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AN APPLICATION OF MATHEMATICA TO A HYDROTHERMAL OPTIMIZATION PROBLEM

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Abstract. In this paper we have developed a simple algorithm that resolves the problem of the optimization of hydrothermal systems. We have set out our problem in terms of optimal control in continuous time, with the Lagrange-type functional. We present two examples employing the algorithm developed for this purpose with the "Mathematica" package.

1 Introduction and Statement

This paper deals with the optimization of hydrothermal systems.

In a previous paper [1], we considered a hydrothermal system with one hydro-plant and m thermal power plants that had been substituted by their thermal equivalent and addressed the $H_\mathbf{1} - T_\mathbf{1}$ problem: minimizing the cost of fuel $F(P)$ during the optimization interval $[0,T]$

$$F(P) = \int_0^T \Psi(P(t))dt \tag{1.1}$$

$$P(t) + H(t, z(t), z'(t)) = P_d(t),\ \forall t \in [0,T] \tag{1.2}$$

$$z(0) = 0, z(T) = b \tag{1.3}$$

where Ψ is the function of thermal cost of the thermal equivalent and $P(t)$ is the power generated by said plant.

The following must be also be verified: the equilibrium equation of active power (1.2), and the boundary conditions (1.3), where $P_d(t)$ is the power demand, $H(t, z(t), z'(t))$ is the power contributed to the system at the instant t by the hydro-plant, $z(t)$ being the volume that is discharged up to the instant t by the plant, $z'(t)$ the rate of water discharge of the plant at the instant t, and b the volume of water that must be discharged during the entire optimization interval.

In said paper, we likewise considered constraints for the admissible generated power:

$$P(t) \geq 0; \;\; H(t, z(t), z'(t)) \geq 0$$

The mathematical problem (P_{Ω_b}) was stated in the following terms:

$$\min_{z\in\Omega_b} J(z) = \min_{z\in\Omega_b} \int_0^T \Psi\left[P_d(t) - H(t, z(t), z'(t))\right] dt = \min_{z\in\Omega_b} \int_0^T L(t, z(t), z'(t))dt$$
$$\Omega_b = \{z \in \widehat{C}^1[0,T] \mid z(0) = 0, z(T) = b, 0 \leq H(t, z(t), z'(t)) \leq P_d(t), \forall t \in [0,T]\}$$

where $(\widehat{C}^1)$ is the set of piecewise C^1 functions.

The problem (P_{Ω_b}) was formulated within the framework of optimal control [2] and

$$\mathbb{Y}_q(t) := -L_{z'}(t, q(t), q'(t)) \cdot \exp\left[-\int_0^t \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))} ds\right] \tag{1.4}$$

was called *the coordination function of* $q \in \Omega_b$, obtaining the following result:

Theorem 1. *If q is a solution of (P_{Ω_b}), then $\exists K \in \mathbb{R}^+$ such that:*

$$\mathbb{Y}_q(t) \;\; is \;\; \begin{cases} \leq K & if \;\; H(t, q(t), q'(t)) = 0 \\ = K & if \;\; 0 < H(t, q(t), q'(t)) < P_d(t) \\ \geq K & if \;\; H(t, q(t), q'(t)) = P_d(t) \end{cases}$$

We now generalize the study to the $H_{\mathbf{n}} - T_{\mathbf{1}}$ problem and let us assume that a hydrothermal system accounts for n hydro-plants. We denote

$$H(t, z_1(t), z_2(t), \ldots, z_n(t), z_1'(t), z_2'(t), \ldots, z_n'(t)) = H(t, \overline{\mathbf{z}}(t), \overline{\mathbf{z}}'(t))$$

the function of effective hydraulic contribution, and is the power contributed to the system at the instant t by the set of hydraulic plants. Under the above notation, let

$$\overline{\mathbf{b}} = (b_1, \ldots, b_n) \in \mathbb{R}^n$$

be the vector of admissible volumes.

