Second International Conference on Multidisciplinary
Design Optimization and Applications,
2-5 September 2008, Gijon, Spain
www.asmdo.com/conference2008/

Influence of the elevation-storage curve in the optimization of hydro-plants

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Abstract
One of the most important physical characteristics of a reservoir is the elevation-storage curve. In most cases, linear simplification is applied to this curve, which is approximated as a linear relationship. However, this simplification can produce serious errors in the optimal solution. In this paper we consider a non-linear elevation-storage curve to be applied to the reservoir and therefore consider a non-linear problem. To obtain the optimal solution, the problem is formulated within the framework of optimal control theory using Pontryagin's Maximum Principle. The advantage of this technique with respect to previous methods lies in the possibility of obtaining theoretical results whose implementation (and computational complexity) is feasible regardless of the non-linear characteristics of the problem. Results of the application of the method to a numerical example are presented and we show the differences between the two approaches.

1 Introduction

This paper deals with the influence of the design of the reservoir in the optimization of hydro-plants, one of the important optimization problems in a hydrothermal power system. We represent the hydro-plant at a high level of detail and include inter-temporal constraints such as hydro reserves. We consider a non-linear elevation-storage curve to be applied to the reservoir and therefore consider a non-linear problem.

In a hydro-plant, power is derived by converting the potential energy of water to electrical energy using a hydraulic turbine which is connected to a generator. The output power P (Mw) is given by

$$P = \frac{qh}{G} \tag{1}$$

where q is the rate of water discharge (m^3/h), h is the effective water head (m), and G is the efficiency $(m^4/\text{h}\,.Mw)$. One of the most important physical characteristics of a reservoir is the elevation-storage curve (Fig. 1). Elevation y (m) and volume s (m^3) are physical relationships linked to each other by the topography of the surrounding area. A mathematical function of the elevation-storage curve for each reservoir needs to be approximated. In most cases, linear simplification is applied, and y is approximated as a linear relationship:

$$y = \alpha_0 + \alpha_1 s \tag{2}$$

where y is the elevation of the water surface above a given reference level, and α_i are parameters computed from measured elevation-storage data.

However this simplification can produce serious errors in the optimal solution. In this paper we consider the curve to be approximated by second-order polynomial functions:

$$y = \alpha_0 + \alpha_1 s + \alpha_2 s^2 \tag{3}$$

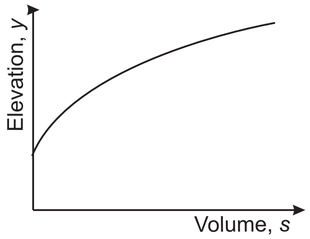


Figure 1. Typical reservoir elevation-storage curve.

Hydraulic optimization problems have been the subject of intensive research and several methods have been widely employed in their solution: dynamic programming (Wood et al. [7]), mixed integer linear programming (Torre et al. [5]), Lagrangian relaxation (Redondo et al. [3]), genetic algorithms (Zoumas et al. [8]), evolutionary programming (Sinha et al. [4]), among others. However, most of these approaches require substantial simplifying assumptions and present certain disadvantages. The main drawback to the majority of these methods is the difficulty of treating non-linear large-scale systems.

Of the many problems concerned with ascertaining optimal hydro power, we shall focus in the present paper on the problem faced by a generation company in a deregulated electricity market when preparing its offers for the day-ahead market for one hydro-plant. Our model of the spot market represents the price of electricity $\pi(t)$ as a known exogenous variable. To obtain the optimal solution, the problem is formulated within the framework of optimal control theory (OCT) (Vinter [6]) using Pontryagin's Maximum Principle (PMP). This feature distinguishes our work from all the aforementioned studies. The advantage of this technique with respect to previous methods lies in the possibility of obtaining theoretical results whose implementation (and computational complexity) is feasible regardless of the non-linear characteristics of the problem.

