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Nonsmooth Optimization of Hydrothermal Problems

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Abstract

In this paper the authors present a necessary condition for minimum of a functional $J(z) := \int_0^T L(t, z(t), z'(t))dt$ in the case in which the function L is continuous but not of class C^1. This situation arises in problems of optimization of hydrothermal systems with pumped-storage plants. In such problems, the function $L_{z'}(t, z, \cdot)$ is discontinuous in $z' = 0$, which is the borderline point between the power generation zone ($z' > 0$) and the pumping zone ($z' < 0$). The problem can be naturally formulated in the framework of nonsmooth analysis, using the generalized (or Clarke's) gradient.

Key words: Nonsmooth, Optimization, Clarke's Gradient, Hydrothermal
MSC 2000: 49J52

1 Introduction

Many problems in pure and applied mathematics deal with nondifferentiable data. In this paper, we present a necessary condition for minimum of a functional J

$$J(z) := \int_0^T L(t, z(t), z'(t))dt \tag{1.1}$$

where the Lagrangian $L(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $L_z(\cdot, \cdot, \cdot)$ are the class C^0 and the function $L_{z'}(t, z, \cdot)$ is piecewise continuous.

This situation arises in a variety of problems of hydrothermal optimization [1] in which the hydroplants have a pumping capacity [2]. The problem consists in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0,T]$. Said cost may be represented by the functional

$$J(z) := \int_0^T \Psi\left[P_d(t) - H(t, z(t), z'(t))\right] dt \tag{1.2}$$

on

$$\Omega := \{z \in AC[0,T] \mid z(0) = 0 \wedge z(T) = b\}$$

For $AC[0,T]$ we denote the set of absolutely continuous functions from $[0,T]$ to $\mathbb{R}$, P_d is the power demand, H is the function of effective hydraulic generation, $z(t)$ the volume that is discharged up to the instant t by the hydroplant, $z'(t)$ the rate of water discharge at the instant t by the hydraulic plant, b is the volume of water that must be discharged during the entire optimization interval and Ψ is the cost function of the equivalent thermal plant [3].

In this kind of problem, the derivative of H with respect to z' ($H_{z'}$) presents discontinuity at $z' = 0$, which is the point at which a sudden change of $H_{z'}$ is produced, as it is the border between the power generation zone (positive values of z') and the pumping zone (negative values of z').

By classical results of Calculus of Variations, if $L \in C^1$ then a minimizer $q \in C^1$ (strong or weak) satisfies, $\forall t \in [0,T]$, the Euler–Lagrange equation

$$L_{z'}(t, q(t), q'(t)) = Const. + \int_0^t L_z(x, q(x), q'(x))dx$$

It is natural to extend classical necessary conditions for minimizers to the case with integrands having low regularity. Over the last quarter century there has been remarkable progress in the theoretical analysis of nonsmooth functions, primarily motivated by optimization. Clarke's introduction of his generalized gradient in 1973 (see [4]) pioneered a rapid development, recently presented in detail in Loewen and Rockafellar [5].

Here we show that our problem can be naturally formulated in the framework of nonsmooth analysis. The main contribution of our work is the introduction, by the first time, of a necessary condition for minimum for the resolution of the problem of hydrothermal optimization, using the Clarke's gradient. Moreover, we have developed a simple algorithm for resolving the problem. Said algorithm was implemented using the Mathematica package and as an example of its practical application, we resolve a real problem of hydrothermal optimization that involve pumped-storage plants.

2 Statement of the Problem

Consider a function $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a point $x \in \mathbb{R}^n$. The classical gradient of f at x is defined only when f is differentiable at x, but nondifferentiable objective functions arise naturally and frequently in optimization problems.

We introduce some preliminary ideas of a new generalized theory of differentiation, the main ideas of which are inspired by the work of Clarke [4].

The nonsmooth analysis works with locally Lipschitz functions that are almost everywhere differentiable (the set of points at which f fails to be differentiable is denoted Ω_f). Let $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschitz near x, and suppose S is any set of Lebesgue measure 0 in $\mathbb{R}^n$. Consider any sequence x_i converging to x while avoiding both S and points at which f is not differentiable, and such that the sequence of the gradients $\nabla f(x_i)$ converges.

The generalized (or Clarke's) gradient ∂f can be calculated as a convex hull of (almost) all converging sequences of the gradients

$$\partial f(x) = \operatorname{co}\{\lim \nabla f(x_i) : x_i \longrightarrow x,\ x_i \notin S,\ x_i \notin \Omega_f\} \tag{2.1}$$

It is essential that at the points of smoothness of $f(x)$ the generalized gradient coincides with gradient, and for a convex function with its subgradient.

