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A Bolza's Problem in Hydrothermal Optimization

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Abstract: This paper studies the optimization of hydrothermal systems. We shall use Pontryagin's Minimum Principle as the basis for proving a necessary condition for the stationary functions of the functional, setting out our problem in terms of optimal control in continuous time, with the Bolza-type functional. This theorem allows us to elaborate the optimization algorithm that leads to determination of the optimal solution of the hydrothermal system. Finally, we present a example employing the algorithm developed for this purpose with the "Mathematica" package.

Keywords: Optimal Control, Bolza's Problem, Pontryagin's Principle, Hydrothermal

Mathematics Subject Classification: 49J24

1 Introduction

In this paper we propose Pontryagin's Minimum Principle (PMP) to solve the optimum scheduling problem of hydrothermal systems. Several applications of optimal control theory (OCT) in hydrothermal optimization have been reported in the literature. These range from the initial studies corresponding to El-Hawary and Christensen [1], to more recent works such as [2] or [3]. In a previous study [4], it was proven that the problem of optimization of the fuel costs of a hydrothermal system with m thermal power plants may be reduced to the study of a hydrothermal system made up of one single thermal power plant, called the thermal equivalent. We will call this problem: the $(H_\mathbf{1} - T_\mathbf{1})$ Problem, and in Section 2 we shall see that this problem consists in the minimization of a functional

$$F(z) = \int_0^T L(t, z(t), z'(t))dt + S[z(T)]$$

within the set of piecewise C^1 functions ($\widehat{C}^1$) that satisfy $z(0) = 0$, $z(T) \leq b$ and the constraints $0 \leq H(t, z(t), z'(t)) \leq P_d(t), \forall t \in [0, T]$. Hence, the problem involves non-holonomic inequality constraints (differential inclusions). Using classic mathematical methods (see for example [5]), we shall focus in the present paper on the development of the applications of optimal control theory (OCT) to the specific problem of hydrothermal optimization.

In Section 3 we shall establish a necessary condition for the stationary functions of the functional and we shall use PMP as the basis for proving this theorem. We shall see that the treatment of the constraints of the problem using this new approach will be very simple. The development enables the construction, in Section 4, of the optimization algorithm that leads to determination of the optimal solution of the hydrothermal system. Finally, in Section 5, we present a example employing the Algorithm developed for this purpose with the "Mathematica" package.

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2 Statement of the Bolza's Problem: Water Cost

The $(H_\mathbf{1} - T_\mathbf{1})$ problem (one Hydraulic plant - Thermal equivalent) consists in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0,T]$. Said cost may be represented by

$$\int_0^T \Psi(P(t))dt + S[z(T)] \tag{2.1}$$

where Ψ is the function of thermal cost of the thermal equivalent, $P(t)$ is the power generated by said plant, and $S[z(T)]$ is the cost assigned to the water discharged. Moreover, the following equilibrium equation of active power will have to be fulfilled

$$P(t) + H(t, z(t), z'(t)) = P_d(t), \forall t \in [0,T]$$

where $P_d(t)$ is the power demand and $H(t, z(t), z'(t))$ is the power contributed to the system at the instant t by the hydro-plant, $z(t)$ being the volume that is discharged up to the instant t by the plant, and $z'(t)$ the rate of water discharge of the plant at the instant t. In this paper, we propose to study the problem when the final instant T is given and the final state has an upper boundary: $z(T) \leq b$. The following boundary conditions will have to be fulfilled $z(0) = 0$, $z(T) \leq b$. Taking into account the equilibrium equation, our objective functional in the Bolza's form is

$$F(z) = \int_0^T L(t, z(t), z'(t))dt + S[z(T)] \tag{2.2}$$

with L having the form $L(t, z(t), z'(t)) = \Psi(P_d(t) - H(t, z(t), z'(t)))$ over the set Θ_b

$$\Theta_b = \{z \in \widehat{C}^1[0,T] \mid z(0) = 0, z(T) \leq b, 0 \leq H(t, z(t), z'(t)) \leq P_d(t), \forall t \in [0,T]\}$$

If z satisfies Euler's equation for the functional F we have that, $\forall t \in [0,T]$

$$L_z(t, z(t), z'(t)) - \frac{d}{dt}\left(L_{z'}(t, z(t), z'(t))\right) = 0 \tag{2.3}$$

If we divide Euler's equation (2.3) by $L_{z'}(t, z(t), z'(t)) < 0$, $\forall t$, and integrating we have that

$$-L_{z'}(t, z(t), z'(t)) \cdot \exp\left[-\int_0^t \frac{H_z(s, z(s), z'(s))}{H_{z'}(s, z(s), z'(s))}ds\right] = -L_{z'}(0, z(0), z'(0)) = K \in \mathbb{R}^+ \tag{2.4}$$

We shall call relation (2.4) the coordination equation, and the positive constant K will be termed the coordination constant of the extremal. Let us now see the fundamental result (The Main Coordination Theorem), which enables us to characterize the extremals of the problem and which is also the basis for elaborating the optimization algorithm that leads to determination of the optimal solution of the hydrothermal system. We shall use the above coordination equation (2.4) in the development of the proof of the theorem.

3 The Main Coordination Theorem

We shall use PMP as the basis for proving this theorem, setting out our problem in terms of optimal control in continuous time, with the Bolza-type functional. In this paper we generalize a previous study [6] and we present the problem considering the state variable to be $z(t)$ and the control variable $u(t) = H(t, z(t), z'(t))$. Moreover, as $H_{z'} > 0$, the equation $u(t) - H(t, z(t), z'(t)) = 0$ allows the state equation $z' = f(t, z, u)$ to be explicitly defined.