The paper is organized as follows. In Section 2, we pay special attention to the mathematical modeling of hydro power, considering two approximations for the elevation-storage curve: one linear and the other quadratic. In Section 3, we set out our non-linear problem in terms of optimal control in continuous time, with the Lagrange-type functional and use PMP to obtain the optimal solution. We present the optimization algorithm that leads to determination of the optimal solution of the hydro-plant. The algorithm is implemented in the commercial program Mathematica. The results of the application of the method to a numerical example are presented in Section 4, in which the differences between the two approaches are highlighted. Finally, Section 5 summarizes the main conclusions of our research.

2 Mathematical models

The appropriate choice of mathematical models for representing the physical system is a crucial aspect when addressing any optimization problem. Many models for hydro-plant performance exist, for example: Glimn-Kirchmayer, Hildebrand, Hamilton-Lamont and Arvanitidis-Rosing. We consider the approximation presented by El-Hawary [2] to be the most appropriate due to its precision and flexibility. Let us now see the chosen modeling for each element of the problem.

2.1 Effective head model

The effective hydraulic head h at the hydro-plant is equal to the difference between the gross head h_g and the head losses in the penstock h_p

$$h = h_g - h_p \tag{4}$$

The gross head h_g is defined as the difference between the forebay elevation y and the tailrace elevation y_T

$$h_g = y - y_T \tag{5}$$

The tailrace elevation is a function of the discharge q as well as the spillage σ. We assume a linear relationship between the two variables expressed by the following relationship:

$$y_T = y_{T_0} + B_T(q + \sigma) \tag{6}$$

The forebay elevation is a function of the geometry of the reservoir, natural water inflow, spillage and water discharge. It is thus necessary to consider reservoir modeling in the case of variable head hydro-plants.

A typical linear variation between the head loss characteristic and discharge may be assumed:

$$h_p = h_{p_0} + B_p q \tag{7}$$

The resulting expression for the effective head is thus

$$h(t) = y(t) - [(y_{T_0} + h_{p_0}) + B_T\sigma(t) + (B_T + B_p)q(t)] \tag{8}$$

2.2 Reservoir model

A reservoir model of interest in our optimization problem is a realistic one that relates the plant's forebay elevation y to the discharge q. These parameters determine the active power generation available from the hydro-plant. The reservoir's dynamics may be suitably described by the equation

$$\frac{ds(t)}{dt} = i(t) - q(t) - \sigma(t) \tag{9}$$

where $s(t)$ is the reservoir storage, $i(t)$ the rate of natural inflow adjusted for evaporation and seepage losses, $q(t)$ the rate of water release through the hydro-plant and $\sigma(t)$ the rate of water spillage.

The variation in storage in a reservoir of regular shape with elevation can be computed using formulas for the volumes of solids. For example, if we assume a trapezoidal reservoir representation (Fig. 2), the forebay elevation $y(t)$ is related to the forebay volume of stored water $s(t)$ by the relation

$$s(t) = b_0 \cdot l \cdot y(t) + l \cdot \tan\phi \cdot y^2(t) \tag{10}$$

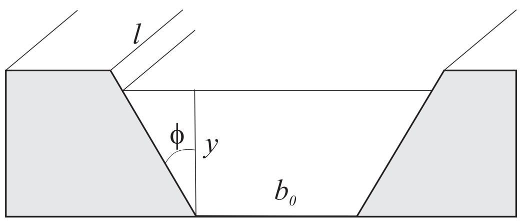


Figure 2. Trapezoidal reservoir configuration.

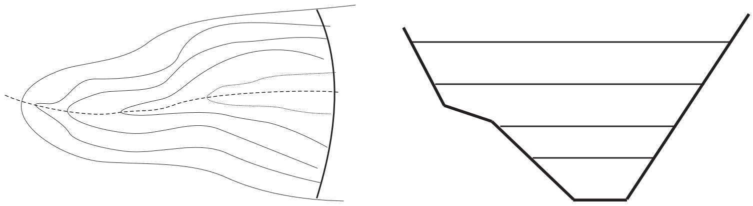
If we consider vertical-sided reservoirs, we have

$$y(t) = B_y s(t) \tag{11}$$

where $B_y = 1/b_0 \cdot l$ denote the inverse of the reservoir surface.