We now extend that study to integral functionals, which will be taken over the σ-finite positive measure space $(\mathbb{T}, \Im, \mu) = [0, T]$ with Lebesgue measure. $L^\infty(\mathbb{T}, Y)$ denotes the space of measurable essentially bounded functions mapping $\mathbb{T}$ to Y, equipped with the usual supremum norm, with Y the separable Banach space $Y = \mathbb{R} \times \mathbb{R}$.

We are also given a closed subspace X of $L^\infty(\mathbb{T}, Y)$

$$X = \left\{ (s, v) \in L^\infty(\mathbb{T}, Y) \ \text{ for some } \ c \in \mathbb{R},\ s(t) = c + \int_0^t v(\tau) d\tau \right\}$$

and a family of functions $f_t : Y \longrightarrow \mathbb{R}$ $(t \in \mathbb{T})$ with $f_t(s, v) = L(t, s, v)$. We define a function f on X by the formula

$$f(s, v) = \int_0^T L(t, s(t), v(t)) dt$$

Note that for any (s, v) in X, we have $f(s, v) = J(s)$. With $(\widehat{s}, \widehat{v})$ a given element of X (so that $\widehat{v} = (d/dt)\widehat{s}$), we assume that the integrand L is measurable in t, and so that for some $\varepsilon > 0$ and some function $k(\cdot)$ in $L^1[0, T]$ one has

$$|L(t, s_1, v_1) - L(t, s_2, v_2)| \leq k(t) \, ||(s_1 - s_2, v_1 - v_2)||$$

for all (s_i, v_i) in $(\widehat{s}(t), \widehat{v}(t)) + \varepsilon B$. Then the next formula holds.

Theorem 1. *Under the hypotheses described above, f is Lipschitz in a neighborhood of $(\widehat{s}, \widehat{v})$ and one has*

$$\partial f(\widehat{s}, \widehat{v}) \subset \int_0^T \partial L(t, \widehat{s}(t), \widehat{v}(t)) dt \tag{2.2}$$

If in addition L is regular, then equality holds. So that if $\xi \in \partial f(\widehat{s}, \widehat{v})$, we deduce the existence of a measurable function $\xi_t = (r(t), p(t))$ such that

$$(r(t), p(t)) \in \partial L(t, \widehat{s}(t), \widehat{v}(t)) \ a.e.$$

(where ∂L denotes generalized gradient with respect to (s, v)) and where, for any $(s, v) \in X$, one has

$$< \xi, (s,v) >= \int_0^T < \xi_t, (s,v) > dt = \int_0^T [r(t)s(t) + p(t)v(t)]\, dt$$

If $\xi = 0$ (as when J attains a local minimum at $\widehat{s}$) then $0 \in \partial f(\widehat{s}, \widehat{v})$, it then follows easily (lemma Dubois-Reimond [6]) that $p(\cdot)$ is absolutely continuous and that $r = p'$ $a.e.$ In this case then we have a nonsmooth version (generalized subgradient version) of the Euler-Lagrange equation

$$(p'(t), p(t)) \in \partial L(t, \widehat{s}(t), \widehat{s}'(t)) \ a.e. \tag{2.3}$$

3 A Necessary Condition

We assume the following notations throughout the paper:

$$L^+_{z'}(t, z, z') := L_{z'}(t, z, z'_+); \quad L^-_{z'}(t, z, z') := L_{z'}(t, z, z'_-)$$
$$¥^+_z(t) = L^+_{z'}(t, z(t), z'(t)) - \int_0^t L_z(\tau, z(\tau), z'(\tau)) d\tau$$
$$¥^-_z(t) = L^-_{z'}(t, z(t), z'(t)) - \int_0^t L_z(\tau, z(\tau), z'(\tau)) d\tau$$

With the above definitions we can demonstrate the next result (necessary condition for minimum).

Theorem 2. *Let $q \in \Omega := \{z \in AC[0,T] \mid z(0) = 0 \wedge z(T) = b\}$. If q is minimum of J on Ω then $\exists K \in \mathbb{R}$ such that*

$$\begin{cases} ¥^+_q(t) = ¥^-_q(t) = K & \text{if} \quad q'(t) \neq 0 \\ ¥^+_q(t) \geq K \geq ¥^-_q(t) & \text{if} \quad q'(t) = 0 \end{cases} \tag{3.1}$$

Proof.

