classical problem, the so-called *Returning Secretary problem*, was introduced in 2012 by Garrod et al. in [1] and by Garrod in [9]. In this variant, every candidate has an identical copy and the goal is still to select the best candidate. In 2015, Vardi [18,19] independently addressed the same problem. Both Garrod and Vardi approach the problem from the perspective of partially ordered sets. Very recently, Grau [11] introduced a new method based on solving differential equations.

The previous variant can be further generalized to consider the *Multi-returning Secretary problem*, in which every candidate has m identical copies or, equivalently, in which every candidate is inspected m times. Garrod [9] shows that the optimal strategy in the Multi-returning Secretary problem is a threshold strategy (just like in the classical problem). He also provides explicit formulas for the optimal threshold $\mathbf{k}_n^m$ [9, Theorem 2.2] and for the probability of success $\mathbf{P}_n^m$ [9, Theorem 2.16]. However, his formulas are very inefficient from the computational point of view. In fact, for a fixed m, the formula for $\mathbf{k}_n^m$ requires a number of operations of order $O(n^2)$ while the formula for $\mathbf{P}_n^m$ requires a number of operations of order $O(n^{m-1})$. Regarding the asymptotic behavior, Garrod is able to prove for every fixed n that $\lim_m \mathbf{k}_n^m = \lceil n/2 \rceil$ and that $\lim_m \mathbf{P}_n^m = 1$. However, the limitations of Garrod's approach for $m > 2$ are clearly shown in the following table [9, p. 51].

Motivated by these limitations, Garrod [9, p. 52] presents a series of open problems:

(1) Provide alternative formulas for $\mathbf{P}_n^m$.
(2) For fixed m, prove that $\lim_n \left(\frac{\mathbf{k}_n^m}{n}\right)$ exists.
(3) If $\lim_n \left(\frac{\mathbf{k}_n^m}{n}\right)$ exists, find its value as the root of an equation or as a function of $\lim_n \left(\frac{\mathbf{k}_n^{m-1}}{n}\right)$.
(4) For fixed m, prove that $\lim_n \mathbf{P}_n^m$ exists.
(5) If $\lim_n \mathbf{P}_n^m$ exists, find its value as the root of an equation or as a function of $\lim_n \mathbf{P}_n^{m-1}$.

In the present paper we address and partially solve the open problems stated above. In particular, we give efficient algorithms that compute $\mathbf{k}_n^m$ and $\mathbf{P}_n^m$ as well as a method, based on the techniques introduced in [11] to compute their asymptotic values. We must note that the main results on which our methods are based remain "conjectural" in the sense that, in order to prove them, we assume the uniform convergence of certain sequences of functions. These assumptions remain unproven. However, computational and experimental evidence strongly suggest that they are most probably true.

The paper is organized as follows. In Section 2 we present some technical results. Section 3 revisits the m-returning secretary problem using a dynamic programming approach, providing a method to compute $\mathbf{P}_n^m$. In Sections 4 and 5, using the ideas and techniques from [11], we give methods to compute the asymptotic values of $\mathbf{k}_n^m$ and $\mathbf{P}_n^m$ under suitable uniform convergence assumptions. Finally, Section 6 concludes the paper relating our results to Garrod's open problems.

2. Some technical results

In this section we present some technical results that will be used extensively in forthcoming sections. The first proposition was already introduced in [11, Proposition 1] and, in some sense, it extends [2, Proposition 1].

Proposition 1. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of functions with $F_n : \{0, \dots, n\} \to \mathbb{R}$ and let $\mathcal{M}(n) \in \{0, \dots, n\}$ be a value at which the function F_n reaches its maximum. Assume that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined by $f_n(x) := F_n(\lfloor nx \rfloor)$ for every $x \in [0, 1]$ converges uniformly on $[0, 1]$ to a continuous function f and that θ is the only global maximum of f in $[0, 1]$. Then,*

(i) $\lim_n \mathcal{M}(n)/n = \theta$.
(ii) $\lim_n F_n(\mathcal{M}(n)) = f(\theta)$.

Proof. It is identical to the proof of [2, Proposition 1]. $\square$

The following result, which is rather similar to the previous one, will also turn out to be useful in the sequel.

Proposition 2. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of functions with $F_n : \{0, \dots, n\} \to \mathbb{R}$ and let $\mathcal{N}(n) \in \{0, \dots, n-1\}$ be such that*

$$\frac{\mathcal{N}(n)}{n} < F_n(\mathcal{N}(n)),$$
$$\frac{\mathcal{N}(n)+1}{n} \geq F_n(\mathcal{N}(n)+1).$$

Assume that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined by $f_n(x) := F_n(\lfloor nx \rfloor)$ for every $x \in [0, 1]$ converges uniformly on $[0, 1]$ to a continuous function f and that θ is the only solution of $x = f(x)$. Then, $\lim_n \mathcal{N}(n)/n = \theta$.

Proof. Let us consider the sequence $\{\mathcal{N}(n)/n\} \subset [0, 1]$ and let $\{\mathcal{N}(s_n)/s_n\}$ be a convergent subsequence. Denote its limit by $\alpha \in [0, 1]$. Then,

$$\alpha = \lim_n \frac{\mathcal{N}(s_n)}{s_n} \leq \lim_n F_{s_n}(\mathcal{N}(s_n)) = \lim_n F_{s_n}\left(\frac{\mathcal{N}(s_n)}{s_n} s_n\right) = \lim_n f_{s_n}\left(\frac{\mathcal{N}(s_n)}{s_n}\right) = f(\alpha),$$
$$f(\alpha) = \lim_n f_{s_n}\left(\frac{\mathcal{N}(s_n)+1}{s_n}\right) = \lim_n F_{s_n}(\mathcal{N}(s_n)+1) \leq \lim_n \frac{\mathcal{N}(s_n)+1}{s_n} = \alpha.$$

Consequently, $\alpha = f(\alpha)$ and since θ is the only solution of $x = f(x)$ it follows that $\theta = \alpha$.

Thus, we have proved that every convergent subsequence of $\{\mathcal{N}(n)/n\}$ must converge to θ. Since $\{\mathcal{N}(n)/n\}$ is included in the compact set $[0, 1]$, this implies that $\{\mathcal{N}(n)/n\}$ itself must also converge to θ. $\square$

In this work, we will follow an approach similar to that in [11]. Namely, we will assume the uniform convergence of the sequence $\{f_n\}$ to a continuous function. Under this assumption the following results show that the limit function f can be easily found provided the functions F_n are recursively defined.

Proposition 3. *Let $\{F_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ and $\{H_n\}_{n\in\mathbb{N}}$ be sequences of functions with $F_n, G_n, H_n : \{0, \dots, n\} \to \mathbb{R}$ which satisfy*

$F_n(k) = G_n(k) + H_n(k)F_n(k-1)$, $k > 0$;
$F_n(0) = \mu$.