Now we will call $P_{\Omega_{\overline{\mathbf{b}}}}$, the problem of minimizing the functional

$$J(\overline{\mathbf{z}}) = \int_0^T L(t, \overline{\mathbf{z}}(t), \overline{\mathbf{z}}'(t))dt = \int_0^T \Psi\left(P_d(t) - H(t, \overline{\mathbf{z}}(t), \overline{\mathbf{z}}'(t))\right) dt \tag{1.5}$$

over the set

$$\Omega_{\overline{\mathbf{b}}} := \left\{ \overline{\mathbf{z}} \in \left(\widehat{C}^1[0,T]\right)^n \;\middle|\; \begin{array}{c} z_i(0) = 0, z_i(T) = b_i, \forall i = 1, \ldots, n \\ 0 \leq H(t, \overline{\mathbf{z}}(t), \overline{\mathbf{z}}'(t)) \leq P_d(t), \;\; \forall t \in [0,T] \end{array} \right\}$$

In this paper we have developed an algorithm of its numerical resolution prompted by the so-called Gauss-Southwell method of coordinate descent.

2 Optimization Algorithm

2.1 $H_1 - T_1$ problem

For one hydro-plant, from the computational point of view, the construction of the solution can be performed with the use of a discretized version of Theorem 1. The problem will consist in finding for each K the function q_K that satisfies conditions of Theorem 1, and from among these functions, an admisible function $q_K \in \Omega_b$.

In general, the construction of q_K cannot be carried out all at once over the entire interval $[0, T]$. The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs, until completing the interval $[0, T]$, where:

$0 < H(t, q_K(t), q'_K(t)) < P_d(t)$ (free extremal arcs),or
$H(t, q_K(t), q'_K(t)) = 0$ (the hydro-plant is on shut-down),or
$H(t, q_K(t), q'_K(t)) = P_d(t)$ (the hydro-plant generates all the demanded power)

Varying K, we would search for the extremal that fulfils the second boundary condition $q_K(T) = b$. The procedure is similar to the shooting method used to resolve second-order differential equations with boundary conditions.

We will denote by M the rate of water discharge at the instant $t = 0$ that is needed for the hydro-plant to satisfy the power demand:

$$H(0, 0, M) = P_d(0)$$

and we will denote by m the rate of water discharge at the instant $t = 0$ that is needed for:

$$H(0, 0, m) = 0$$

We also set

$$K_m = -L_{z'}(0, 0, m); \quad K_M = -L_{z'}(0, 0, M)$$

To construct q_K, we proceed by the steps shown next:

Step 1 (the first arc).

i) If $K \geq K_m$, we set $q_K(t) = \omega(t)$, the solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = 0$$

with $\omega(0) = 0$ in the maximal interval $[0, t_1]$, where $K \geq \mathbb{Y}_\omega(t)$.

ii) If $K \leq K_M$, we set $q_K(t) = \omega(t)$, the solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = P_d(t)$$

with $\omega(0) = 0$ in the maximal interval $[0, t_1]$, where $K \leq \mathbb{Y}_\omega(t)$.

iii) $K_M < K < K_m$.

Now q_K will be the arc of the interior extremal (with $q_K(0) = 0$) which satisfies Euler's equation in its maximal domain $[0, t_1]$ and therefore *the coordination equation*

$$K = \mathbb{Y}_{q_K}(t)$$

i-th Step (i-th arc).

A) If q_K has an interior arc in $[t_{i-1}, t_i]$, there are two possibilities:

I) If $H(t_i, q_K(t_i), q'_K(t_i)) = 0$, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K \geq -L_{z'}(t, \omega(t), \omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q'_K(s))}{H_{z'}(s, q_K(s), q'_K(s))} ds - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))} ds\right]$$

$\omega(t)$ being a solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = 0 \text{ with } \omega(t_i) = q_K(t_i)$$

If this is the case, we set $q_K(t) = \omega(t)$, $\forall t \in [t_i, t_{i+1}]$.

II) If $H(t_i, q_K(t_i), q'_K(t_i)) = P_d(t_i)$, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K \leq -L_{z'}(t, \omega(t), \omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q'_K(s))}{H_{z'}(s, q_K(s), q'_K(s))} ds - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))} ds\right]$$

$\omega(t)$ being a solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = P_d(t) \text{ with } \omega(t_i) = q_K(t_i)$$

If this is the case, we set $q_K(t) = \omega(t)$, $\forall t \in [t_i, t_{i+1}]$.

