The Operation of Infimal/Supremal Convolution in Mathematical Economics

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November 30, 2013

1 Introduction

The infimal convolution (IC) operation is a fundamental fact in discrete convex analysis that is often usefully applied in mathematical economics. A excellent review of the literature on IC within the context of optimal risk exchange and optimal allocation problems can be found in [1]. In two previous papers the authors of the present paper presented two new applications of the IC: in [2] the firm's cost-minimization (FCM) problem with the Cobb-Douglas production function, and in [3] the FCM problem with the linear production function in economies of scale.

In this paper we present a new application to demonstrate the enormous potential of this mathematical tool in the field of economics: the analytical solution of the utility maximization problem. We shall address this problem in an exact way in this paper, transforming it into the constrained supremal convolution problem of the log-concave functions. Moreover we do not solve a particular problem for a particular level of budget level ξ, but for a family of problems: all those posed when considering all the permissible levels of budget level ξ.

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2 The Utility Maximization Problem

In this paper we shall consider the Utility Maximization (UM) problem for the case of the utility function following the Cobb-Douglas model, $\prod_{i=1}^{m} x_i^{\alpha_i}$, and the availability of the commodities having upper constraints. In the UM problem the aim is to choose the best among all possible options subject to the budget constraint: $\sum_{i=1}^{m} p_i x_i = \xi$ and to the available amount of commodities: $0 \le x_i \le N_i$, such that the utility is maximized, where $\mathbf{p} = (p_1, \ldots, p_m)$ is the price vector of the different commodities, i.e.:

$$\begin{aligned} u(\mathbf{p}, \xi) &= \max \prod_{i=1}^{m} x_i^{\alpha_i} \\ \text{s.t.} \quad \sum_{i=1}^{m} p_i x_i &= \xi; \quad 0 \le x_i \le N_i \end{aligned} \tag{1}$$

This problem is equivalent to a new problem:

$$\begin{aligned} \widetilde{u}(\mathbf{p}, \xi) &= \max \sum_{i=1}^{m} \alpha_i \ln\left(\tfrac{y_i}{p_i}\right) \\ \text{s.t.} \quad \sum_{i=1}^{m} y_i &= \xi; \quad 0 < y_i \le p_i N_i = M_i \end{aligned} \tag{2}$$

with $\alpha_i, p_i > 0, i = 1, ..., m$, in which only the following change in the variables needs to be taken into account: $p_i x_i = y_i$. The function $\widetilde{u}(\mathbf{p}, \cdot)$ is in fact the supremal convolution of the log-concave functions:

$$F_i(y_i) := \alpha_i \ln\left(\frac{y_i}{p_i}\right)$$

3 Solution of the Problem

In this section we shall calculate the supremal convolution for the functions $F_i(y_i)$ and then go on to prove that it belongs to the class C^1. The demonstration of the results not shown will be analogous to those developed in a previous paper [2] for exponential functions. Let C_ξ be the set:

$$C_\xi := \{(y_1, \ldots, y_m) \in (0, M_1] \times ... \times (0, M_m] \ / \ \sum_{i=1}^{m} y_i = \xi\}$$

The supremal convolution of the $\{F_i\}_{i=1}^m$is:

$$(F_1 \circledast \cdots \circledast F_m)(\xi) := \max_{C_\xi} \sum_{i=1}^{m} F_i(y_i)$$

Definition 1. Let us call the function $\Psi_i : \left(0, \sum_{j=1}^m M_j\right] \longrightarrow (-\infty, M_i]$ *the i-th distribution function,* defined by:

$$\Psi_i(\xi) = y_i, \quad \forall i = 1, \dots, m$$

where $(y_1, \dots, y_m)$ is the unique maximum of F on the set C_ξ, i.e.:

$$\sum_{i=1}^{m} \Psi_i(\xi) = \xi \quad \text{and} \quad \sum_{i=1}^{m} F_i(\Psi_i(\xi)) = (F_1 \circledast \cdots \circledast F_m)(\xi)$$

Theorem 1. *For every $k = 1, \dots, m$ the k-th distribution function is*

$$\Psi_k(\xi) = \begin{cases} \dfrac{\alpha_k}{\sum\limits_{i=1}^{m} \alpha_i} \xi & if \quad \xi < \theta_m \\ \dfrac{\alpha_k}{\sum\limits_{i=1}^{j-1} \alpha_i} \left[\xi - \sum\limits_{i=j}^{m} M_i \right] & if \quad \theta_j \le \xi < \theta_{j-1} \le \theta_k \\ M_k & if \quad \xi \ge \theta_k \end{cases}$$

with the coefficients:

$$\theta_k = \sum_{i=k}^{m} M_i + \frac{M_k}{\alpha_k} \sum_{i=1}^{k-1} \alpha_i$$

Theorem 2. *The supremal convolution of the log functions $F_i(y_i)$ is a logarithmic piecewise function:*

$$(F_1 \circledast F_2 \circledast \cdots \circledast F_m)(\xi) = \begin{cases} \widetilde{\beta}_{m+1} + \sum\limits_{i=1}^{m} \alpha_i \ln \left(\frac{\xi}{p_i} \right) & if \quad \xi < \theta_m \\ \widetilde{\beta}_k + \sum\limits_{i=1}^{k-1} \alpha_i \ln \left(\xi - \sum\limits_{i=k}^{m} M_i \right) & if \quad \theta_k \le \xi < \theta_{k-1} \end{cases}$$

with the coefficients:

$$\widetilde{\alpha}_k = \sum_{i=1}^{k} \alpha_i; \quad \widetilde{\beta}_k = \sum_{i=1}^{k-1} \alpha_i \ln\left(\frac{\alpha_i}{\widetilde{\alpha}_{k-1}}\right) + \sum_{i=k}^{m} \alpha_i \ln\left(\frac{M_i}{p_i}\right)$$

Moreover, it belongs to the class C^1.

Having calculated the function $\widetilde{u}(\mathbf{p}, \xi)$, and considering the fact that $u(\mathbf{p}, \xi) = \mathbf{e}^{\widetilde{u}(\mathbf{p}, \xi)}$, the solution of problem (1) is evident.

4 Example

We shall now consider an example with $m = 20$ commodities that has also been used in [2]. Figure 1 shows the graph of the utility function, $u(\mathbf{p}, \xi)$, obtained for each budget level ξ.

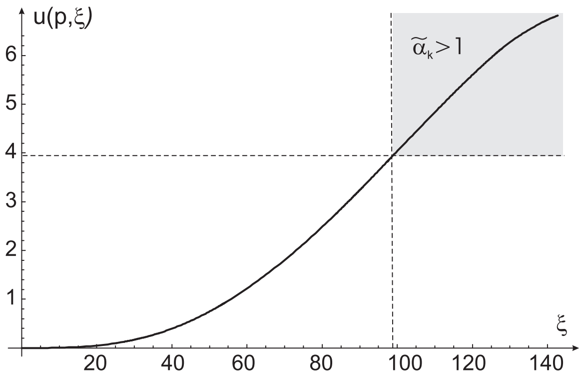

Figure 1. Utility function, $u(\mathbf{p}, \xi)$.

References

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