Moreover, for every $x \in [0, 1]$, let us define $f_n(x) := F_n(\lfloor nx \rfloor)$, $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$. If the following conditions hold:

(i) *Both sequences $\{h_n\}$ and $\{g_n\}$ converge on $(0, 1)$ and uniformly on $[\varepsilon, \varepsilon']$ for every $0 < \varepsilon < \varepsilon' < 1$ to continuous functions $h(x)$ and $g(x)$, respectively.*
(ii) *The sequence $\{f_n\}$ converges uniformly on $[0, 1]$ to a continuous function f.*

Then, $f(0) = \mu$ and f satisfies the equation $f'(x) = -f(x)h(x) + g(x)$ for every $x \in (0, 1)$.

Proof. See [11, Theorem 1]. $\square$

Proposition 4. *Let $\{F_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ and $\{H_n\}_{n\in\mathbb{N}}$ be sequences of functions with $F_n, G_n, H_n : \{0, \dots, n\} \to \mathbb{R}$ which satisfy*

$F_n(k) = G_n(k) + H_n(k)F_n(k+1)$, $k < n$;
$F_n(n) = \mu$.

Moreover, for every $x \in [0, 1]$, let us define $f_n(x) := F_n(\lfloor nx \rfloor)$, $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$. If the following conditions hold:

(i) *Both sequences $\{h_n\}$ and $\{g_n\}$ converge on $(0, 1)$ and uniformly on $[\varepsilon, \varepsilon']$ for every $0 < \varepsilon < \varepsilon' < 1$ to continuous functions $h(x)$ and $g(x)$, respectively.*
(ii) *The sequence $\{f_n\}$ converges uniformly on $[0, 1]$ to a continuous function f.*

Then, $f(1) = \mu$ and f satisfies the equation $f'(x) = f(x)h(x) - g(x)$ for every $x \in (0, 1)$.

Proof. See [11, Theorem 2]. $\square$

Remark 1. Note that the two previous propositions are almost identical. The only difference is that in Proposition 3 the function F_n is defined by a *forward* recursion, while in Proposition 4 it is defined by a *backward* recursion. Accordingly, in Proposition 3 we are given an initial condition and in Proposition 4 we are given a final condition.

Example 1. In the case of the classical Secretary problem, the probability of success using the threshold k is given by a function $F_n(k)$ which satisfies the following recurrence relation:

$$F_n(k) = \frac{1}{n} + \frac{k}{k+1}F_n(k+1),$$
$$F_n(n) = 0.$$

If we consider $G_n(k) = \frac{1}{n}$ and $H_n(k) = \frac{k}{k+1}$ we get that $g_n(x) = 1$ and $h_n(x) = n\frac{\lfloor nx+1\rfloor - \lfloor nx\rfloor}{\lfloor nx+1\rfloor}$ so it is easy to check that condition i) in Proposition 4 holds with $g(x) = 1$ and $h(x) = \frac{1}{x}$. Moreover, since the functions $f_n(x)$ can be expressed in terms of the digamma function (see [2, p. 706] for details), condition ii) also holds; i.e., the sequence $\{f_n\}$ converges uniformly on $[0, 1]$ to a certain continuous function f. Then due to Proposition 4, this function f must satisfy the ODE $f'(x) = \frac{f(x)}{x} - 1$ for every $x \in (0, 1)$ and the condition $f(1) = 0$.

This leads to the well-known function $f(x) = -x\log(x)$, whose maximization in $[0, 1]$ together with Proposition 1 provide the asymptotic value of the optimal threshold n/e as well as the asymptotic probability of success e^{-1}.

3. A dynamic programming approach to the *m*-returning secretary problem

Let us assume that there are n candidates that arrive sequentially and that there are exactly m identical copies of each candidate. The order in which they are inspected is uniform random (of course, there are $(mn)!$ possibilities). At any given step it is only possible to know who is the best candidate so far and how many copies of this candidate have been inspected. Once a candidate is accepted, the process ends and, as usual, to succeed means to select the best candidate. We seek to maximize the probability of success.

For the sake of clarity, let us consider the following equivalent situation.

(1) There is an urn with mn objects, namely m copies of n different (rankable) objects. We want to select one of the m copies of the best object.
(2) At each step, one object is randomly and uniformly extracted from the urn. Once it is inspected, it is decided either to select or to reject it.
(3) When an object is inspected, we remove from the urn all the copies of those worse objects that have been previously inspected.

In this setting it is clear that, at any given step, the relevant information is just the number of different inspected objects (that we will denote by k) and the number of appearances of the maximal object so far (that we will denote by i). Furthermore, at each step, the set of remaining items contains all copies of non-inspected candidates ($(n-k)m$ elements) plus the remaining $m - i$ copies of the current maximal candidate. In what follows, a maximal candidate appearing for the mth time (i.e. for the last time) will be called a *nice candidate*.

First let us introduce a lemma in which we compute the probability of certain events that will play an important role later on.

Lemma 1. *Let us assume, with the previous notation, that k represents the number of different inspected candidates and i represents the number of times the current maximal candidate has been inspected. We consider the following events:*

$A =$ *"The next inspected object is a copy of the current maximal candidate"*,

$B =$ *"The next inspected object is a new maximal candidate"*.

Then,

$$p(A) = \frac{m-i}{(n-k)m + m - i},$$
$$p(B) = \frac{mn - mk}{(k+1)\big((n-k)m + m - i\big)}.$$

Proof. As we already pointed out, in the conditions of the statement, there are $(n-k)m+m-i$ candidates left, all of them with the same probability of being the next inspected one. Among them, there are exactly $m - i$ copies of the current maximal candidate, so the value of $p(A)$ readily follows. On the other hand, each of the remaining $(n - k)m$ candidates has probability $1/(k + 1)$ to be maximal, so the value of $p(B)$ follows. □

Remark 2. Note that, at a given step, $\{A, B, C = \overline{A \cup B}\}$ is a complete system of events. This fact will be useful in the sequel.

In what follows we will consider the events

$X^{k,i}_{m,n}$ ="succeed following the optimal strategy after having rejected k different candidates among which the maximal candidate has appeared i times"

and we will denote $\Psi^i_{m,n}(k) = p\left(X^{k,i}_{m,n}\right)$. Recall that $\mathbf{P}^m_n$ denotes the probability of success under the optimal strategy. Hence, with our notation, we have that $\mathbf{P}^m_n = p(X^{1,1}_{m,n}) = \Psi^1_{m,n}(1)$ (note that due to the conditions of the problem we will always reject the first candidate if $mn > 1$).

The following proposition is devoted to provide recursive relations for the function $\Psi^i_{m,n}$ that will ultimately allow us to effectively compute $\mathbf{P}^m_n$.