It is easy to see that the hypotheses of the theorem 1 are satisfied for the functional (1.1). Bearing in mind that the function $L_{z'}(t, z, \cdot)$ is discontinuous in z', we have, using (2.1), that the Clarke's gradient is

$$\partial L(t, q(t), q'(t)) = \left(L_z, [L^-_{z'}, L^+_{z'}]\right) \ a.e.$$

so the equation (2.3) is

$$(p'(t), p(t)) \in \left(L_z, [L^-_{z'}, L^+_{z'}]\right) \; a.e.$$
$$\begin{cases} p'(t) = L_z(t, q(t), q'(t)) \Longrightarrow p(t) = K + \int_0^t L_z(\tau, q(\tau), q'(\tau))d\tau \\ p(t) \in [L^-_{z'}, L^+_{z'}] \end{cases}$$

Then, we have

$$L^-_{z'} \leq K + \int_0^t L_z(\tau, q(\tau), q'(\tau))d\tau \leq L^+_{z'}$$
$$L^-_{z'} - \int_0^t L_z(\tau, q(\tau), q'(\tau))d\tau \leq K \leq L^+_{z'} - \int_0^t L_z(\tau, q(\tau), q'(\tau))d\tau$$
$$\yen^+_q(t) \geq K \geq \yen^-_q(t)$$

If $q'(t) \neq 0$, then $L^+_{z'} \equiv L^-_{z'}$ and $\yen^+_q(t) = \yen^-_q(t)$ and in such a case

$$\yen^+_q(t) = \yen^-_q(t) = K$$

□

This theorem 2 allows the extremals q_K to be constructed in a simple way:

i) For each K we construct q_K, where q_K satisfies the conditions (3.1) of theorem 2 and the initial condition $q_K(0) = 0$. In general, the construction of q'_K cannot be carried out all at once over all the interval $[0, T]$. The construction must necessarily be carried out by constructing and successively concatenating the extremal arcs ($q'(t) \neq 0$) and arcs where the plant neither it generates neither it pumps ($q'(t) = 0$) until completing the interval $[0, T]$. This is relatively simple to implement, with the use of a discretized version of the equations (3.1).

ii) K is calculated such that $q_K \in \Omega$. The procedure is similar to the shooting method used to resolve second-order differential equations with boundary conditions. Effectively, we may consider the function $\varphi(K) := q_K(T)$ and calculate the root of $\varphi(K) - b = 0$, which may be realized approximately using elemental procedures like the secant method.

In some cases, for example, with functionals $L(t, z')$ with $L_{z'}(t, \cdot)$ strictly increasing, the functional is convex, and the above condition is also sufficient for minimum. This situation arises in a hydrothermal system (see functional (1.2)) with a hydroplant (fixed head) whose power production H is a lineal function of the rate of water discharge ($m \cdot z'(t)$) and whose power consumption during pumping is also a lineal function of the amount of water pumped ($M \cdot z'(t)$). The proof of the next theorem is easy, using the previous theorem 2.

Theorem 3. *Let* $\Psi \in C^1[\mathbb{R}]$, Ψ' *strictly increasing,* $P_d \in C[0,T]$, *and* $L(t, z'(t)) := \Psi(P_d(t) - H(z'(t)))$, *with*

$$H(x) := \begin{cases} m \cdot x & \text{if} \quad x \geq 0 \\ M \cdot x & \text{if} \quad x \leq 0 \end{cases} \; ; \quad (0 < m < M)$$

If $K > 0$ *and* $\left[\frac{K}{M}, \frac{K}{m}\right] \subset \Psi'[\mathbb{R}]$, *then* $q_K(t) := \int_0^t \omega_K(s)ds$ *with*

$$\omega_K(t) := \begin{cases} 0 & \text{if} \quad \Psi'^{-1}(\frac{K}{m}) > P_d(t) > \Psi'^{-1}(\frac{K}{M}) \\ \dfrac{\Psi'^{-1}(\frac{K}{m}) - P_d(t)}{-m} & \text{if} \quad \Psi'^{-1}(\frac{K}{m}) \leq P_d(t) \\ \dfrac{\Psi'^{-1}(\frac{K}{M}) - P_d(t)}{-M} & \text{if} \quad P_d(t) \leq \Psi'^{-1}(\frac{K}{M}) \end{cases}$$

provides the minimum value of J *on*

$$\Omega_K := \{z \in AC[0,T] | z(0) = 0 \wedge z(T) = q_K(T)\}$$

The meaning of this proposition can be seen in figure 1. We call $P_{th}(t) := P_d(t) - H(t, z'(t))$ the optimal power generated by the thermal equivalent. It is easy to see that $P_{th}(t)$ is constant in the interior arcs of the extremal, and

$$P_{th}(t) = \begin{cases} \Psi'^{-1}(\frac{K}{m}) & \text{if} \quad \Psi'^{-1}(\frac{K}{m}) \leq P_d(t) \\ \Psi'^{-1}(\frac{K}{M}) & \text{if} \quad P_d(t) \leq \Psi'^{-1}(\frac{K}{M}) \end{cases}$$

