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A General Algorithm for Control Problems with Variable Parameters and Quasi-linear Models

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Abstract. This paper presents an algorithm that is able to solve optimal control problems in which the modelling of the system contains variable parameters, with the added complication that, in certain cases, these parameters can lead to control problems governed by quasi-linear equations. Combining the techniques of Pontryagin's Maximum Principle and the shooting method, an algorithm has been developed that is not affected by the values of the parameters, being able to solve conventional problems as well as cases in which the optimal solution is shown to be bang-bang with singular arcs.

Keywords: Optimal Control, Pontryagin Maximum Principle, quasi-linear, shooting method, bang-bang
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INTRODUCTION

When an optimal control problem arises, the first step is to obtain a suitable model of the problem. Simplifications are frequently used to make the problem mathematically tractable and problems can often be found that are considered always quadratic, or always linear. In fact, what usually happens is that the model parameters present a range of variation (to say nothing of the uncertainty associated with them) and this variation can lead to problems of a very different nature. This is the case of a great many examples, which can in turn be found in a great variety of fields such as biology, chemistry, economics, etc. In this abstract, we shall use the classic problem of the economic study of the extraction of non-renewable resources [1], due to the simplicity of this approach.

Faced with the complication of having to use different techniques when the functional is linear or nonlinear in the control variable, our method presents the contribution of being valid in cases that range between quasi-linearity to singular arcs. It is also valid, of course, in conventional solutions that can be solved using Pontryagin's Maximum Principle (PMP) [2], [3]. We have used the combined techniques of PMP and the shooting method to build this optimization algorithm.

STATEMENT OF THE PROBLEM

In the classic problem introduced in Gray in 1914 [1], the problem for the resource owner is to choose an extraction path $R(t)$ for the resource stock $S(t)$ that maximizes total discounted profits:

$$\max_{R(t)} \int_0^{t_f} \left[pR(t) - b(R(t), S(t))\right] e^{-rt} dt \tag{1}$$

where the exogenously given price, p, is given, t_f is the final instant, which we assume is given, the rate of depletion is $R(t)$, the cost of extracting the resource is given by $b = b(R(t), S(t))$, the discount rate is r, and the initial stock of the resource is also known $S(0)$. It is then straightforward to verify that:

$$\dot{S}(t) = -R(t) \tag{2}$$

We see that the cost of extraction, b, depends not only on the depletion rate, $R(t)$, but also on the stock $S(t)$. This is a realistic description of the extraction technology for many exhaustible natural resources, and implies increasing extraction costs as the stock is depleted. A common model for b is to consider it inversely proportional to the stock, $S(t)$ and directly proportional to the square of the depletion rate $R(t)$, where k is the proportionality constant. However, we shall use a more general model of the form:

$$b(R(t), S(t)) = k\frac{R^{\alpha}(t)}{S(t)}; \text{ with } 1 \simeq \alpha < 2 \tag{3}$$

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The presence of the parameter α is responsible for the variability of the model, for the reason that, if $\alpha \simeq 1$, the model becomes quasi-linear.

OPTIMIZATION ALGORITHM

A optimal control problem (OCP) in Lagrange form can be formulated as:

$$\max_{u(t)} J = \int_0^{t_f} F\left(t, x(t), u(t)\right) dt \tag{4}$$

subject to satisfying:

$$\dot{x}(t) = f\left(t, x(t), u(t)\right),\ 0 \le t \le t_f \tag{5}$$

$$x(0) = x_0 \tag{6}$$

$$u(t) \in [u_{\min}, u_{\max}] = U(t),\ 0 \le t \le t_f \tag{7}$$

where J is the functional, F is the objective function, x is the state variable, with initial conditions x_0, u is the control variable bounded by $u_{\min}$ and $u_{\max}$, U denotes the set of admissible control values, and t is the operation time that starts from 0 and ends at t_f. The state variable must satisfy the state equation (5) with given initial conditions (6). In this statement, for the sake of simplicity, we consider the final instant to be fixed and the final state to be free.

Let H be the Hamiltonian function associated with the problem

$$H(t, x, u, \lambda) = F\left(t, x, u\right) + \lambda \cdot f\left(t, x, u\right) \tag{8}$$

where λ is called the costate variable. The classical approach involves the use of PMP [2], which results in a two-point boundary value problem (TPBVP). In order for $u^* \in U$ to be optimal, a nontrivial function λ must necessarily exist, such that for almost every $t \in [0, t_f]$:

$$\dot{x} = H_\lambda = f;\ \ x(0) = x_0 \tag{9}$$

$$\dot{\lambda} = -H_{\mathbf{x}};\ \ \lambda(t_f) = 0 \tag{10}$$

$$H(t, x, u^*, \lambda) = \max_{u(t) \in U} H(t, x, u, \lambda) \tag{11}$$

In virtue of PMP and Equation (10), there exists a piecewise C^1 function λ that satisfies:

$$\dot{\lambda}(t) = -H_x = -F_x - \lambda(t) \cdot f_x \tag{12}$$

and hence:

$$\lambda(t) = \left[K - \int_0^t F_x e^{\int_0^s f_x dz} ds\right] e^{-\int_0^t f_x ds} \tag{13}$$

denoting $K = \lambda(0)$. From (11), it follows that for each t, $u(t)$ maximizes H. Hence, in accordance with the Kuhn-Tucker Theorem, for each t, there exists two real non negative numbers, β_1 and β_2, such that $u(t)$ is a critical point of:

$$\mathbb{H}(u) = F + \lambda(t) \cdot f + \beta_1 \cdot (u_{\min} - u) + \beta_2 \cdot (u - u_{\max}) \tag{14}$$

it being verified that if $u^* > u_{\min}$, then $\beta_1 = 0$ and if $u^* < u_{\max}$, then $\beta_2 = 0$. We thus have $\dot{\mathbb{H}} = 0$ and the following cases:

Case 1) $u_{\min} < u^* < u_{\max}$. In this case, $\beta_1 = \beta_2 = 0$ and hence

$$F_u + \lambda(t) \cdot f_u = 0 \tag{15}$$

From (13) and (15), we have:

$$K = -\frac{F_u}{f_u} \cdot e^{\int_0^t f_x ds} + \int_0^t F_x \cdot e^{\int_0^s f_x dz} ds \tag{16}$$

If we denote by $\mathbb{Y}(t)$ (the *coordination function*) the second member of the above equation (the *coordination equation*), the following relation is fulfilled:

$$\mathbb{Y}(t) = K \tag{17}$$

Case 2) $u^* = u_{\max}$, then $\beta_2 \geq 0$ and $\beta_1 = 0$. By analogous reasoning, we have:

$$\mathbb{Y}(t) \geq K \tag{18}$$

Case 3) $u^* = u_{\min}$, then $\beta_1 \geq 0$ and $\beta_2 = 0$. By analogous reasoning, we have:

$$\mathbb{Y}(t) \leq K \tag{19}$$

Thus, the problem consists in finding for each K the function that satisfies (6), the condition (17), and from among these functions, the one that satisfies:

$$\lambda(t_f) = 0 \tag{20}$$

From the computational point of view, the construction can be performed with the same procedure as the simple shooting method, with the use of a discretized version of the coordination equation (16). When the values obtained do not obey the constraints (7), we force the solution to belong to the boundary until the moment established by conditions (18) and (19).

We now consider the particular case of scalar control appearing linearly:

$$\begin{aligned} &\max \int_0^T [f_1(t,x) + u f_2(t,x)]dt \\ &\dot{x}(t) = g_1(t,x) + u g_2(t,x);\;\; x(0) = x_0 \\ &u(t) \in [u_{\min}, u_{\max}] = U(t),\; 0 \leq t \leq t_f \end{aligned} \tag{21}$$

The Hamiltonian is linear in u and can be written as:

$$H(t,x,u,t) := f_1(t,x) + \lambda g_1(t,x) + [f_2(t,x) + \lambda g_2(t,x)]u \tag{22}$$

The optimality condition (maximize H w.r.t. u) leads to:

$$u^*(t) = \begin{cases} u_{\max} & \text{if} \quad H_u > 0 \\ u_{\text{sing}} & \text{if} \quad H_u = 0 \\ u_{\min} & \text{if} \quad H_u < 0 \end{cases} \tag{23}$$

and u^* is undetermined if:

$$\Phi(x,\lambda) \equiv H_u = f_2(t,x) + \lambda g_2(t,x) = 0 \tag{24}$$

The function Φ is called the switching function. If $\Phi = 0$ only at isolated points in time, then the optimal control switches between its upper and lower bounds, which is known as a bang-bang type control (i.e. the problem is not singular). The times when the OC switches from $u_{\max}$ to $u_{\min}$ or vice-versa are called switching times. If $\Phi = 0$ for every t in some subinterval, then the original problem is called a singular control problem and the corresponding trajectory, a singular arc.

EXAMPLE

In this section, we shall see the excellent behaviour of our approach by means of various examples. For the sake of brevity, we now present only the case posed in Section 2: the extraction of a non-renewable resource. The data can be seen in Table 1:

TABLE 1. Parameters of the model

p	k	r	$S(0)$	T	$u_{\min}$	$u_{\max}$
1.	1.	0.1	10.	1	0	10

The optimal results obtained for the depletion profile, $R(t)$, can be seen in Fig. 1 for different values of the parameter α, with $1 \simeq \alpha < 2$.

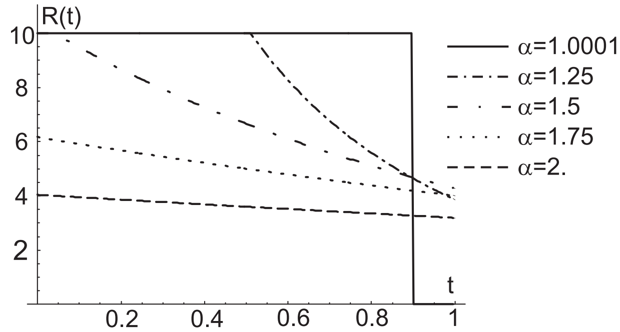

FIGURE 1. Optimal Solution

TABLE 2. Results

α	2	1.75	1.5	1.25	1.0001
$\max J$	1.91903	2.59944	3.7513	5.18001	6.46151
Iteration	5	11	11	11	101
CPU t(s)	0.98	3.57	3.4	3.71	77.2

The profits obtained are shown in Table 2. In the extreme case of $\alpha \simeq 1$, we obtain a final value of the stock $S(T) = 1$. As the Hamiltonian is linear in the control variable in this case, the problem presents a solution of the bang-bang type, jumping from $u_{\max}$ $u_{\min}$ for switching time $t = 0.9$. We used a discretization of 200 subintervals and the algorithm ran very quickly. In the worst example, with $\alpha = 1.0001$ we achieve the prescribed tolerance in (20): $tol = 1 \cdot 10^{-5}$ in only 101 iterations and the CPU time required by the program was 77.2 sec on a personal computer (Intel Core 2/2.66 GHz).

CONCLUSIONS

In this paper, we have presented a completely general algorithm for solving optimal control problems. The scope of the paper comprises all those problems in which mathematical modelling leads to models with variable parameters. Moreover, this variability often makes the type of solution to the optimal control problem vary. This means that many conventional algorithms become invalid, as they are not able to address conventional solutions, bang-bang solutions or solutions containing singular arcs simultaneously.

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