Proposition 5. *With the previous notation $\Psi^m_{m,n}(n) = 0$, and $\Psi^i_{m,n}(n) = 1$ for every $i \in \{1, \dots, m-1\}$. Furthermore, for every $1 \le k < n$, we have that*

$$\Psi^m_{m,n}(k) = \frac{1}{k+1}\Psi^1_{m,n}(k+1) + \frac{k}{k+1}\Psi^m_{m,n}(k+1),$$

$$\Psi^{m-1}_{m,n}(k) = \frac{1}{mn-mk+1}\max\left\{\frac{k}{n}, \Psi^m_{m,n}(k)\right\} + \frac{mn-mk}{(k+1)(mn-mk+1)}\Psi^1_{m,n}(k+1) + \frac{k(mn-mk)}{(k+1)(mn-mk+1)}\Psi^{m-1}_{m,n}(k+1),$$

and, for every $i \in \{1, \dots, m-2\}$

$$\Psi^i_{m,n}(k) = \frac{m-i}{mn-mk+m-i}\Psi^{i+1}_{m,n}(k) + \frac{mn-mk}{(k+1)(mn-mk+m-i)}\Psi^1_{m,n}(k+1) + \frac{k(mn-mk)}{(k+1)(mn-mk+m-i)}\Psi^i_{m,n}(k+1).$$

Proof. First of all, it is obvious by definition that $\Psi^m_{m,n}(n) = 0$. In fact, if we have inspected n different candidates (all the possible ones), and the maximal candidate has already appeared m times, then this maximal candidate is in fact the best candidate and it will not appear again. Thus, we can no longer select it and we cannot succeed.

In the same way it is also obvious by definition that $\Psi^i_{m,n}(n) = 1$ for every $i \in \{1, \dots, m-1\}$. If we have inspected n different candidates (all the possible ones), and the maximal candidate has appeared i times (with $1 \le i \le m-1$), then this maximal candidate is in fact the best candidate and we just have to wait until it appears again in order to guarantee the success.

Now, let us focus on $\Psi^m_{m,n}(k)$ for $1 \le k < n$; i.e., $i = m$ with the previous notation. Due to Remark 2, and taking into account that in this case $A = \emptyset$ and $p(B) = 1/(k+1)$, the law of total probability leads to

$$\Psi^m_{m,n}(k) = p\left(X^{k,m}_{m,n}\right) = p\left(X^{k,m}_{m,n}|B\right)p(B) + p\left(X^{k,m}_{m,n}|\overline{B}\right)p(\overline{B}).$$

By the very definition it is straightforward to see that $p\left(X^{k,m}_{m,n}|B\right) = \Psi^1_{m,n}(k+1)$ and that $p\left(X^{k,m}_{m,n}|\overline{B}\right) = \Psi^m_{m,n}(k+1)$ and the result follows.

Now, let us focus on $\Psi^{m-1}_{m,n}(k)$. Another application of the law of total probability leads to

$$\Psi^{m-1}_{m,n}(k) = p\left(X^{k,m-1}_{m,n}\right) = p\left(X^{k,m-1}_{m,n}|A\right)p(A) + p\left(X^{k,m-1}_{m,n}|B\right)p(B) + p\left(X^{k,m-1}_{m,n}|C\right)p(C).$$

Just like above, the very definition leads to the fact that $p\left(X^{k,m-1}_{m,n}|B\right) = \Psi^1_{m,n}(k+1)$ and $p\left(X^{k,m-1}_{m,n}|C\right) = \Psi^{m-1}_{m,n}(k+1)$. The last remaining probability is slightly more tricky. We claim that $p\left(X^{k,m-1}_{m,n}|A\right) = \max\left\{\frac{k}{n}, \Psi^m_{m,n}(k)\right\}$. This happens because in this case the optimal strategy will select the next candidate (which is a nice candidate) if and only if the probability of success choosing it (which is k/n) is greater than the probability of success if we reject the nice candidate and keep going (which is $\Psi^m_{m,n}(k)$). In any case, the final probability of success is the maximum of both values, as claimed.

Finally, let us focus on $\Psi^i_{m,n}(k)$ for $i \in \{1, \dots, m-2\}$. The law of total probability leads again to

$$\Psi^i_{m,n}(k) = p\left(X^{k,i}_{m,n}\right) = p\left(X^{k,i}_{m,n}|A\right)p(A) + p\left(X^{k,i}_{m,n}|B\right)p(B) + p\left(X^{k,i}_{m,n}|C\right)p(C).$$

In this situation, and like above, the very definition leads to the fact that $p\left(X^{k,i}_{m,n}|A\right) = \Psi^{i+1}_{m,n}(k)$, $p\left(X^{k,i}_{m,n}|B\right) = \Psi^1_{m,n}(k+1)$ and $p\left(X^{k,i}_{m,n}|C\right) = \Psi^i_{m,n}(k+1)$ and the result follows. □

Once we have established the previous recursive relations, we are in a position to determine the computational complexity of the associated dynamic program.

Proposition 6. *For any fixed m, the computational complexity of the dynamic program defined by the previous recurrences is $O(n)$.*

Table 2
Computation time (in seconds) of $\mathbf{P}_n^4$ using Proposition 5.

$n/10$	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}
Exact	0.06	0.12	0.28	0.85	3.29	17.09	98.26	536.32	3047.77
100 digit precision	0.04	0.09	0.17	0.37	0.73	1.42	2.93	6.14	11.78

Proof. It is enough to observe that the number of required operations to compute $\{\Psi^i_{m,n}(k)\}_{i=1}^m$ from $\{\Psi^i_{m,n}(k+1)\}_{i=1}^m$ is independent of n. □

We have already mentioned that if a maximal candidate is accepted after inspecting k different candidates, the probability of success is k/n, just like in the classical Secretary problem. Also recall that it is always preferable to reject a maximal candidate unless it is a nice candidate. Consequently, it is clear that the optimal strategy consists in accepting a nice candidate whenever the number of different inspected candidates belongs to the so-called stopping set

$$\mathcal{S} := \{k : k/n \geq \Psi^m_{m,n}(k)\}.$$

If the stopping set consists of a single stopping island, $\mathcal{S} = \{k, k+1, \dots, n\}$, then we say that the optimal strategy is a *threshold strategy*. In such a case, $\min \mathcal{S}$ is called the *optimal threshold*. We now prove that like in the classic Secretary problem, the optimal strategy for the m-returning Secretary problem is a threshold strategy. Although this was already proved by Garrod, we provide a simpler proof based on the previous dynamic program.

Theorem 1. *In the m-returning Secretary problem, let n be the number of different objects. Then, there exists $\mathbf{k}_n^m$ such that the following strategy is optimal:*

(1) *Reject the $\mathbf{k}_n^m$ first different inspected objects.*
(2) *After that, accept the first nice candidate.*

Proof. We have to prove that for every k the following holds

$$\frac{k}{n} \geq \Psi^m_{m,n}(k) \Longrightarrow \frac{k+1}{n} \geq \Psi^m_{m,n}(k+1).$$

To do so, it suffices to see that $\Psi^m_{m,n}(k) \geq \Psi^m_{m,n}(k+1)$. In fact, the very definition implies that $\Psi^m_{m,n}(k+1) \leq \Psi^1_{m,n}(k+1)$ so, applying Proposition 5 we get that

$$\Psi^m_{m,n}(k) = \frac{k}{1+k}\Psi^m_{m,n}(k+1) + \frac{1}{1+k}\Psi^1_{m,n}(k+1) \geq \Psi^m_{m,n}(k+1)$$

and the result follows. □

As we already pointed out, $\mathbf{P}_n^m = \Psi^1_{m,n}(1)$. Consequently, it is possible to compute the probability of success using Proposition 5. In the following table we show the time required to compute $\mathbf{P}_n^m$ for $m = 4$ and $n = 10 \cdot 2^i$ with $7 \leq i \leq 15$ using our method. It is noteworthy (second row of Table 2) that the time does not seem to follow the linear behavior stated in Proposition 6. This is because, although the number of required operations is linear, the exact computations involve rational numbers with a high number of digits and hence the computation time and the number of operations behave quite differently. However, if we use 100 digits precision (third row of Table 2) we see that the computations time clearly follows a linear behavior.

