Optimal Control of Counter-Terrorism Tactics

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1 Introduction

Modeling a "stock" of terrorists, is not common, but has precedents, especially after September 11, 2001 [1]. In this sense [2] presents an intelligent ecological metaphor to analyze actions by Governments and citizens against terror. In [3] a model for the transmission dynamics of extreme ideologies in vulnerable populations is presented. In [4] the authors propose a terror-stock model that treats the suicide bombing attacks in Israel. In other countries like, for example, Spain or Ireland, the problem has also been analyzed.

Several papers develop dynamical models of terrorism. In [5] the authors incorporate the effects of both military/police and nonviolent/persuasive intervention to reduce the terrorist population. This idea is widely developed in [6] where the controls are two types of counter-terror tactics: "water" and "fire", which is the model we shall consider in this paper.

In this context we present in this work a new approach to analyze the efficacy of counter-terrorism tactics. We state an optimal control problem that attempts to minimize the total cost of terrorism. An excellent summary of optimal control application in terrorism issues can be consulted in [7].

The optimization criterion is to minimize the discounted damages created by terror attacks plus the costs of counterterror efforts. The underlying mathematical problem is complicated. It constitutes a multi-dimensional,

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constrained problem where the optimization interval is infinite. An important feature is that the time t is not explicitly present in the problem (hence, it is a time-autonomous problem), except in the discount factor. Using the Minimum Principle of Pontryagin, the shooting method and the cyclic descent of coordinates we develop an optimization algorithm. We alse present a method (based upon [8]) for computing the optimal steady-state in multi-control, infinite-horizon, autonomous models. This method does not require the solution of the dynamic optimization problem. Using it, we can choose parameters that reach a desirable steady-state solution.

2 Mathematical Model

In this work we use the excellent model provided by [6], which classifies counter-terrorism tactics into two categories:

- "Fire" strategies are tactics that involve significant collateral damage. They include, for example, the killing of terrorists through drones, the use of indiscriminate checkpoints or the aggressive blockade of roads.
- "Water" strategies, on the other hand, are counter-measures that do not affect innocent people, like intelligence arrests against suspects individuals.

The fire and water strategies will be denoted by the control variables $v(t)$ and $u(t)$, respectively. Both controls have their advantages, and their drawbacks. For example $v(t)$ have the direct benefit of eliminating current terrorists but the undesirable indirect effect of stimulating recruitment rates (and the possible harm to innocent bystanders). On the other hand, $u(t)$ is more expensive and more difficult to be applied than $v(t)$.

The strength or size of the terrorists is represented by the state variable $x(t)$. This includes not only the number of active terrorists, but also the organization's total resources including financial resources, weapons, etc. [2]. Its value changes over time and we distinguish two inflows and three outflows in it:

$$\dot{x} = \tau + I(v,x) - O_1(x) - O_2(u,x) - O_3(v,x) \tag{1}$$

We include first of all a small constant recruitment term τ, accounting for a small constant recruitment rate. Second, following [3], the model considers that new terrorists are recruited by existing terrorists. So the inflow $I(v, x)$ is increasing in proportion to the current number of terrorists x. But this growth is bounded and should also slow down. Moreover, the aggressive control v, also increases recruitment. In summary the form of the model is:

$$I(v, x) = (1 + \rho v)kx^{\alpha} \tag{2}$$

with $\tau, \rho \geq 0$, $k > 0$ and $0 \leq \alpha \leq 1$.

On the other hand, we consider three outflows: The first one, $O_1(x)$, represents the rate at which people leave the organization by several reasons not related with the controls. This natural outflow is assumed linear in x:

$$O_1(x) = \mu x \tag{3}$$

with $\mu > 0$. The second outflow, $O_2(u, x)$, reflects the effects of water strategies. This outflow is assumed to be concave in x because there is a limited number of units that conduct water operations:

$$O_2(u, x) = \beta(u)x^{\theta} \tag{4}$$

with $\theta \leq 1$. The third outflow $O_3(v, x)$ is due to fire strategies. This is modeled as linear in x, because the methods are perceived to be "direct attack":

$$O_3(v, x) = \gamma(v)x \tag{5}$$

The functions $\beta(u)$ and $\gamma(v)$ should be concave; Culkins [6] uses the same functional form for both: a logarithmic function. The water function is premultiplied by a constant β smaller than the corresponding constant γ for fire operations. These two constants reflect the "efficiency" of the two types of operations.