B) If $[t_{i-1}, t_i]$ is the boundary interval, we consider the maximal interval $[t_i, t_{i+1}]$ such that, $\forall t \in [t_i, t_{i+1}]$

$$K = -L_{z'}(t, \omega(t), \omega'(t)) \cdot \exp\left[-\int_0^{t_i} \frac{H_z(s, q_K(s), q'_K(s))}{H_{z'}(s, q_K(s), q'_K(s))} ds - \int_{t_i}^{t} \frac{H_z(s, \omega(s), \omega'(s))}{H_{z'}(s, \omega(s), \omega'(s))} ds\right]$$

$\omega(t)$ being an interior arc of the extremal, with $\omega(t_i) = q_K(t_i)$, which satisfies the coordination equation in its maximal domain $[t_i, t_{i+1}]$. Now, we set

$$q_K(t) = \omega(t), \forall t \in [t_i, t_{i+1}]$$

2.2 $H_n - T_1$ problem

Having resolved the $H_\mathbf{1} - T_\mathbf{1}$ problem, we now generalize the study to the $H_\mathbf{n} - T_\mathbf{1}$ problem.

Let us assume that a hydrothermal system accounts for n hydro-plants and we present an algorithm of its numerical resolution using a particular strategy related to the Gauss-Southwell method of coordinate descent [3]. With this method, a problem of the type $H_\mathbf{n} - T_\mathbf{1}$ could be solved, under certain conditions, if we start out from the resolution of a sequence of problems of the type $H_\mathbf{1} - T_\mathbf{1}$.

Let the function

$$G : \mathbb{R}^n \to \mathbb{R}, G \in C^1(\mathbb{R}^n)$$

and $\overline{\mathbf{x}} = (x_1, \ldots, x_j, \ldots, x_n)$.

The idea of the coordinate descent method is to use the coordinate axes as descent directions. The method sequentially searches for the minimum of G in all the directions $\overline{\mathbf{e}}_j$. Descent with respect to the x_j coordinate means that $G(x_1, \ldots, x_j, \ldots, x_n)$ is minimized with respect to

x_j, while the rest remain fixed. There exists a number of different selection strategies for the coordinates. However, we are specifically interested in the *Gauss-Southwell*-type selection scheme, which selects the coordinate that has the largest absolute value in the gradient vector. Now we adapt the finite-dimensional version of this algorithm to our functional (1.5).

The algorithm for the $H_{\mathbf{n}} - T_{\mathbf{1}}$ problem carries out several iterations and at each *k-th* iteration calculates n stages, one for each hydro-plant. At each stage, it calculates the optimal functioning of a hydro-plant, while the behavior of the rest is assumed fixed. For every $\overline{\mathbf{q}} = (q_1, \dots, q_n) \in \Omega_{\overline{\mathbf{b}}}$, we consider the functional $J^i_{\overline{\mathbf{q}}}$ defined by

$$J^i_{\overline{\mathbf{q}}}(z_i) = \int_0^T \Psi\left(P_d(t) - H^i_{\overline{\mathbf{q}}}(t, z_i(t), z_i'(t))\right) dt, \quad \text{with}$$
$$H^i_{\overline{\mathbf{q}}}(t, z_i, z_i') = H(t, q_1, \dots, q_{i-1}, z_i, q_{i+1}, \dots, q_n, q_1', \dots, q_{i-1}', z_i', q_{i+1}', \dots, q_n')$$

where $H^i_{\overline{\mathbf{q}}}$ represents the power generated by the hydraulic system as a function of the rate of water discharge and the volume turbined by the i-th plant, under the assumption that the rest of the plants behave in a definite way. We call the i-th minimizing mapping the mapping $\phi_i : \Omega_{\overline{\mathbf{b}}} \longrightarrow \Omega_{\overline{\mathbf{b}}}$, defined in the following way: for every $\overline{\mathbf{q}} \in \Omega_{\overline{\mathbf{b}}}$

$$\phi_i(q_1, \dots, q_i, \dots, q_n) = (q_1, \dots, q^*, \dots, q_n)$$

where q^* minimizes $J^i_{\overline{\mathbf{q}}}$. Beginning with some admissible $\overline{\mathbf{q}}^0 = (q_1^0, \dots, q_n^0)$, we construct a sequence of $\overline{\mathbf{q}}^k$ via successive applications of $\{\phi_i\}_{i=1}^n$. In order to select the "coordinate" q_i at which we carry out the descent at each stage, instead of calculating the coordinate that has the largest absolute value in the gradient vector, which now does not make sense, we consider the function $\mathbb{Y}^i_{\overline{\mathbf{q}}}(t)$, and the set of instants $\chi^i_{\overline{\mathbf{q}}}$ where the solution is a free extremal:

$$\mathbb{Y}^i_{\overline{\mathbf{q}}}(t) = -L_{z_i'}(t, \overline{\mathbf{q}}(t), \overline{\mathbf{q}}'(t)) \cdot \exp\left[-\int_0^t \frac{H_{z_i}(s, \overline{\mathbf{q}}(s), \overline{\mathbf{q}}'(s))}{H_{z_i'}(s, \overline{\mathbf{q}}(s), \overline{\mathbf{q}}'(s))} ds\right]$$
$$\chi^i_{\overline{\mathbf{q}}} = \{t \in [0, T] \mid 0 < H(t, \overline{\mathbf{q}}(t), \overline{\mathbf{q}}'(t)) < P_d(t)\}$$

We give the name *imbalance* in the i-th plant at $\overline{\mathbf{q}}$ to the positive number

$$\delta^i_{\overline{\mathbf{q}}} = \max_{t \in \chi^i_{\overline{\mathbf{q}}}} \mathbb{Y}^i_{\overline{\mathbf{q}}}(t) - \min_{t \in \chi^i_{\overline{\mathbf{q}}}} \mathbb{Y}^i_{\overline{\mathbf{q}}}(t)$$

The algorithm, at each stage of the *k-th* iteration, it selects the coordinate i-th with largest $\delta^i_{\overline{\mathbf{q}}}$. If we set

$$\Phi_{\sigma_k} = (\phi_{\sigma_k(n)} \circ \phi_{\sigma_k(n-1)} \circ \cdots \circ \phi_{\sigma_k(2)} \circ \phi_{\sigma_k(1)})$$

σ_k being the permutation that at the k-th iteration establishes the above mentioned order, and

$$\overline{\mathbf{q}}^k = \Phi_{\sigma_k}(\overline{\mathbf{q}}^{k-1})$$

the algorithm will search

$$\lim_{k \to \infty} \overline{\mathbf{q}}^k$$

The analysis of the convergence of the minimizing sequence $\{\overline{\mathbf{q}}^k\}$, like the verification of the minimizing character of its limit, is a non trivial problem of Functional Analysis that exceeds the goals of this paper. It is, however, simple to justify the convergence of the algorithm in a finite number of steps, simply by considering the following solution set:

$$\{\overline{\mathbf{q}} \mid \exists\sigma \in \Sigma_n \ \text{ such that } \ F[\overline{\mathbf{q}}] - F[\Phi_\sigma(\overline{\mathbf{q}})] < \varepsilon\}$$

i.e. the set of admissible elements over which, after one iteration of the algorithm, the funcional has not decreased more than ε. It need only be borne in mind that the value of the functional is lower limited by zero and hence an infinite sequence of descents greater than ε cannot occur.

The implementation of the algorithm with Mathematica package is very simple. For example, next we present the models, the coordination equation and the optimization subroutine for the $H_1 - T_1$ problem in the case of free extremal (without restrictions). It's very easy to identify the shooting method, the Euler's method for differential equation and the discretized version of Theorem 1.

```
(*Models*)
ph[1]=A[1]*q[1]-B[1]*q[1]*v[1];
h[1]=ph[1]-bll[1]*ph[1]^2;
P=pd[t]-h[1];
in[1]=D[h[1],v[1]] / D[h[1],q[1]];

(*Coordination equation*)
eq[1]:=-D[L[P],q[1]]e^(-sumin[1])==k[1];

(*Optimization subroutine:  Shooting method*)
Do[
If[iter==1,k=kmin[1]];
If[iter==2,k=kmax[1]];
If[iter≥3,k=((vs-b[1])*r-(vr-b[1])*s)/(vs-vr)];
Do[
int=in[1]/.{v[i]->(vk+q[1]*T/n)};
eqt=eq[1]/.{sumin[1]->(sk+int*T/n),v[1]->(vk+q[1]*T/n),k[1]->k};
eqt=eqt/.q[1]->qq;

(*Theorem 1*)
sol=FindRoot[eqt,{qq,b[1]/T}];
q[1,t]=qq/.sol;

(*Euler's method*)
v[1,t]=vk+q[1,t]*T/n;vk=v[1,t];
int=in[1]/.{q[1]->q[1,t],v[1]->vk};
sk=sk+int*T/n;
,{t,0,T,T/n}];
If[iter==1,vr=vk;r=k;];
If[iter==2,vs=vk;s=k;];
If[iter≥3,If[(vr-b[i])*(vk-b[1])≤0,s=k;vs=vk,r=k;vr=vk]];
err=Abs[vk-b[1]];If[err≤tol,Return["END"]];
,{iter,1,maxiter,1}];
```