On the other hand, Table 3 shows the time required to compute $\mathbf{P}_n^m$ for $m = 4$ and $n = 2^i$ with $3 \leq i \leq 8$ using Garrod's method. A comparison with Table 2 makes our improvement clear.

The software and hardware used to perform these computations were Mathematica 11, running in an Intel Core i5 9th gen. processor.

4. A conjecture about the asymptotic behavior of the optimal stopping threshold

In the previous section we proved the existence of an optimal stopping threshold $\mathbf{k}_n^m$. Now, we study its asymptotic behavior. In fact, under certain likely uniform convergence hypotheses, we will be able to compute $\lim_n \mathbf{k}_n^m/n$. To do so, we need to consider the events

$$Y^{k,i}_{m,n} = \text{"succeed accepting the first nice candidate after having rejected } k \text{ different candidates among which the maximal candidate has appeared } i \text{ times"}$$

and we will denote $\Phi^i_{m,n}(k) = p\left(Y^{k,i}_{m,n}\right)$.

These functions $\Phi^i_{m,n}$ satisfy nearly the same recursive relations that were satisfied by the functions $\Psi^i_{m,n}$ (because $\max\left\{\frac{k}{n}, \Psi^m_{m,n}(k)\right\}$ happens to be $\frac{k}{n}$) as we see in the proposition below.

Table 3
Computation time (in seconds) of $\mathbf{P}_n^4$ using Garrod's work.

n	8	16	32	64	128	556
Garrod's method	0.015	0.18	2.31	32.71	483.80	7489.23

Proposition 7. *With the previous notation $\Phi_{m,n}^m(n) = 0$, and $\Phi_{m,n}^i(n) = 1$ for every $i \in \{1, \dots, m-1\}$. Furthermore, for every $1 \le k < n$, we have that*

$$\Phi_{m,n}^m(k) = \frac{1}{k+1}\Phi_{m,n}^1(k+1) + \frac{k}{k+1}\Phi_{m,n}^m(k+1),$$

$$\begin{aligned}\Phi_{m,n}^{m-1}(k) =& \frac{1}{mn-mk+1}\frac{k}{n} + \frac{mn-mk}{(k+1)(mn-mk+1)}\Phi_{m,n}^1(k+1) + \\ &+ \frac{k(mn-mk)}{(k+1)(mn-mk+1)}\Phi_{m,n}^{m-1}(k+1),\end{aligned}$$

and, for every $i \in \{1, \dots, m-2\}$

$$\begin{aligned}\Phi_{m,n}^i(k) =& \frac{m-i}{mn-mk+m-i}\Phi_{m,n}^{i+1}(k) + \frac{mn-mk}{(k+1)(mn-mk+m-i)}\Phi_{m,n}^1(k+1) + \\ &+ \frac{k(mn-mk)}{(k+1)(mn-mk+m-i)}\Phi_{m,n}^i(k+1).\end{aligned}$$

Proof. Just reason in the same way as in Proposition 5. □

As a consequence of their very similar definition and since they satisfy nearly the same recursive relations, it is no surprise that the functions $\Phi_{m,n}^i$ and $\Psi_{m,n}^i$ are closely related. In fact, we have the following result showing that the behavior seen in Fig. 1 is general.

Proposition 8. *The following relations hold.*

(i) *If $k \ge \mathbf{k}_n^m$, then $\Phi_{m,n}^i(k) = \Psi_{m,n}^i(k)$ for every $1 \le i \le m$.*
(ii) *$\Phi_{m,n}^m(\mathbf{k}_n^m - 1) = \Psi_{m,n}^m(\mathbf{k}_n^m - 1)$.*
(iii) *If $1 \le k < \mathbf{k}_n^m$, then $\Phi_{m,n}^i(k) \le \Psi_{m,n}^i(k)$ for every $1 \le i \le m$.*

Proof. It is enough to recall that $\mathbf{k}_n^m$ is the optimal threshold. Hence, if $k \ge \mathbf{k}_n^m$, the optimal strategy (recall Theorem 1) that defines $\Psi_{m,n}^i(k)$ coincides with the strategy that defines $\Phi_{m,n}^i(k)$ and both functions are equal as claimed. □

Proposition 9. *Let $\mathbf{k}_n^m$ be the optimal threshold. Then, $\Phi_{m,n}^m(\mathbf{k}_n^m) \le \mathbf{k}_n^m/n$ and $\Phi_{m,n}^m(\mathbf{k}_n^m - 1) > (\mathbf{k}_n^m - 1)/n$.*

Proof. Recall that, by definition $\mathbf{k}_n^m = \min\{k : k/n \ge \Psi_{m,n}^m(k)\}$. Thus, it is enough to apply this fact and Proposition 8. □

Now, for every $1 \le i \le m$, let us define $\phi_{m,n}^i(x) := \Phi_{m,n}^i(\lfloor xn \rfloor)$. Under suitable assumptions about the uniform convergence of the sequence $\{\phi_{m,n}^i\}_n$, we will be able to apply Propositions 2 and 4 in order to determine the asymptotic behavior of $\mathbf{k}_n^m$.

Proposition 10. *Let us assume that, for every $1 \le i \le m$, the sequence of functions $\{\phi_{m,n}^i\}_n$ converges uniformly on $[0, 1]$ to a function Y_m^i which is continuous and differentiable on $(0, 1]$. Then, the functions $\{Y_m^i\}$ satisfy the following system of ODEs on the interval $(0, 1]$*

$$\begin{cases} m(1-x)y_1'(x) = (m-1)y_1(x) - (m-1)y_2(x) \\ mx(1-x)y_i'(x) = m(x-1)y_1(x) + (m-ix)y_i(x) - x(m-i)y_{i+1}(x), \;\; 2 \le i \le m-2 \\ mx(1-x)y_{m-1}'(x) = -m(1-x)y_1(x) + (m-(m-1)x)y_{m-1}(x) - x^2 \\ xy_m'(x) = -y_1(x) + y_m(x) \end{cases}$$

with the conditions $y_i(1) = 1$ for every $1 \le i \le m-1$ and $y_m(1) = 0$.