Finally, the costs of terrorism are assumed to be linear in the number of terrorists, that is, of the form cx. We also model the control cost function as separable, and the costs of employing the water and fire strategies are modeled as quadratic. Over a infinite planning horizon, the objective is to minimize the sum of both costs (terrorism and counter-terror operations). We also assume that outcomes are discounted by a constant rate r. In brief, the control problem we pose can be written as:

$$\min_{u,v\geq 0} J = \min_{u,v\geq 0} \int_0^{\infty} (cx + u^2 + v^2)e^{-rt}dt \tag{6}$$

$$\dot{x} = \tau + (1 + \rho v)kx^{\alpha} - \mu x - \beta \ln(1 + u)x^{\theta} - \gamma \ln(1 + v)x; \quad x(0) = x_0$$

$$u(t) \geq 0; \quad v(t) \geq 0$$

where x_0 is the initial stock level and we impose also control constraints.

3 Optimization Algorithm

The above problem (6), is an Optimal Control Problem (OCP) where the total costs have to be minimized, given the state dynamics and the constraints on the controls. Denoting $\mathbf{u}(t) = (u(t), v(t)) = (u_1(t), u_2(t))$, we wish to compute:

$$\min_{\mathbf{u}(t)} J = \int_0^\infty F\left(t, x(t), \mathbf{u}(t)\right) dt \tag{7}$$

subject to satisfying:

$$\dot{x}(t) = f\left(t, x(t), \mathbf{u}(t)\right), \ 0 \leq t \leq \infty; \quad x(0) = x_0 \tag{8}$$

$$\mathbf{u}(t) \in \mathbf{U}(t), \ 0 \leq t \leq \infty \tag{9}$$

The problem presents several noteworthy features. First, the optimization interval is infinite. Second, the time t is not explicitly present in the problem (it is a time-autonomous problem), except in the discount factor. Third, we impose constraints on the control and finally, it constitutes a multi-dimensional problem.

To solve the multi-control variational problem, we propose a numerical algorithm which uses a particular strategy related to the cyclic coordinate descent (CCD) method [9]. The classic CCD method minimizes a function of n variables cyclically with respect to the coordinates. With our method, the problem can be solved like a sequence of problems whose error functional converges to zero. The algorithm (with $i = 1, 2$) carries out several iterations and at each *j-th* iteration it calculates 2 stages, one for each i. At each stage, it computes the optimal of $u_i(t)$, assuming the other variable is fixed.

Beginning with some admissible $\mathbf{u}^0$, we construct a sequence of $(\mathbf{u}^j)$ and the algorithm will search:

$$\lim_{j \to \infty} \mathbf{u}^j \tag{10}$$

It is easy to justify the convergence of the algorithm taking into account Zangwill's global convergence Theorem [10].

Based on the above, we present the solution for the unidimensional case, using Pontryagin's Minimum Principle (PMP) [11]. Our integrand takes the form:

$$F(x(t), u(t), t) = G(x(t), u(t), t)e^{-rt} \tag{11}$$

where r is the positive rate of discount, and G is a function bounded from above. Under these conditions, the integral is found to be convergent for each admissible control. Let H be the associated Hamiltonian:

$$H(t, x, u, \lambda) = F(t, x, u) + \lambda \cdot f(t, x, u) \tag{12}$$

where λ is thecostate variable. Using PMP, the optimal solution can be obtained from a two-point boundary value problem. In order for $u^* \in U$ to be optimal, a nontrivial function λ must exist, such that for almost every $t \in [0, \infty)$:

$$\dot{x} = H_\lambda = f; \quad x(0) = x_0 \tag{13}$$

$$\dot{\lambda} = -H_{\mathbf{x}}; \quad \lim_{t\to\infty} \lambda(t) = 0 \tag{14}$$

$$H(t, x, u^*, \lambda) = \min_{u(t)\in U} H(t, x, u, \lambda) \tag{15}$$

Due to the nonlinearity of the system dynamics, the optimal solution can only be computed numerically. In this paper we propose an efficient method which adapts the shooting method, Euler's method, and numerical integration. All the calculations are carried out in the Mathematica environment.

4 Steady-state Solution

In [8] a method for computing the optimal steady-state in infinite-horizon one-dimensional problems is presented which does not require the solution of the dynamic optimization problem, in which the bounds $U(t)$ do not play any role. Tsur considers:

$$\min_{u(t)} J = \int_0^\infty G(x(t), u(t)) e^{-rt} dt \tag{16}$$

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{17}$$

For the steady-state solution, $u = R(x)$, the *evolution function* is defined by:

$$L(x) = r\left(\frac{G_u(x, R(x))}{f_u(x, R(x))} + \dot{W}(x)\right) \text{ with } W(x) = \frac{1}{r} G(x, R(x)) \tag{18}$$

The function $L(x)$ allows to formulate the following necessary condition for the optimal steady state x_s:

$$L(x_s) = 0 \tag{19}$$

We also extend the method to multi-dimensional problems.

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