3 Application to a Hydrothermal Problem

A program that resolves the optimization problem was elaborated using the Mathematica package and was then applied to two examples of hydrothermal systems made up of 8 thermal plants and 10 and 20 hydro-plants respectively. The cost function Ψ that has been used is a quadratic model

$$\Psi(P) = \alpha + \beta P + \gamma P^2$$

and we consider Kirchmayer's model for the transmission losses: $b_{ll} \cdot P^2$, where b_{ll} is termed the loss coefficient. The units for the coefficients are: α in $(\$/h), \beta$ in $(\$/h.Mw), \gamma$ in $(\$/h.Mw^2)$, and the loss coefficients b_{ll} in $(1/Mw)$.

We use a variable head model and the hydro-plant's active power generation P_h (variable head) is a function of $z(t)$ and $z'(t)$

$$P_h(t, z(t), z'(t)) := A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t)$$

with

$$A(t) := \frac{B_y}{G}(S_0 + t \cdot i);\ B = \frac{B_y}{G}$$

We consider that the transmission losses for the hydro-plant are also expressed by Kirchmayer's model. Hence, the function of effective hydraulic generation is

$$H(t, z(t), z'(t)) := P_h(t, z(t), z'(t)) - b_{ll} P_h^2(t, z(t), z'(t))$$

The units for the coefficients of the hydro-plant are: the efficiency G in $(m^4/h.Mw)$, the constraint on the volume b in (m^3), the loss coefficient b_{ll} in $(1/Mw)$, the natural inflow i in (m^3/h), the initial volume S_0 in (m^3) and the coefficient B_y (a parameter that depends on the geometry of the tanks) in (m^{-2}).

To verify the convergence of the algorithm, two tests were conducted considering in both the same 8 thermal plants and 10 and 20 hydro-plants respectively. Fig. 1 presents the obtained results. We can see how the Gauss-Southwell-type method presents a rapid convergence.

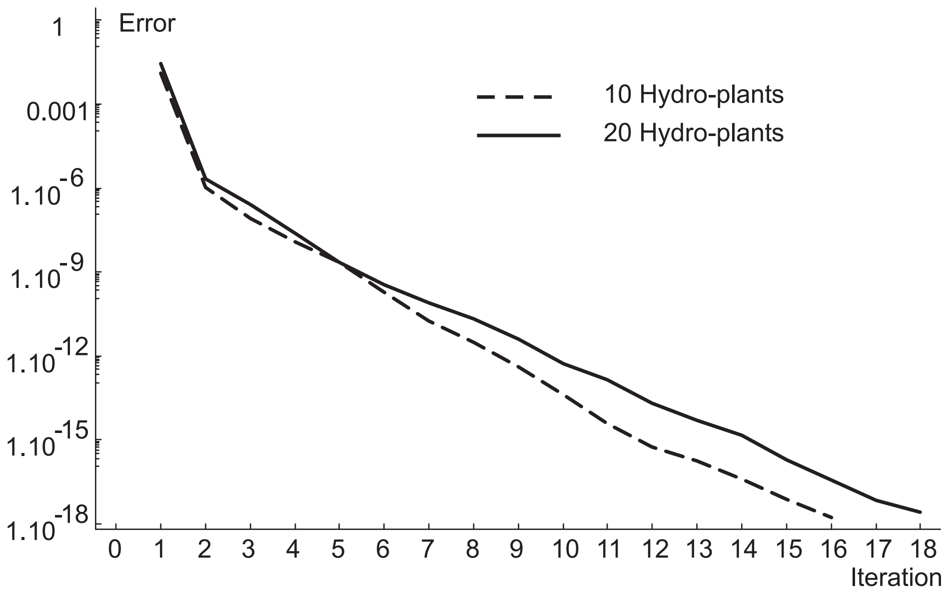

Fig. 1. Convergence of the Algorithm.

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