An algorithm for quasi-linear control problems in the economics of renewable resources: The steady state and end state for the infinite and long-term horizon.

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1 Introduction

This paper presents the problem of finding the optimal harvesting strategy (see [1], [2] and [3]), maximizing the expected present value of total revenues. The problem is formulated as an optimal control problem [4]. Combining the techniques of Pontryagin's Maximum Principle and the shooting method, an algorithm has been developed that is not affected by the values of the parameter. The algorithm is able to solve conventional problems as well as cases in which the optimal solution is shown to be bang-bang with singular arcs. In addition, we present a result that characterizes the optimal steady-state in infinite-horizon, autonomous models (except in the discount factor) and does not require the solution of the dynamic optimization problem. We also present a result that, under certain additional conditions, allows us to know a priori the final state solution when the optimization interval is finite. Finally, several numerical examples are presented to illustrate the different possibilities of the method.

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2 Statement of the problem

For the study of the economics of a renewable resource [2], we shall first see the pattern of biological growth of the resource. In this paper, we consider the growth function for a population of some species of fish. We assume that this fishery has a *intrinsic growth rate* denoted by r, which represents the difference between the population's birth and natural mortality rates. Let us assume that the *population stock* is x, and the *rate of change of the population is* $\dot{x}$. A commonly used functional form is the *simple logistic function*:

$$\dot{x}(t) = f_l(x) = rx(t)\left(1 - \frac{x(t)}{k}\right) \tag{1}$$

where k is the carrying capacity of the species. In this paper, and in line with [3], we model the dynamics of the fish stock biomass (x) when human harvesting is included in the problem, in the more general form as:

$$\dot{x}(t) = f_l(x) - h(t) \tag{2}$$

where $h(t)$, the rate of biomass harvest, will be considered as a independent variable. Let us now see how to model the cost functions.

Let $\pi(x, h)$ be the instantaneous net revenue from the harvest of the stock biomass:

$$\pi(x, h) = p(h)h - c(x, h) \tag{3}$$

where $p(h)$ is the inverse demand function and $c(x, h)$, the cost function associated with the harvest. The functional forms for the demand and cost functions adopted in this paper are:

$$p(h) = p_0 - p_1 h \tag{4}$$

$$c(x, h) = \frac{ch^\alpha}{x} \tag{5}$$

where h represents landings of fish and p_0 and p_1 are coefficients. Substituting (4) and (5) in (3), the profit function is:

$$\pi(x, h) = p_0 h - p_1 h^2 - \frac{ch^\alpha}{x} \tag{6}$$

where the meaning of the parameters is: p_0 is the price of the stock, p_1 is the strength of demand, c is the cost of exploitation and α is the harvest cost parameter.

Our model of renewable resource exploitation is an open-access fishery model, in which each firm takes the market price of landed fish as given. The firm's objective is to maximize profits from the harvest schedule over an infinite time horizon, subject to the dynamic constraint equation (2) and other natural and policy restrictions that involve limits (or bounds) for the harvest, $h(t)$, and stock, $x(t)$. Hence, our objective is to maximize profit from the harvest schedule over an infinite time horizon:

$$\max_{h(t)} \int_0^\infty \pi(x,h) e^{-\delta t} dt = \max_{h(t)} \int_0^\infty (p_0 h - p_1 h^2 - \frac{ch^\alpha}{x}) e^{-\delta t} dt \tag{7}$$

subject to:

$$\dot{x}(t) = f_l(x) - h(t);\ x(0) = x_0 \tag{8}$$

$$h(t) \in H(t);\ x \in [0,k] \tag{9}$$

where $\delta > 0$ is the discount rate, i.e. the marginal returns on capital for the company, and x_0 is the initial stock level.

As can be seen, the stated problem (7), (8), (9) is one of Optimal Control (OC) that presents a number of noteworthy features. First, the optimization interval is infinite. Second, the time t is not explicitly present in the problem (time-autonomous problem), except in the discount factor. Third, we impose constraints on the control and, fourth, it constitutes a problem which is quasi-linear when real values are considered for the parameters.

3 Optimization Algorithm

Faced with the complication of having to use different techniques when the functional is linear or nonlinear in the control variable, the contribution of our method is that it is valid in cases ranging between quasi-linearity and singular arcs. We have used the combined techniques of Pontryagin's Maximum Principle (PMP) [4] and the shooting method to build this optimization algorithm. If we denote by $\mathbb{Y}_x(t)$ the *coordination function*:

$$\mathbb{Y}_x(t) = -\frac{F_u}{f_u} \cdot e^{\int_0^t f_x ds} + \int_0^t F_x \cdot e^{\int_0^s f_x dz} ds \tag{10}$$

the theoretical development carried out allows us to present a necessary maximum condition.

Theorem 1. A necessary maximum condition

Let u^* be the optimal control, let $x^* \in \widehat{C}^1$ be a solution of the above problem. Then there exists a constant $K \in \mathbb{R}$ such that:

$$\begin{array}{lll} \text{If } u_{\min} < u^* < u_{\max} & \Longrightarrow & \mathbb{Y}_{x^*}(t) = K \\ \text{If } u^* = u_{\max} & \Longrightarrow & \mathbb{Y}_{x^*}(t) \leq K \\ \text{If } u^* = u_{\min} & \Longrightarrow & \mathbb{Y}_{x^*}(t) \geq K \end{array} \tag{11}$$

Thus, the problem consists in finding for each K the function x_K that satisfies: $x_K(0) = x_0$, the conditions of Theorem 1 and, from among these functions, the one that satisfies the transversality condition:

$$\lim_{t\to\infty} \lambda(t) = 0 \tag{12}$$

The algorithm consists of two fundamental steps:

Step 1) *The construction of x_K.* The construction of x_K can be performed using a discretized version of the *coordination equation:* $\mathbb{Y}_x(t) = K$. For each K, we construct the x_K, using this equation and when the values obtained do not obey the constraints, we force the solution to belong to the boundary until the moment established by conditions of Theorem 1.

Step 2) *The calculation of the optimal K.* The calculation of the optimal K could be achieved by means of an adaptation of the shooting method. Varying the coordination constant, K, we search for the extremal that fulfils the second boundary condition (12). Starting out from two values for the coordination constant, K: $K_{\min}$ and $K_{\max}$ and using a conventional method such as the secant method, our algorithm converges satisfactorily.

4 Steady-state solution

For time-autonomous problems, where the time, t, is not explicitly present in the problem, except in the discount factor, the optimal solution is time invariant in the long term and converges to an equilibrium state. The method developed in [5] characterizes the optimal steady-state in single-state, infinite-horizon problems, by means of a simple function of the state variable, called the *evolution function*.

The method consider the one-dimensional, infinite-horizon problems of

Based on the Euler's equation, that can be rewritten for autonomous systems as follows:

$$F - \dot{x}F_{\dot{x}} = cte \tag{20}$$

and given that the solution for the steady state, (x_s, h_s) (with $\dot{x} = 0$), may be known a priori by means of the method explained in the previous section, the value of the constant, cte, present in (20) can be obtained straightforwardly:

$$cte = F(x_s) = \pi(x_s, h_s) \tag{21}$$

If we now consider the final moment, T, the two following conditions must be simultaneously verified at that moment, given that the end state is free:

$$\begin{cases} F_{\dot{x}}(x(T), h(T)) = 0 \\ F(x(T), h(T)) = cte \end{cases} \tag{22}$$

The first is the transversality condition corresponding to the free end state, and the second the simplified Euler equation (20). Simply solving this system, the end state $(x(T), h(T))$ can be obtained straightforwardly.

References

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