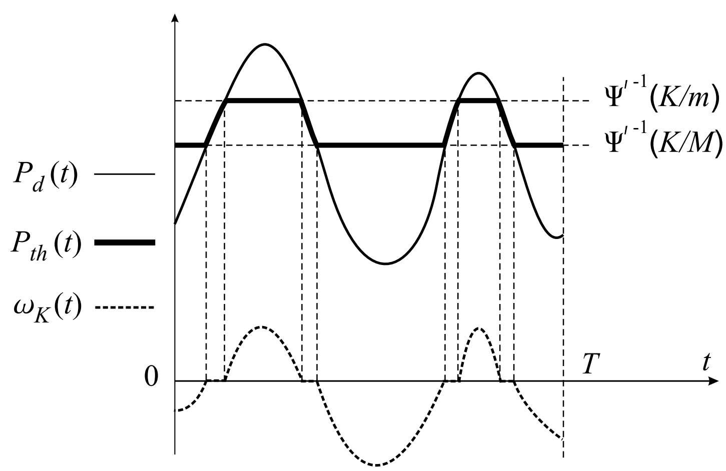

Figure 1. Meaning of Theorem 3.

4 Application to a Hydrothermal Problem

Let us now see a problem of a hydrothermal nature whose solution may be constructed in a simple way taking into account the above theorem 1. A program that resolves the optimization problem was elaborated using the Mathematica package and was then applied to one example of hydrothermal system made up of 8 thermal plants and a hydraulic pumped-storage plant of variable head.

We consider the functional (1.2). The cost function that has systematically been used is a second-order polynomial

$$\Psi_i(x) = \alpha_i + \beta_i x + \gamma_i x^2$$

Table I: Coefficients of the thermal plants.

Plant i	α_i	β_i	γ_i	b_{ii}
1 (*Aboño* 1)	1227.83	17.621	0.01325	0.000103
2 (*Aboño* 2)	743.78	20.842	0.00211	0.000072
3 (*Soto* 2)	77.72	21.277	0.00286	0.000172
4 (*Soto* 3)	1615.35	16.676	0.01659	0.000100
5 (*Narcea* 2)	2248.16	-7.984	0.17026	0.000353
6 (*Narcea* 3)	1459.44	21.569	0.01489	0.000121
7 (*Lada* 3)	1625.43	6.347	0.09803	0.000220
8 (*Lada* 4)	2155.62	17.745	0.01982	0.000097

It is also usual to consider the function of losses $l_i(x) = b_{ii} \cdot x^2$, (Kirchmayer's model) where b_{ii} is termed the loss coefficient.

As an example, we shall use the thermal system of the company *HC* in Asturias (Spain), which is made up of 8 thermal plants. The data of the plants is summarized in Table I. The units for the coefficients are: α_i in $(\$/h), \beta_i$ in $(\$/h.Mw), \gamma_i$ in $(\$/h.Mw^2)$, and the loss coefficients b_{ii} in $(1/Mw)$. For the fuel cost model of the equivalent thermal plant, we use the quadratic model

$$\Psi(P(t)) = \alpha_{eq} + \beta_{eq} P(t) + \gamma_{eq} P(t)^2$$

We construct the equivalent thermal plant as we saw in [3], obtaining

$$\alpha_{eq} = 9377.2(\$/h);\ \beta_{eq} = 19.2616(\$/h.Mw);\ \gamma_{eq} = 0.00175314(\$/h.Mw^2)$$

We use a variable head model and the hydro-plant's active power generation P_h is different depending on the positivity or negativity (pumping) of the rate of water discharge. The power production P_h of the hydroplant (variable head) is function of $z(t)$ and $z'(t)$ and its power consumption during pumping is a lineal function of the amount of water pumped ($M \cdot z'(t)$). Hence the function P_h is defined piecewise as

$$P_h(t, z(t), z'(t)) := \begin{cases} A(t) \cdot z'(t) - B \cdot z(t) \cdot z'(t) & \text{if} \quad z'(t) > 0 \\ M \cdot z'(t) & \text{if} \quad z'(t) \leq 0 \end{cases}$$