The variation of storage with elevation for reservoirs at natural sites is determined from the elevation-storage curve. This curve can be calculated from the topography of the surrounding area using commercial software by simply introducing the elevation and the area enclosed within each contour within the reservoir site (see Fig. 3). Thus, if we know a number of points of the elevation-storage curve, it can be approximated (ordinary least-squares polynomial regression) by a general mathematical model of the form:

$$y(t) = \sum_{i=0}^{N} \alpha_i s^i(t) \tag{12}$$



(a) Contour map of the reservoir. (b) Side view.

Figure 3. Determination of the elevation-storage curve.

It should be noted that natural factors (for example sediment accumulation) will change the configuration of the reservoir over time and that the reservoir model needs to be updated periodically.

2.3 Hydro-plant model

As can be seen in (1), the hydro-plant's active power generation is given by

$$P(t) = \frac{q(t)h(t)}{G} \tag{13}$$

We shall use (8) to calculate the effective hydraulic head h and shall consider two approximations for $y(t)$: (a) the classic linear relationship, and (b) the second-order approximation, to then compare the results. For the sake of simplicity, we assume the following to be negligible: $\sigma(t)$ the rate of water spillage, and h_p the penstock head losses. Thus, we have

$$h(t) = y(t) - [y_{T_0} + B_T q(t)] \tag{14}$$

(a) **Linear elevation-storage curve.** If we choose

$$y(t) = y_0 + B_y s(t) \tag{15}$$

expression (13) can be written as

$$P(t) = \frac{q(t)}{G}[y_0 - y_{T_0} + B_y s(t) - B_T q(t)] \tag{16}$$

From (9), we have that

$$\frac{ds(t)}{dt} = i(t) - q(t) \tag{17}$$

In general, as the natural inflow i is assumed constant,

$$s(t) = s(0) + i \cdot t - \int_0^t q(r)dr = S_0 + i \cdot t - Q(t) \tag{18}$$

$Q(t)$ being the volume discharged up to the instant t by the plant and S_0 the initial volume.

$$P(t) = \frac{q(t)}{G}[(y_0 - y_{T_0}) + B_y(S_0 + i \cdot t - Q(t)) - B_T q(t)] \tag{19}$$

For convenience of formulation, we introduce this new notation: $q(t) \equiv \dot{z}(t); Q(t) \equiv z(t)$ and we have that

$$P(t, z(t), \dot{z}(t)) := A(t) \cdot \dot{z}(t) - B \cdot z(t) \cdot \dot{z}(t) - C \cdot \dot{z}^2(t) \tag{20}$$

with

$$A(t) = \frac{(y_0 - y_{T_0}) + B_y(S_0 + i \cdot t)}{G}; \; B = \frac{B_y}{G}; \; C = \frac{B_T}{G} \tag{21}$$

This is a *variable-head* model and the hydro-plant's hydraulic generation P is a function of $z(t)$ and $\dot{z}(t)$.

(b) **Quadratic elevation-storage curve.**
If we choose

$$y(t) = y_0 + B_y s(t) + C_y s^2(t) \tag{22}$$

following the same steps as in (a), we have that

$$P(t, z(t), \dot{z}(t)) := A(t) \cdot \dot{z}(t) - B \cdot z(t) \cdot \dot{z}(t) - C \cdot \dot{z}^2(t) + D \cdot z^2(t) \cdot \dot{z}(t) \tag{23}$$

with

$$A(t) = \frac{(y_0 - y_{T_0}) + B_y(S_0 + i \cdot t) + C_y(S_0 + i \cdot t)^2}{G};\ B = \frac{B_y + 2C_y(S_0 + i \cdot t)}{G};\ C = \frac{B_T}{G};\ D = \frac{C_y}{G} \tag{24}$$