The optimal control problem is thus:

$$\min_{u(t)} \int_0^T L(t, z(t), u(t))dt + S[z(T)] \quad \text{with} \quad \begin{cases} z' = f(t, z, u) \\ z(0) = 0, \quad z(T) \le b \\ u(t) \in \Omega(t) = \{x \mid 0 \le x \le P_d(t)\} \end{cases}$$

with $L(t, z(t), u(t)) = \Psi(P_d(t) - u(t))$. We shall see that with this approach we shall arrive at the coordination equation (2.4). It can be seen that from the relations $u(t) - H(t, z(t), z'(t)) = 0$ and $z' = f(t, z, u)$, we easily obtain $f_z = -\frac{H_z}{H_{z'}}$; $f_u = \frac{1}{H_{z'}}$. We define the following function.

Definition 1. Let us term the coordination function of $q \in \Theta_b$ the function in $[0, T]$, defined as follows

$$\mathbb{Y}_q(t) = -L_{z'}(t, q(t), q'(t)) \cdot \exp\left[-\int_0^t \frac{H_z(s, q(s), q'(s))}{H_{z'}(s, q(s), q'(s))} ds\right]$$

Theorem 1. The Main Coordination Theorem.
If $q \in \widehat{C}^1$ is a solution of problem $(H_1 - T_1)$, then $\exists K$ such that
i) If $0 < H(t, q(t), q'(t)) < P_d(t) \Longrightarrow \mathbb{Y}_q(t) = K$.
ii) If $H(t, q(t), q'(t)) = P_d(t) \Longrightarrow \mathbb{Y}_q(t) \ge K$.
iii) If $H(t, q(t), q'(t)) = 0 \Longrightarrow \mathbb{Y}_q(t) \le K$.
and

$$K \ge \frac{\partial S[q(T)]}{\partial z} \cdot \frac{-\mathbb{Y}_q(T)}{L_{z'}(T, q(T), q'(T))} \tag{3.1}$$

4 Construction of the Optimal Solution

From the computational point of view, the construction of the optimal solution can be performed with the next procedure:

i) For each K we construct q_K, where q_K satisfies the conditions i), ii) and iii) of theorem 1 and the initial condition $q_K(0) = 0$.

In general, the construction of q'_K cannot be carried out all at once over all the interval $[0, T]$. The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs $(0 < H(t, q_K(t), q'_K(t)) < P_d(t))$ and boundary arcs $(H(t, q_K(t), q'_K(t)) = P_d(t)$ or $H(t, q_K(t), q'_K(t)) = 0)$ until completing the interval $[0, T]$. This is relatively simple to implement, with the use of a discretized version of the equations.

ii) Varying the coordination constant K, we would search for the extremal that fulfils the second boundary condition $z(T) \le b$ and (3.1).

Firstly, we search for the value of K whose associated extremal satisfies $q_K(T) = b$. The procedure is similar to the shooting method used to resolve second-order differential equations with boundary conditions. Effectively, we may consider the function $\varphi(K) := q_K(T)$ and calculate the root of $\varphi(K) - b = 0$, which may be realized approximately using elemental procedures like the secant method.

If the relation (3.1) is fulfilled then $q_K(t)$ is the optimal solution and all the available water, b, is consumed. If the encountered K does not verify (3.1), the value of K that fulfills the equality in (3.1) is the optimal solution, and the optimal final volume in this case is $q_K(T) < b$.

5 Application to a Hydrothermal Problem

Let us now see a hydrothermal problem whose solution may be constructed in a simple way taking into account the above theorem 1. A program that resolves the optimization problem was elaborated using the Mathematica package and was then applied to one example of hydrothermal system made up of 8 thermal plants and one hydraulic plant of variable head. We consider the functional (2.1).

For the fuel cost model of the equivalent thermal plant Ψ, we use the quadratic model

$$\Psi(P(t)) = \alpha_{eq} + \beta_{eq} P(t) + \gamma_{eq} P(t)^2$$

We use a variable head model and the hydro-plant's active power generation P_h is function of $z(t)$ and $z'(t)$. Hence the function P_h is defined as

$$P_h(t, z(t), z'(t)) := A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t)$$

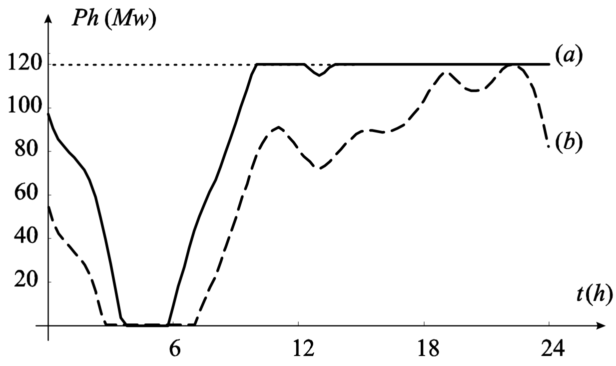
We consider that the transmission losses for the hydro-plant are expressed by Kirchmayer's model, where b_{ll} is the loss coefficient. So, the function of effective hydraulic generation is

$$H(t, z(t), z'(t)) := P_h(t, z(t), z'(t)) - b_{ll} P_h^2(t, z(t), z'(t))$$

Furthermore, we shall consider a linear model for the associated water cost

$$S[z(T)] = \nu \cdot z(T)$$

where ν is a water conversion factor, which accounts for the unit conversion from (m^3) to (\$). In this example we present two cases: (a) $\nu = 0.00375(\$/m^3)$ and (b) $\nu = 0.00475(\$/m^3)$. The optimal power for the hydro-plant, $P_h(t)$, for both cases is shown in the next figure.



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