The cyclic coordinate descent in hydrothermal optimization problems with non-regular Lagrangian

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Abstract. In this paper we present an algorithm, inspired by the cyclic coordinate descent method, which allows the resolution of hydrothermal optimization problems involving pumped-storage plants. The proof of the convergence of the succession generated by the algorithm was based on the use of an appropriate adaptation of Zangwill's global theorem of convergence.

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INTRODUCTION

The coordinate descent method enjoys a long history in convex differentiable minimization. Surprisingly, very little is known about the convergence of the iterates generated by this method. Convergence typically requires restrictive assumptions such as that the cost function has bounded level sets and is in some sense strictly convex. The problem of minimizing a strictly convex function subject to linear constraints is considered in [1]; a convex function of the Legendre type subject to linear constraints is considered in [2]; while in [3], the author considers the objective to be pseudoconvex in every pair of the coordinate blocks and regular in some natural sense.

In a prior study [4], it was proven that the problem of optimization of the fuel cost of a hydrothermal system with several thermal plants may be reduced to the study of a hydrothermal system made up of one single thermal plant, called the thermal equivalent. A necessary minimum condition was established in [5] for the optimization of hydrothermal problems involving one single hydraulic pumped-storage plant, thereby considering non-regular Lagrangian and non-holonomic inequality constraints. The present paper addresses the generalization of this problem to several hydro-plants with pumping capacity. We introduce a relaxation numerical method for its resolution. The proof of the convergence of the succession generated by the algorithm was based on the use of an appropriate adaptation of Zangwill's global theorem of convergence [6]. Finally, we present the solution of a hydrothermal optimization problem in which the potential of the proposed algorithm is evidenced.

STATEMENT OF THE PROBLEM

Let us consider a hydrothermal system comprised of n thermal plants and m hydro-plants, assuming, with no loss in generality, that of the m hydro-plants, the first k are of the pumped-storage type (non-regular Lagrangian). The problem consists in minimizing the cost needed to satisfy a certain power demand during the optimization interval $[0,T]$. Said cost may be represented by the functional

$$J(\mathbf{z}) = \int_0^T L(t,\mathbf{z}(t),\dot{\mathbf{z}}(t))dt \tag{1}$$

$L(\cdot,\cdot,\cdot)$ is the class $C^2\left([0,T]\times\mathbb{R}^{2m} - \bigcup_{i=1}^{k} S_i\right)$ and $L(\cdot,\cdot,\dot{\mathbf{z}})$ is the class $C^2([0,T]\times\mathbb{R}^{2m})$, such that

$$L(t,\mathbf{z}(t),\dot{\mathbf{z}}(t)) = \Psi\left(P_d(t) - H(t,\mathbf{z}(t),\dot{\mathbf{z}}(t))\right)$$

over the set

$$\Theta := \{\mathbf{z} \in \left(C^1[0,T]\right)^m \ / \ \mathbf{z}(0) = \mathbf{0}\ ,\ \mathbf{z}(T) = \mathbf{b}\ ,\ H_{i\min} \leq H_i(t,\mathbf{z}(t),\dot{\mathbf{z}}(t)) \leq H_{i\max}\ \}$$

where Ψ is the function of thermal equivalent cost, $P_d(t)$ is the power demand, $\mathbf{z} = (z_1,\cdots,z_m)$ is the vector of admissible volumes, $z_i(t)$ being the volume that is discharged up to the instant t by the i-th hydro-plant, $\dot{\mathbf{z}} = (\dot{z}_1,\cdots,\dot{z}_m)$

is the vector of admissible rates, $\dot{z}_i(t)$ being the rate of water discharge at the instant t by the i-th hydro-plant and $\mathbf{b} = (b_1, \cdots, b_m)$ is the vector of admissible volumes, b_i being the volume that must be discharged up to the instant T by the i-th hydro-plant. $H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$ is the power contributed to the system at the instant t by the set of hydro-plants, $H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$ the function of effective hydraulic contribution by the i-th hydro-plant, being

$$H(t, \mathbf{z}(t), \dot{\mathbf{z}}(t)) = \sum_{i=1}^{m} H_i(t, \mathbf{z}(t), \dot{\mathbf{z}}(t))$$

Furthermore, S_i, for every $i \in \{1, \ldots, k\}$, is the set of points where $L_{\dot{z}_i}(t, z, \dot{z}_1, \ldots, \dot{z}_{i-1}, \cdot, \dot{z}_{i+1}, \ldots, \dot{z}_m)$ presents its only discontinuity (in $\dot{z}_i = 0$, the stoppage zone of the i-th hydro-plant). That is,

$$S_i := \left\{(t, \mathbf{z}, \dot{z}_1, \ldots, \dot{z}_{i-1}, 0, \dot{z}_{i+1}, \ldots, \dot{z}_m) \in [0, T] \times \mathbb{R}^{2m}\right\}$$