3 Statement of the problem and optimal solution

Let $P(t, z(t), \dot{z}(t))$ be the function of the hydro-plant's hydraulic generation, $z(t)$ being the volume that is discharged up to the instant t by the plant, and $\dot{z}(t)$ the rate of water discharge of the plant at the instant t. If we assume that b is the volume of water that must be discharged during the entire optimization interval $[0, T]$, the following boundary conditions will have to be fulfilled

$$z(0) = 0, z(T) = b \tag{25}$$

For the sake of convenience, throughout the paper we assume that the function of effective hydraulic generation $P(t, z, \dot{z}) : \Omega_P = [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is strictly increasing with respect to the rate of water discharge $\dot{z}$, i.e., $P_{\dot{z}} > 0$. Let us also assume that $P(t, z, \dot{z})$ is concave with respect to $\dot{z}$, i.e., $P_{\dot{z}\dot{z}} < 0$. The real models meet these two constraints; the former means more power to a higher rate of water discharge. It can be seen that we shall only admit non-negative volumes, $z(t)$, and rates of water discharge, $\dot{z}(t)$. Besides the previous statement, we consider $P(t, z(t), \dot{z}(t))$ to be bounded by technical restrictions

$$P_{\min} \leq P(t, z(t), \dot{z}(t)) \leq P_{\max}, \forall t \in [0, T] \tag{26}$$

In our problem, the objective function is given by revenue during the optimization interval $[0, T]$

$$F(z) = \int_0^T L(t, z(t), \dot{z}(t))dt = \int_0^T \pi(t)P(t, z(t), \dot{z}(t))dt \tag{27}$$

Revenue is obtained by multiplying the hydraulic production of the company by the clearing price $\pi(t)$ at each hour t. In keeping with the previous statement, our objective functional in continuous time form is

$$\max_z F(z) = \max_z \int_0^T \pi(t)P(t, z(t), \dot{z}(t))dt \tag{28}$$

on the set

$$\Omega = \left\{ z \in \widehat{C}^1[0, T] | \begin{array}{c} z(0) = 0, z(T) = b \\ P_{\min} \leq P(t, z(t), \dot{z}(t)) \leq P_{\max}, \forall t \in [0, T] \end{array} \right\} \tag{29}$$

where $\widehat{C}^1$ is the set of piecewise C^1 functions.

To obtain the optimal solution, the problem is formulated in this paper within the framework of Optimal Control Theory (OCT).

If z satisfies Euler's equation for the functional F, we have that, $\forall t \in [0, T]$, Euler's equation is fulfilled

$$L_z(t, z(t), \dot{z}(t)) - \frac{d}{dt} L_{\dot{z}}(t, z(t), \dot{z}(t)) = 0 \tag{30}$$

If we divide Euler's equation by $L_{\dot{z}}(t, z(t), \dot{z}(t)) > 0$, $\forall t$, and integrate, we have that

$$L_{\dot{z}}(t, z(t), \dot{z}(t)) \cdot \exp\left[-\int_0^t \frac{P_z(s, z(s), \dot{z}(s))}{P_{\dot{z}}(s, z(s), \dot{z}(s))} ds \right] = K \in \mathbb{R}^+, \forall t \in [0, T] \tag{31}$$

We shall call this relation the coordination equation for $z(t)$, and the positive constant $K \in \mathbb{R}^+$ will be termed the coordination constant of the extremal. Let us term the coordination function of $z \in \Omega$ the function in $[0, T]$, defined as follows

$$\mathbb{Y}_z(t) = L_{\dot{z}}(t, z(t), \dot{z}(t)) \cdot \exp\left[-\int_0^t \frac{P_z(s, z(s), \dot{z}(s))}{P_{\dot{z}}(s, z(s), \dot{z}(s))} ds\right] \tag{32}$$

We present the problem considering the control variable $u(t) = P(t, z(t), \dot{z}(t))$ and the state variable to be $z(t)$. Moreover, as $P_{\dot{z}} > 0$, the state equation $\dot{z} = f(t, z, u)$ can be explicitly defined. The optimal control problem is thus:

$$\max_{u(t)} \int_0^T L(t, u(t))dt \quad \text{with} \quad \begin{cases} \dot{z} = f(t, z, u) \\ z(0) = 0, \quad z(T) = b \\ u(t) \in \{x \mid P_{\min} \leq x \leq P_{\max}\} \end{cases} \tag{33}$$

Using Pontryagin's Minimum Principle (PMP) (Vinter [6]), it is easy to prove (Bayón et al. [1]) the following theorem:

Theorem. *If $z^* \in \widehat{C}^1$ is a solution of our problem, then $\exists K \in \mathbb{R}^+$ such that:*

$$\mathbb{Y}_{z^*}(t) \;\; is \;\; \begin{cases} \leq K & if \;\; P(t, z^*(t), \dot{z}^*(t)) = P_{\min} \\ = K & if \;\; P_{\min} < P(t, z^*(t), \dot{z}^*(t)) < P_{\max} \\ \geq K & if \;\; P(t, z^*(t), \dot{z}^*(t)) = P_{\max} \end{cases} \tag{34}$$

On the basis of this theorem, we now present the optimization algorithm that leads to the determination of the optimal solution of the hydro-plant.

To obtain the optimum operating conditions of the hydro-plant, we shall use the coordination equation

$$\mathbb{Y}_z(t) = K, \forall t \in [0, T] \tag{35}$$

The problem will consist in finding for each K the function z_K that satisfies $z_K(0) = 0$ and the conditions of the theorem, and from among these functions, the one that gives rise to an admissible function ($z_K(T) = b$). From the computational point of view, the construction of z_K can be performed using the same procedure as in the shooting method, employing a discretized version of the coordination equation. The exception is that at the instant when the values obtained for z and $\dot{z}$ do not obey the constraints, we force the solution z_K to belong to the boundary until the moment when the conditions of leaving the domain (established in the theorem) are fulfilled. A more detailed explanation of this algorithm can be consulted in Bayón et al. [1].

4 Example

A program was written using the Mathematica package to apply the results obtained in this paper to an example of a hydrothermal system made up of one variable-head hydro-plant. The hydro-plant data are summarized in Table I.

Table I: Hydro-plant coefficients.

$G(m^4/\text{h}.Mw)$	$b(m^3)$	$i(m^3/\text{h})$	$S_0(m^3)$	$y_{T_0}(m)$	$B_T(\text{h}\,m^{-2})$
319840	$50\ 10^6$	133200	$2.395\ 10^8$	5	$2.94\ 10^{-7}$

We shall also consider the technical restrictions: $P_{\min} = 0; P_{\max} = 100$. In order for the comparison of the two models (see Fig. 4) to be reliable, we shall consider the real elevation-storage curve to fit the following quadratic model perfectly ($r^2 = 1$) :

$$y(t) = y_0 + B_y s(t) + C_y s^2(t) \tag{36}$$

with $y_0 = 5(m); B_y = 4.3407910^{-8}(m^{-2})$; $C_y = -2.8938610^{-17}(m^{-5})$. We now perform a least-squares polynomial regression to a linear model:

$$y(t) = y_0 + B_y s(t) \tag{37}$$

obtaining for the fit: $y_0 = 6.18166$; $B_y = 2.8938610^{-8}(m^{-2})$ with $r^2 = 0.983$.

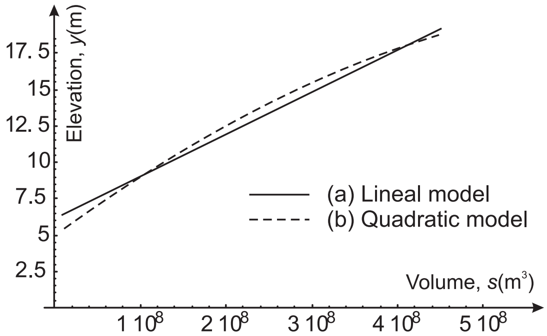


Figure 4. Two approximations for $y(t)$.

An optimization interval of $T = 24$ h. was considered, with a discretization of 24 subintervals. The solution may be constructed in a simple way by taking into account the above theorem. The secant method was used to calculate the approximate value of K for which $z_K(T) - b = 0$. We obtain $K = 0.001869288935030247$ in 3 iterations for the lineal model and $K = 0.002045271714939654$ in 10 iterations for the quadratic model.

Table II presents the optimal solution (optimal rate $q(10^6 m^3/h)$ and optimal hydro power P(MW)), together with the clearing price $\pi(euro/h{\cdot}\mathrm{MW})$ for $t = 1, \ldots, 24$ (h) corresponding to the Spanish market. We use the superscript l to denote the solution obtained using the linear model and the superscript q the solution obtained with the quadratic model.