Proof. Due to Proposition 7, we have that $\Phi_{m,n}^1(n) = 1$ and

$$\Phi_{m,n}^1(k) = G_n(k) + H_n(k)\Phi_{m,n}^1(k+1),$$

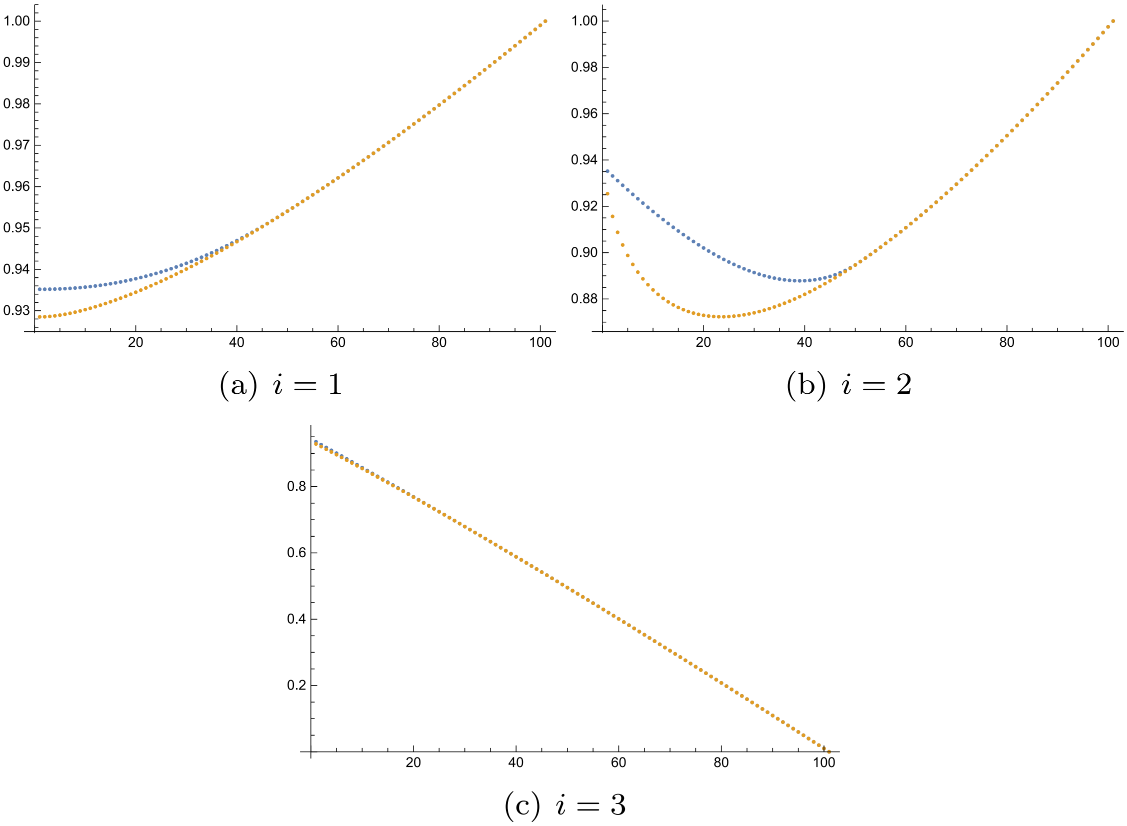

Fig. 1. $\Phi^{i}_{3,300}(k)$ (orange) and $\Psi^{i}_{3,300}(k)$ (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where

$$G_n(k) = \frac{m-1}{m\,n - m\,k + m - 1}\Phi^{2}_{m,n}(k),$$
$$H_n(k) = \frac{m\,n - m\,k}{m\,n - m\,k + m - 1}.$$

If we denote $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$, we are in a position to apply Proposition 4 with $g(x) = \dfrac{m-1}{m(1-x)}y_2(x)$ and $h(x) = \dfrac{m-1}{m(1-x)}$ and we can conclude that $y_1(1) = 1$ and that $y_1(x)$ satisfies the following ODE

$$y_1'(x) = h(x)y_1(x) - g(x) = \frac{m-1}{m(1-x)}y_1(x) - \frac{m-1}{m(1-x)}y_2(x).$$

The remaining equations arise in the same way just using Propositions 4 and 7 repeatedly. □

Remark 3. To prove the analyticity of the solution of the system of ODEs of the previous proposition it is enough to consider the autonomous system given by the equations above together with $x' = 1$. This system possesses a regular singularity at $x = 1, y_i = 1$. Its linear part is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1-m}{m} & \frac{1-m}{m} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{2-m}{m} & \frac{2-m}{m} & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{3-m}{m} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & \frac{-2}{m} & \frac{-2}{m} \\ -1 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{m} \end{pmatrix}$$

whose eigenvalues are clearly 1 (whose eigenspace is transverse to $x = 1$) and $\frac{i-m}{m}$ for $i = 1, \dots, m-1$, which are all negative and whose eigenspaces are included in $x = 1$. Applying the Hadamard–Perron Theorem [20, Chapter 1, Section 7], one obtains the existence of a unique invariant analytic manifold transverse to $x = 1$.

Theorem 2. *Let us assume that, for every $1 \le i \le m$, the sequence of functions $\{\phi^i_{m,n}\}_n$ converges uniformly on $[0, 1]$ to a continuous function $Y^m_m(x)$. Also, let $\mathbf{k}^m_n$ be the optimal threshold. Then, $\lim_n \mathbf{k}^m_n/n = \vartheta_m$, where ϑ_m is the solution to the equation $x = Y^m_m(x)$.*

Proof. Propositions 9 and 10 imply that we are in the conditions to apply Proposition 2 (taking $\mathcal{N}(n) = \mathbf{k}_n$) and the result follows immediately. □

Theorem 2 can be used to compute $\lim_n \mathbf{k}^m_n/n$ with arbitrary precision because, due to Proposition 10, we can obtain $Y^m_m(x)$ as a power series centered at $x = 1$. As an example we work out the case $m = 3$, but the reasoning would be essentially the same for any other value of m.

Corollary 1. *With the previous notation, and under the suitable assumptions about uniform convergence,*

$$\lim_n \mathbf{k}^3_n/n = \vartheta_3 = 0.49263576026053198177870853577593\dots$$

Proof. For the sake of simplicity, let us denote $f(x) = Y^3_3(x)$. First of all, we will take into account that (see Remark 3) $f(x)$ is analytic at $x = 1$ and for every $i > 1$,

$$\left|\frac{f^{(i+1)}(1)}{f^{(i)}(1)}\right| = \left|\frac{-9i^3 + 9i^2 - 2i + 2}{9i^2 + 9i + 2}\right| < i.$$

Consequently, the following power series has radius of convergence greater or equal than 1:

$$f(x) = \sum_{i=1}^{\infty} \frac{(x-1)^i f^{(i)}(1)}{i!}.$$

Now, consider the truncated series

$$\overline{f}(x) = \sum_{i=1}^{1000} \frac{(x-1)^i f^{(i)}(1)}{i!}.$$

Since $\left|\frac{f^{(i+1)}(1)}{f^{(i)}(1)}\right| < i$ and $f'(1) = -1$, it follows that $|f^{(i)}(1)| < i!$ so

$$|f(x) - \overline{f}(x)| = \left|\sum_{i=1001}^{\infty} \frac{(x-1)^i f^{(i)}(1)}{i!}\right| < \left|\sum_{i=1001}^{\infty} (1-x)^i\right| = \frac{(1-x)^{1001}}{x}.$$

Thus, for every $x \in [1/4, 1]$, we have that $|f(x) - \overline{f}(x)| < 4 \cdot 10^{-124}$ and in the exact same way we obtain that $|f'(x) - \overline{f}'(x)| < 4 \cdot 10^{-124}$.