We shall assume that the functions Ψ, H, H_i, z_i, $\dot{z}_i$ and L verify the conditions satisfied in the real problems of hydrothermal optimization. We consider Θ equipped with the topology induced by the norm

$$||\mathbf{p}||^* := \max\{||\mathbf{p}||_\infty, ||\dot{\mathbf{p}}||_\infty\} = \max\{\max_{i=1,\ldots,m} ||p_i||_\infty, \max_{i=1,\ldots,m} ||\dot{p}_i||_\infty\}$$

MINIMUM NECESSARY CONDITION

At this point, we shall prove a result that will allow us to characterize the minimum candidates of the proposed problem. We define the following function.

Definition 1. *If* $(t, \overline{\mathbf{q}}(t), \overline{\mathbf{q}}'(t)) \notin S_i$, $\forall t$, *we define the "i-th coordination function" of* $\mathbf{q} \in \Theta$ *in* $[0, T]$ *as*

$$\mathbb{Y}^i_{\mathbf{q}}(t) = -L_{\dot{z}_i}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \cdot \exp\left[-\int_0^t \frac{H_{z_i}(s, \mathbf{q}(s), \dot{\mathbf{q}}(s))}{H_{\dot{z}_i}(s, \mathbf{q}(s), \dot{\mathbf{q}}(s))} ds\right]$$

We denote by $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^+(t)$ and $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^-(t)$ the expressions obtained when considering the lateral derivatives with respect to $\dot{z}$. The fundamental result is the following.

Theorem 1. *If* $\mathbf{q} \in \Theta$ *is solution of the problem (1), then there exists* $\{C_i\}_{i=1}^m \subset \mathbb{R}^+$ *satisfying:*

i) *If* $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$, $\left(\mathbb{Y}^i_{\mathbf{q}}\right)^+(t) \leq C_i \leq \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-(t)$.

ii) *If* $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i$, $\mathbb{Y}^i_{\mathbf{q}}(t)$ *is* $\begin{cases} \leq C_i & \text{if} \quad H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\min} \\ = C_i & \text{if} \quad H_{i\min} < H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) < H_{i\max} \\ \geq C_i & \text{if} \quad H_i(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = H_{i\max} \end{cases}$

DEFINITION OF THE DESCENT ALGORITHM

The solution algorithm that we shall present is based on the resolution of a problem with m hydro-plants, subsequent to solving a succession of problems with one single hydro-plant. Let $\mathbf{q} \in \Theta$. Let

$$L^i_{\mathbf{q}}(t, z_i, \dot{z}_i) := L(q_1(t), \cdots, q_{i-1}(t), z_i, q_{i+1}(t), \cdots, q_m(t), \dot{q}_1(t), \cdots, \dot{z}_i, \cdots, \dot{q}_m(t))$$

and the functional $F^i_{\mathbf{q}} : \Theta^i_{\mathbf{q}} \longrightarrow \mathbb{R}$,

$$F^i_{\mathbf{q}}(z_i) := F(q_1, \cdots, q_{i-1}, z_i, q_{i+1}, \cdots, q_m) = \int_0^T L^i_{\mathbf{q}}(t, z_i(t), \dot{z}_i(t)) dt$$

$$\Theta^i_{\mathbf{q}} := \{z \in C^1[0, T] \ / \ z(0) = 0, \ z(T) = b_i, \ H_{i\min} \leq H_i(t, q_1(t), .., q_{i-1}(t), z, q_{i+1}(t), .., q_n(t), \dot{q}_1(t), .., \dot{z}, .., \dot{q}_m(t)) \leq H_{i\max}\}$$

Definition 2. *We define the i-th minimizing map as the map* $\Phi_i : \Theta \longrightarrow \Theta$ *that satisfies for every* $\mathbf{q} = (q_1, \ldots, q_m) \in \Theta$

$$\Phi_i(q_1, \ldots, q_i, \ldots, q_m) = (q_1, \ldots, q^*, \ldots, q_m), \ \text{ where } F^i_{\mathbf{q}}(q^*) < F^i_{\mathbf{q}}(z_i), \ \forall z_i \in \Theta^i_{\mathbf{q}} - \{q^*\}$$

We shall denote by Φ the map associated with the descent algorithm, which will be the composition of the i-th minimizing map:

$$\Phi := \Phi_m \circ \cdots \circ \Phi_1$$

In every k-th iteration of the algorithm, "the m hydro-plants will have been minimized" through the i-th minimizing applications in the established order, thus obtaining the new, admissible element, q_k,

$$\mathbf{q}_k = \Phi(\mathbf{q}_{k-1}) = (\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_2 \circ \phi_1)(\mathbf{q}_{k-1})$$

The limit of this descending succession will be provided by the sought after minimum.
We denote by $\mathbf{q}_i^* = (q_1, \ldots, q_i^*, \ldots, q_m)$ and $\dot{\mathbf{q}}_i^* = (\dot{q}_1, \cdots, \dot{q}_i^*, \cdots, \dot{q}_m)$. The following proposition is verified.