Table II: Optimal solution and clearing price.

t	q^l	P^l	q^q	P^q	π	t	q^l	P^l	q^q	P^q	π
1	0.5861	14.52	0.4029	10.84	76.93	13	4.1436	81.41	4.2913	91.69	104.08
2	0.	0.	0.	0.	68.20	14	3.8006	74.61	3.9145	83.57	100.00
3	0.	0.	0.	0.	68.20	15	1.6811	36.04	1.5830	36.96	80.50
4	0.	0.	0.	0.	60.00	16	1.3738	29.68	1.2447	29.31	78.23
5	0.	0.	0.	0.	55.01	17	1.1935	25.87	1.0459	24.73	76.93
6	0.	0.	0.	0.	56.28	18	1.2002	25.89	1.0532	24.80	76.93
7	0.	0.	0.	0.	69.47	19	2.8251	56.05	2.8457	61.57	90.00
8	0.4337	10.82	0.2325	6.31	75.79	20	4.3167	78.09	4.4924	88.48	106.89
9	4.2426	89.50	4.3555	100.	105.90	21	4.0357	72.62	4.1837	81.90	103.00
10	4.3044	88.93	4.4507	100.	106.50	22	3.8114	68.10	3.9377	76.47	100.00
11	4.6031	91.98	4.5545	100.	110.00	23	1.4749	29.34	1.3675	29.62	76.93
12	4.4909	88.43	4.6684	100.	108.46	24	1.4817	29.29	1.3752	29.60	76.93

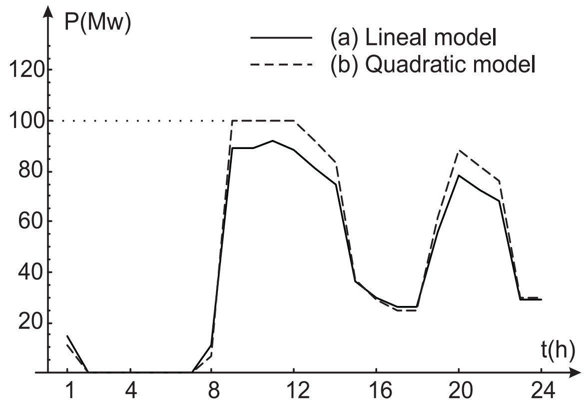


Figure 5. Optimal hydro-power P.

Figure 5 presents the optimal hydro-power P for the two approaches. This figure allows a better appreciation of the difference between the optimum solutions corresponding to the two models. This difference is of up to $10 - 15\%$ at peak hours, despite the fact that the linear approximation of $y(t)$ (in the light of its r^2) may be considered very good. It can also be appreciated that the minimum constraint affects the optimum solution in both cases, while the maximum constraint (100 MW) only does so in the quadratic case P^q.

Furthermore, it should be noted how the different behavior of the plan produces a discrepancy in the optimum profits depending on the model employed. The greater degree of approximation of the elevation-storage curve will enable optimum power generation to be achieved at the plant, thus guaranteeing increased profits. The benefits of the optimal solution is 97936 *euro* using the linear model and 107021 *euro* with the quadratic model. The algorithm runs very quickly, the CPU time employed for both models being 4.0 sec on a personal computer (Pentium IV/$2GHz$).

5 Conclusions and future perspectives

As we have shown in this paper, linear simplification applied to the elevation-storage curve produces serious errors in the optimal solution of the hydraulic problem. Perfect knowledge of the optimum power on the part of electricity generation companies is very important, since in the new pool-based electricity market, the drawing up of next-day prices is based (among other aspects) on this datum. It is therefore recommendable to model this curve in detail and even to update the model periodically in order to take into account the effect of sediments, for instance, which is another crucial aspect not usually considered.

We believe the results presented in this paper open up many future lines of research, such as for example: higher-degree models for y, the consideration of the tailrace elevation y_T also as a quadratic function of the discharge q, or the analysis of the influence of the rate of water spillage $\sigma(t)$ and penstock head losses h_p on the optimum solution.

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