Let us denote $\overline{\vartheta}$ such that $\overline{f}(\overline{\vartheta}) = \overline{\vartheta}$ and recall that ϑ_3 satisfies, by Theorem 2, that $f(\vartheta_3) = \vartheta_3$. Now, Lagrange's mean value theorem (applied to the function $g(x) := f(x) - x$) implies that, for some c between ϑ and ϑ_3 it holds that

$$|\vartheta_3 - \overline{\vartheta}| = \left|\frac{\overline{f}(\vartheta_3) - f(\vartheta_3)}{\overline{f}'(c) - 1}\right|.$$

We can choose c such that $|\overline{f}'(c) - 1| > 1$ because $\overline{f}'$ is negative in $(0, 1]$. Thus, it follows that $|\vartheta_3 - \overline{\vartheta}| < |\overline{f}(\vartheta_3) - f(\vartheta_3)| < 4 \cdot 10^{-124}$. But we can compute $\overline{\vartheta}$ with arbitrary precision so if we consider, for example,

$$\overline{\vartheta} = 0.49263576026053198177870853577593\dots$$

we are done. □

5. The probability of success

In order to compute the probability of success $\mathbf{P}^m_n$, we first consider the following event:

$Y^k_{m,n}$ = "succeed accepting the first nice candidate after having rejected k different candidates"

and we denote $\mathbf{P}^m_n(k) = p\left(Y^k_{m,n}\right)$. Clearly, $\mathbf{P}^m_n = \mathbf{P}^m_n\left(\mathbf{k}^m_n\right)$.

Moreover, let us define now the following events:

$Z^{k,i}_{m,n}$ ="the maximal candidate has been inspected i times when the kth different candidate is inspected for the first time"

and define $\Theta^{i}_{m,n}(k) = p\left(Z^{k,i}_{m,n}\right)$.

Clearly, the family $\{Z^{k,i}_{m,n}\}_{i=1}^{m}$ is a complete system of events. Thus, the law of total probability leads to

$$\begin{aligned}\mathbf{P}^{m}_{n}(k) = p\left(Y^{k}_{m,n}\right) = \sum_{i=1}^{m} p\left(Y^{k}_{m,n}|Z^{k,i}_{m,n}\right) p\left(Z^{k,i}_{m,n}\right) = \\ = \sum_{i=1}^{m} p\left(Y^{k,i}_{m,n}\right) p\left(Z^{k,i}_{m,n}\right) = \sum_{i=1}^{m} \Phi^{i}_{m,n}(k)\Theta^{i}_{m,n}(k).\end{aligned}$$

Consequently, in order to determine $\mathbf{P}^{m}_{n}(k)$ we need to determine the value of $\Theta^{i}_{m,n}(k)$ for every $1 \le i \le m$. We do so in the following series of results.

Proposition 11. *$\Theta^{1}_{m,n}(1) = 1$ and for every $1 < k \le n$, we have that*

$$\Theta^{1}_{m,n}(k) = \frac{1}{k} + \left(\frac{m-1}{k(mn-m(k-1)+m-1)} + \frac{mn-m(k-1)}{mn-m(k-1)+m-1} - \frac{1}{k}\right)\Theta^{1}_{m,n}(k-1).$$

Proof. First of all, $\Theta^{1}_{m,n}(1) = 1$ by definition because when the first candidate is inspected for the first time, it is obviously maximal.

Now, as we have already done before, we apply the law of total probability to get that

$$\Theta^{1}_{m,n}(k) = p\left(Z^{k,1}_{m,n}\right) = p\left(Z^{k,1}_{m,n}|Z^{k-1,1}_{m,n}\right) p\left(Z^{k-1,1}_{m,n}\right) + p\left(Z^{k,1}_{m,n}|\overline{Z^{k-1,1}_{m,n}}\right) p\left(\overline{Z^{k-1,1}_{m,n}}\right).$$

Since $p\left(Z^{k-1,1}_{m,n}\right) = 1 - p\left(\overline{Z^{k-1,1}_{m,n}}\right) = \Theta^{1}_{m,n}(k-1)$, we just have to compute the remaining terms:

- To compute $p\left(Z^{k,1}_{m,n}|\overline{Z^{k-1,1}_{m,n}}\right)$, we assume that the maximal candidate has appeared more than once when the $(k-1)$-th different candidate is inspected for the first time. Then, the only possible way in which the maximal candidate can appear once when the kth different candidate is inspected is that this kth candidate is in fact a maximal candidate. Since this happens with probability $1/k$, we have just seen that

$$p\left(Z^{k,1}_{m,n}|\overline{Z^{k-1,1}_{m,n}}\right) = \frac{1}{k}.$$

- To compute $p\left(Z^{k,1}_{m,n}|Z^{k-1,1}_{m,n}\right)$, we assume that the maximal candidate has appeared once when the $(k-1)$-th different candidate is inspected for the first time. Then, there are two possible ways in which the maximal candidate can appear once when the kth different candidate is inspected:
  - The maximal element appears again before the inspection of the kth different candidate and the kth different candidate is a new maximal. This happens with probability

$$\frac{m-1}{mn-m(k-1)+m-1} \cdot \frac{1}{k}.$$

  - The maximal element does not appear again before the inspection of the kth different candidate and the kth different candidate is non maximal. This happens with probability

$$1 - \frac{m-1}{mn-m(k-1)+m-1} = \frac{mn-m(k-1)}{mn-m(k-1)+m-1}.$$

  Consequently, we have just seen that

$$p\left(Z^{k,1}_{m,n}|Z^{k-1,1}_{m,n}\right) = \frac{m-1}{k(mn-m(k-1)+m-1)} + \frac{mn-m(k-1)}{mn-m(k-1)+m-1},$$

  and it is enough to combine all the previous computations to get the result. □

Proposition 12. *Let $i \geq 2$. Then, $\Theta^i_{m,n}(1) = 0$ and for every $1 < k \leq n$, we have that*

$$\Theta^2_{m,n}(k) = \frac{(k-1)\big(mn - m(k-1)\big)}{k(mn - m(k-1) + m - 2)}\Theta^2_{m,n}(k-1) + \\ + \frac{(k-1)(m-1)\big(mn - m(k-1)\big)}{k(mn - m(k-1) + m - 2)(mn - m(k-1) + m - 1)}\Theta^1_{m,n}(k-1)$$

and

$$\Theta^i_{m,n}(k) = \frac{k-1}{k}\frac{mn - m(k-1)}{mn - m(k-1) + m - i}\Theta^i_{m,n}(k-1) + \frac{m - i + 1}{mn - m(k-1) + m - i}\Theta^{i-1}_{m,n}(k)$$

for every $i \geq 3$

Proof. First of all, if $i \geq 2$, $\Theta^i_{m,n}(1) = 0$ by definition because if the first candidate has been inspected only once, it is impossible that the maximal element has appeared twice or more.