Proposition 1. *If* $\mathbf{q} \in \Theta$*, then* $\Phi_i(\mathbf{q}) = \mathbf{q}_i^*$ *is of class* C^1 *and there exists* $\{C_i\}_{i=1}^m \subset \mathbb{R}^+$ *satisfying:*

i) If $\dot{q}_i^*(t)$ *is a point of discontinuity of* $\left(L_{\mathbf{q}}^i\right)_{\dot{z}_i}(t, q_i^*(t), \cdot)$, $\left(\mathbb{Y}^i_{\Phi_i(\mathbf{q})}\right)^+(t) \le C_i \le \left(\mathbb{Y}^i_{\Phi_i(\mathbf{q})}\right)^-(t)$.

ii) If $\left(L_{\mathbf{q}}^i\right)_{\dot{z}_i}(t, q_i^*(t), \cdot)$ *is continuous in* $\dot{q}_i^*(t)$, $\mathbb{Y}^i_{\Phi_i(\mathbf{q})}(t)$ *is* $\begin{cases} \le C_i & \text{if} \quad H_i(t, \mathbf{q}_i^*(t), \dot{\mathbf{q}}_i^*(t)) = H_{i\min} \\ = C_i & \text{if} \quad H_{i\min} < H_i(t, \mathbf{q}_i^*(t), \dot{\mathbf{q}}_i^*(t)) < H_{i\max} \\ \ge C_i & \text{if} \quad H_i(t, \mathbf{q}_i^*(t), \dot{\mathbf{q}}_i^*(t)) = H_{i\max} \end{cases}$

CONVERGENCE OF THE ALGORITHM

We now base the demonstration of the convergence of the proposed algorithm on a topological version of the global convergence theorem of descent algorithms with more general hypotheses that do not affect the correctness of the demonstration given in [6] by Zangwill. The main results obtained are summarized below.

Proposition 2. Let *L in the conditions of the problem (1). If* $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *converges uniformly to* $\mathbf{q}$ *in* $(\Theta, ||\ \ ||^*)$, *then:*
i) If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \notin S_i$, $\{\mathbb{Y}^i_{\mathbf{q}_n}\}_{n\in\mathbb{N}}$ *converges pointwise to* $\mathbb{Y}^i_{\mathbf{q}}$, $\forall i = 1, \ldots m$.
ii) If $(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in S_i$, *then* $\exists \{\mathbf{q}_{n_k}\}_{k\in\mathbb{N}} \subset \{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *and* $\{\mathbf{q}_{n_s}\}_{s\in\mathbb{N}} \subset \{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *such that*

$$\left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_k}}\right)^+\right\}_{k\in\mathbb{N}} \text{converges pointwise to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^+, \text{ and/or } \left\{\left(\mathbb{Y}^i_{\mathbf{q}_{n_s}}\right)^-\right\}_{s\in\mathbb{N}} \text{converges pointwise to } \left(\mathbb{Y}^i_{\mathbf{q}}\right)^-$$

Proposition 3. *If* $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *and* $\{\Phi(\mathbf{q}_n)\}_{n\in\mathbb{N}}$ *converge in* $(\Theta, ||\ \ ||^*)$, *then*

$$\{\Phi(\mathbf{q}_n)\}_{n\in\mathbb{N}} \text{ converges to } \Phi(\lim_{n\to\infty}(\mathbf{q}_n))$$

Proposition 4. *Let* $\mathbb{U} := \Theta \cap \widehat{C}^2$. *Then* $\exists M \in \mathbb{R}$ *such that, being* $\mathbb{U}_M := \{\mathbf{z} \in \mathbb{U} \,/\, ||\ddot{\mathbf{z}}||_\infty < M\}$, *it is verified that:*
i) $\Phi(\mathbb{U}_M) \subseteq \mathbb{U}_M$.
ii) $\mathbb{U}_M$ *is relatively sequentially compact in* $(\Theta, ||\ \ ||^*)$.
iii) $\Phi : (\Theta, ||\ \ ||^*) \longrightarrow (\Theta, ||\ \ ||^*)$ *is sequentially continuous.*
iv) $F : (\Theta, ||\ \ ||^*) \longrightarrow (\mathbb{R}, |\ \ |)$ *is sequentially continuous satisfying* $\Phi(\mathbf{x}) \neq \mathbf{x} \Longrightarrow F(\Phi(\mathbf{x})) < F(\mathbf{x})$.