Note that the family $\{Z^{k-1,j}_{m,n}\}_{j=1}^{k-1}$ is a complete system of events. Thus, the law of total probability leads to

$$\Theta^i_{m,n}(k) = p\left(Z^{k,i}_{m,n}\right) = \sum_{j=1}^{k-1} p\left(Z^{k,i}_{m,n} | Z^{k-1,j}_{m,n}\right) p\left(Z^{k-1,j}_{m,n}\right).$$

Now, we have the following:

- If $j \geq i+1$, $p\left(Z^{k,i}_{m,n} | Z^{k-1,j}_{m,n}\right) = 0$ by definition. Every time we inspect a new different candidate for the first time the number of times the maximal candidate has appeared either stays the same (if no new copies of the maximal candidate appear in between and the new candidate is not maximal), increases (if new copies of the maximal candidate appear in between and the new candidate is not maximal) or decreases to 1 (if the new candidate is a new maximal).
- To compute $p\left(Z^{k,i}_{m,n} | Z^{k-1,i}_{m,n}\right)$, we assume that the maximal candidate has appeared i times when the $(k-1)$-th different candidate is inspected for the first time. Then, the only possible way in which the maximal candidate can appear i times when the kth different candidate is inspected is that the maximal element does not appear again before the inspection of the kth different candidate and the kth different candidate is not a new maximal. This happens with probability
$$\frac{mn - m(k-1)}{mn - m(k-1) + m - i} \cdot \frac{k-1}{k}.$$
- To compute $p\left(Z^{k,i}_{m,n} | Z^{k-1,j}_{m,n}\right)$ for $1 \leq j \leq i-1$, we assume that the maximal candidate has appeared j times when the $(k-1)$-th different candidate is inspected for the first time. Then, the only possible way in which the maximal candidate can appear i times when the kth different candidate is inspected is that the maximal element appears exactly $i-j$ times again before the inspection of the kth different candidate and the kth different candidate is not a new maximal. This happens with probability
$$\left(\frac{mn - m(k-1)}{mn - m(k-1) + m - i} \cdot \prod_{l=j}^{i-1} \frac{m - l}{mn - m(k-1) + m - l}\right) \cdot \frac{k-1}{k}.$$

Recall that, by definition $p\left(Z^{k-1,j}_{m,n}\right) = \Theta^j_{m,n}(k-1)$. Now, for $i = 2$ our previous considerations lead to

$$\Theta^2_{m,n}(k) = \sum_{j=1}^{2} p\left(Z^{k,2}_{m,n} | Z^{k-1,j}_{m,n}\right) \Theta^j_{m,n}(k-1) = \\ = \left(\frac{mn - m(k-1)}{mn - m(k-1) + m - 2}\frac{m-1}{mn - m(k-1) + m - 1}\frac{k-1}{k}\right)\Theta^1_{m,n}(k-1) + \\ + \left(\frac{mn - m(k-1)}{mn - m(k-1) + m - 2}\frac{k-1}{k}\right)\Theta^2_{m,n}(k-1) \tag{1}$$

as claimed.

On the other hand, for the case $i \geq 3$, we will proceed inductively. First, for $i = 3$ we have just seen that

$$\begin{aligned}\Theta^3_{m,n}(k) =& \frac{k-1}{k} \cdot \frac{mn-m(k-1)}{mn-m(k-1)+m-3}\Theta^3_{m,n}(k-1)+ \\ &+ \frac{mn-m(k-1)}{mn-m(k-1)+m-3} \cdot \frac{m-2}{mn-m(k-1)+m-2} \cdot \frac{k-1}{k}\Theta^2_{m,n}(k-1)+ \\ &+ \frac{mn-m(k-1)}{mn-m(k-1)+m-3} \cdot \frac{m-1}{mn-m(k-1)+m-1} \cdot \frac{m-2}{mn-m(k-1)+m-2} \cdot \frac{k-1}{k}\Theta^1_{m,n}(k-1)\end{aligned}$$

and the result follows from a direct computation, using the formula from (1).

Finally, if $i > 3$, it is enough to take into account that

$$\Theta^i_{m,n}(k) = \frac{k-1}{k}\frac{mn-m(k-1)}{mn-m(k-1)+m-i}\Theta^i_{m,n}(k-1) + \sum_{j=1}^{i-1} p\left(Z^{k,i}_{m,n}|Z^{k-1,j}_{m,n}\right)\Theta^j_{m,n}(k-1)$$

an proceed inductively to get the result. □

Now, for every $1 \leq i \leq m$, let us define $\theta^i_{m,n}(x) := \Theta^i_{m,n}(\lfloor xn \rfloor)$. Under suitable assumptions over the uniform convergence of the sequence $\{\theta^i_{m,n}\}_n$, we will be able to apply Proposition 3.

Proposition 13. *Let us assume that, for every $1 \leq i \leq m$, the sequence of functions $\{\theta^i_{m,n}\}_n$ converges uniformly on $[0, 1]$ to a function z^i_m which is continuous and differentiable on $[0, 1)$. Then,*

$$z^1_m(x) = \begin{cases} \frac{m\left(-1+(1-x)^{-\frac{1}{m}}\right)(1-x)}{x}; & \text{if } x \in (0,1], \\ 1; & \text{if } x = 0. \end{cases}$$

$$z^i_m(x) = \begin{cases} \frac{m!\left(1-(1-x)^{\frac{1}{m}}\right)^i (1-x)^{\frac{m-i}{m}}}{i!\ x\ (m-i)!}; & \text{if } x \in (0,1], \quad (2 \leq i \leq m). \\ 0; & \text{if } x = 0. \end{cases}$$

Proof. Taking into account Proposition 11, we have that $\Theta^1_{m,n}(1) = 1$ and

$$\Theta^1_{m,n}(k) = G_n(k) + H_n(k)\Theta^1_{m,n}(k-1),$$

where

$$G_n(k) = \frac{1}{k},$$

$$H_n(k) = \frac{m-1}{k(mn-m(k-1)+m-1)} + \frac{mn-m(k-1)}{mn-m(k-1)+m-1} - \frac{1}{k}.$$

Let $h_n(x) := n(1 - H_n(\lfloor nx \rfloor))$ and $g_n(x) := nG_n(\lfloor nx \rfloor)$, we apply Proposition 3 with $g(x) = \frac{1}{x}$ and $h(x) = \frac{m-x}{mx-x^2}$ to conclude that z^1_m satisfies the following ODE on $(0, 1)$:

$$y'(x) = g(x) - h(x)y(x) = \frac{1}{x} - \frac{m-x}{mx-mx^2}y(x).$$

Since $z^1_m(x)$ is continuous at $x = 0$ with $z^1_m(0) = 1$, it is enough to solve this ODE to get that

$$z^1_m(x) = \frac{m\left(-1+(1-x)^{-\frac{1}{m}}\right)(1-x)}{x}$$

as claimed.

Finally, the remaining cases can we inductively worked out in a similar way using Proposition 3 again but just considering the recursive relations and initial conditions from Proposition 12. In fact, we obtain the following ODE for each $2 \leq i \leq m$:

$$y_i'(x) = \frac{m!(-i+m+1)(1-x)^{\frac{1-i}{m}}\left(1-(1-x)^{\frac{1}{m}}\right)^{i-1}}{mx(i-1)!(-i+m+1)!} + \frac{(m-ix)}{mx-mx^2}y_i(x)$$

and the initial condition $y_i(0) = 0$. We provide no further details. □

Recall that $\mathbf{P}^m_n(k) = \sum_{i=1}^m \Phi^i_{m,n}(k)\Theta^i_{m,n}(k)$. If we now define $\mathbf{p}^m_n(x) := \mathbf{P}^m_n(\lfloor nx \rfloor)$, the following result is straightforward.