Theorem 2. *For every* $\mathbf{q}_0 \in \Theta \cap \widehat{C}^2$, *the sequence generated by the algorithm* $\{\mathbf{q}_n = \Phi(\mathbf{q}_{n-1})\}_{n\in\mathbb{N}}$ *possesses a subsequence that converges in* $(\Theta, ||\ \ ||^*)$ *and the limit is a fixed point of* Φ. *Moreover, any convergent subsequence of* $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ *will converge at a fixed point on* Φ.

EXAMPLE

A program that resolves the optimization problem was written using the Mathematica package and was then applied to one example of a hydrothermal system made up of 8 thermal plants and 5 hydro-plants of variable head, two of which have pumping capacity. For the thermal plants, the cost function Ψ_i that was used is a quadratic model:

$$\Psi_i(x) = \alpha_i + \beta_i x + \gamma_i x^2$$

Furthermore, we consider Kirchmayer's model for the transmission losses: $l_i(x) = b_{ii} \cdot x^2$, where b_{ii} is termed the loss coefficient. We use a *variable head* model, and the i-th hydro-plant's active power generation P_{hi} (in the generation zone) is given by

$$P_{hi}(t, z_i(t), \dot{z}_i(t)) = A_i(t)\dot{z}_i(t) - B_i \dot{z}_i(t)[z_i(t) - Coup_i(t)]; \;\; \dot{z}_i(t) \ge 0,$$

where $A_i(t)$ and B_i are the coefficients

$$A_i(t) = \frac{1}{G_i} B_{y_i}(S_{0i} + t \cdot i_i); \;\; B_i = \frac{B_{y_i}}{G_i},$$

and $Coup_i(t)$ represents the hydraulic coupling between plants. For the pumped-storage plants, P_{hi} is defined piecewise, taking in the pumping zone:

$$P_{hi}(t, z_i(t), \dot{z}_i(t)) = M_i \cdot [A_i(t)\dot{z}_i(t) - B_i\dot{z}_i(t)z_i(t)]\,;\;\; \dot{z}_i(t) < 0,$$

where M is the efficiency of the hydro-plant in the pumping zone. We consider that the transmission losses for the hydro-plants are also expressed by Kirchmayer's model (where b_{ll} is the loss coefficient). Hence, the function of effective hydraulic generation is

$$H_i(t, z_i(t), \dot{z}_i(t)) := P_{hi}(t, z_i(t), \dot{z}_i(t)) - b_{lli}P_{hi}^2(t, z_i(t), \dot{z}_i(t)),\;\forall \dot{z}_i(t).$$

We consider a short-term hydrothermal scheduling (24 hours) with an optimization interval $[0,24]$ and we consider a discretization of 48 subintervals. The vector $\mathbf{C}^n = (C_1, \ldots, C_m)$ was considered as the stopping criterion for the algorithm in each iteration, the components of which are the coordination constants associated with the different hydro-plants, the tolerance being defined as: $Tol(n) = \left\|\mathbf{C}^n - \mathbf{C}^{n-1}\right\|$. For our example, for the case of the 5 hydro-plants, the tolerance was less than 10^{-9} in 10 iterations, and the time required by the program was 194 s on a personal computer (Pentium IV/$2GHz$). We can see how the method presents a rapid convergence. To verify this statement, another test was conducted considering the same 8 thermal plants from the above example and 10 hydro-plants. In this case, the method requires 20 iterations to achieve the established tolerance. Fig. 1 presents the obtained results.

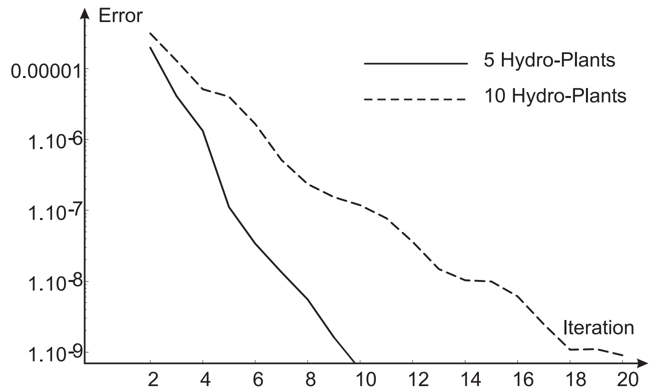

Fig. 1. Convergence with 5 hydro-plants and 10 hydro-plants.

CONCLUSIONS

In this paper we present an algorithm, inspired by the cyclic coordinate descent method, that will allow the solution of hydrothermal optimization problems involving hydraulic pumped-storage plants. We prove the convergence of the succession generated by the algorithm under weak assumptions.

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