Corollary 2. *Under the assumptions from Propositions 10 and 13, the sequence of functions* $\{\mathbf{p}_n^m\}_n$ *converges uniformly on* $[0,1]$ *to the continuous function*

$$\pi_m(x) := \sum_{i=1}^{m} Y_m^i(x) z_m^i(x).$$

This corollary leads immediately to the final result of the paper, that allows us to determine the asymptotic probability of success. Recall that ϑ_m is the solution to the equation $Y_m^m(x) = x$.

Theorem 3. *Under the suitable assumptions about uniform convergence,*

$$\lim_n \mathbf{P}_n^m = \pi_m(\vartheta_m).$$

Proof. Taking into account the definition of $\mathbf{P}_n^m$, the fact that the sequence $\{\mathbf{p}_n^m\}$ converges uniformly to the continuous function π_m, and considering Theorem 2, we have that:

$$\lim_n \mathbf{P}_n^m = \lim_n \mathbf{P}_n^m(\mathbf{k}_n^m) = \lim_n \mathbf{p}_n^m\left(\frac{\mathbf{k}_n^m}{n}\right) = \left(\lim_n \mathbf{p}_n^m\right)\left(\lim_n \frac{\mathbf{k}_n^m}{n}\right) = \pi_m(\vartheta_m),$$

as claimed. □

Example 2. In Corollary 1 we computed ϑ_3. Now, we use the previous results to compute $\lim_n \mathbf{P}_n^3$. To do so, we express $Y_3^i(x)$ as a power series centered at $x = 1$:

$$Y_3^1(x) \approx 1 + \frac{x-1}{10} + \frac{9}{560}(x-1)^2 - \frac{171(x-1)^3}{30800} + \frac{14193(x-1)^4}{5605600} - \frac{1036089(x-1)^5}{762361600} + \cdots$$
$$Y_3^2(x) \approx 1 + \frac{x-1}{4} + \frac{9}{140}(x-1)^2 - \frac{171(x-1)^3}{5600} + \frac{14193(x-1)^4}{800800} - \frac{1036089(x-1)^5}{89689600} + \cdots$$
$$Y_3^3(x) \approx -(x-1) - \frac{1}{20}(x-1)^2 + \frac{19(x-1)^3}{1680} - \frac{1577(x-1)^4}{369600} + \frac{115121(x-1)^5}{56056000} + \ldots$$

On the other hand, Proposition 13 leads to:

$$z_3^1(x) = -\frac{3\left(\frac{1}{\sqrt[3]{1-x}} - 1\right)(x-1)}{x}$$
$$z_3^2(x) = \frac{3\left(\sqrt[3]{1-x} - 1\right)^2 \sqrt[3]{1-x}}{x}$$
$$z_3^3(x) = -\frac{\left(\sqrt[3]{1-x} - 1\right)^3}{x}$$

If we combine the previous expressions with Corollaries 2 and 3 we get that

$$\lim_n \mathbf{P}_n^3 = \pi_3(\vartheta_3) = \sum_{i=1}^{3} Y_3^i(\vartheta_3) z_3^i(\vartheta_3) = 0.93486905929575053539075628931\ldots$$

As we can see in Fig. 2 the value ϑ_3, which is the solution of $Y_3^3(x) = x$, coincides with the maximum of the function $\pi_3(x)$.

6. Final comments

To close the paper, we are to make some comments relating our results to the five open problems proposed by Garrod that were presented in the introduction.

- The first open problem asked for an improvement on the formula to compute $\mathbf{P}_n^m$. Even if we have not given a closed formula, in Section 3 we have obtained a recursive formula that allows its computation in linear time (with respect to the number of candidates n).
- Open problems 2 and 3 have been partially solved, but our proofs are conditioned to assume the uniform convergence of certain sequences of functions. Nevertheless, under these assumptions, we have provided a method to compute them with arbitrary precision.
- Finally, regarding open problems 4 and 5, we have been able to find the value of $\lim_n(\mathbf{k}_n^m/n)$ as the root of the equation $Y_m^m(x) = x$. This root can be approximated using a power-series whose coefficients can be computed with arbitrary precision. However, it does not seem like it can be expressed in terms of elementary or special functions. In addition, $\lim_n(\mathbf{P}_n^m)$ can be computed just using the fact that $\lim_n(\mathbf{P}_n^m) = \pi_m(\vartheta_m)$.

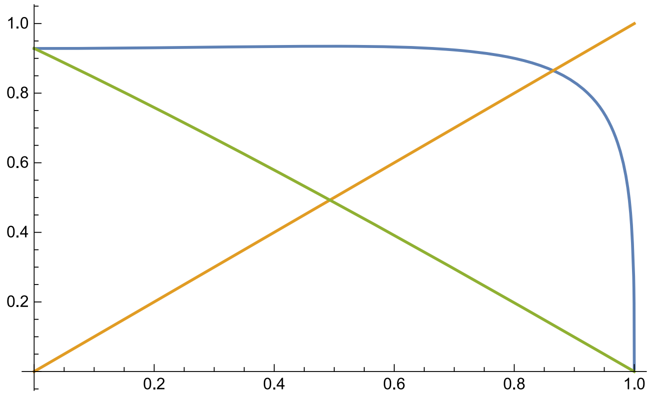

Fig. 2. $f(x) = x$ (orange), $Y_3^3(x)$ (green) and $\pi_3(x)$ (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 4
Optimal threshold, probability of success and asymptotic values using our results.

m	$\mathbf{k}_{100}^m$	$\mathbf{k}_{1000}^m$	$\mathbf{k}_{10000}^m$	$\lim_n \left(\frac{\mathbf{k}_n^m}{n}\right)$	$\mathbf{P}_{100}^m$	$\mathbf{P}_{1000}^m$	$\mathbf{P}_{10000}^m$	$\lim_n \mathbf{P}_n^m$
1	38	369	3679	0.367879441	0.37104277	0.36819561	0.36791104	0.3678794
2	48	471	4710	0.470926543	0.76970661	0.76814759	0.76799160	0.7679742
3	50	493	4927	0.492635760	0.93518916	0.93490075	0.93487222	0.9348690
4	50	499	4981	0.498053032	0.98310787	0.98307710	0.98307411	0.9830737
5	50	500	4995	0.499479760	0.99561947	0.99561715	0.99561693	0.9956169
6	50	500	4999	0.499861014	0.99885461	0.99885447	0.99885446	0.9988544
7	50	500	5000	0.499963006	0.99969900	0.99969899	0.99969899	0.9996989
8	50	500	5000	0.499990198	0.99992082	0.99992082	0.99992082	0.9999208
9	50	500	5000	0.499997415	0.99997920	0.99997920	0.99997920	0.9999792
10	50	500	5000	0.499999321	0.99999455	0.99999455	0.99999455	0.9999945

Finally, we provide the following table showing the optimal threshold $\mathbf{k}_n^m$ and the probability of success $\mathbf{P}_n^m$ for some values of n and $1 \leq m \leq 10$. We also provide the asymptotic values computed using Taylor series of the appropriate degree. It is worth comparing Table 4 with Table 1.

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