where $A(t)$ and B are the coefficients

$$A(t) := \frac{B_y}{G}(S_0 + t \cdot i); \;\; B = \frac{B_y}{G}$$

In the variable-head models, the term $-B \cdot z(t) \cdot z'(t)$ represents the negative influence of the consumed volume, and reflects the fact that consuming water lowers the effective height and hence the performance of the station. So, the function of effective hydraulic generation is

$$H(t, z(t), z'(t)) := \begin{cases} P_h(t, z(t), z'(t)) - b_{ll} P_h^2(t, z(t), z'(t)) & \text{if} \quad z'(t) > 0 \\ P_h(t, z(t), z'(t)) & \text{if} \quad z'(t) \leq 0 \end{cases}$$

where b_{ll} is the loss coefficient.

The values for the coefficients of the hydroplant are:

the efficiency G:	$G = 526315 (m^4/h.Mw)$
the restriction on the volume b:	$b = 1.1 \cdot 10^7 (m^3)$
the natural inflow i:	$i = 3.1313 \cdot 10^5 (m^3/h)$
the loss coefficient b_{ll}:	$b_{ll} = 0.00015 (1/Mw)$
the initial volume S_0:	$S_0 = 200 \cdot 10^8 (m^3)$
the coefficient B_y	$B_y = 149.5 \cdot 10^{-11} (m^{-2})$

where B_y is a parameter that depends on the geometry of the tanks. We also consider $M = (1.1) \cdot A(0)$ $(h.Mw/m^3)$, that is the factor of water-conversion of the pumped-storage unit.

An optimization interval of $T = 24$ h. was considered, with a discretization of $24 \cdot 4$ subintervals. The secant method was used to calculate the approximate value of K for which $q_K(T) - b = 0$. In 8 iterations:

$$|q_K(T) - b| < 10^{-6} (m^3)$$

for $K = 1358.252465 \cdot 10^{-6}$.

The Table II presents the optimal solution and the power demand for $t = 0, 1, \ldots, 24$ (h). The figure 2 presents the optimal hydro-power P_h (Mw) and the figure 3 presents the optimal thermal-power P_{th} (Mw) and the power demand P_d (Mw). We can see that the optimal thermal-power remains constant in all the instants in which pumping takes place (the conditions of theorem 3 are satisfied). The cost of the optimal solution is \$ 918276 and the CPU time used was 10.0 sec.

Table II: Optimal solution and power demand.

t	P_d	P_{th}	P_h	t	P_d	P_{th}	P_h
0	1480	1433.3	47.0	13	1489	1438.2	51.2
1	1316	1316	0	14	1515	1456.4	59.2
2	1171	1171	0	15	1539	1473.1	66.5
3	839	839	0	16	1534	1469.6	65.1
4	388	669.2	-281.2	17	1540	1473.8	66.9
5	410	669.2	-259.2	18	1574	1497.7	77.3
6	765	765	0	19	1616	1527.3	90.0
7	1175	1175	0	20	1584	1504.6	80.4
8	1347	1340.3	6.7	21	1582	1503.2	79.8
9	1430	1397.4	32.8	22	1613	1525.0	89.2
10	1524	1462.8	61.8	23	1590	1508.7	82.3
11	1560	1488.0	72.8	24	1480	1431.6	48.8
12	1522	1461.3	61.2				

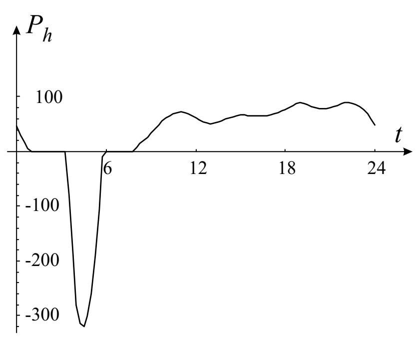

Figure 2. Optimal hydro-power P_h.

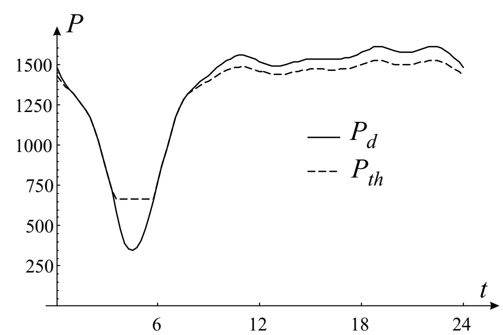

Figure 3. Optimal P_{th} and P_d.

5 Conclusions

In this paper we present the resolution of a problem of hydrothermal optimization with pumped-storage plants. The problem can be naturally formulated in the framework of nonsmooth analysis. We use, by the first time, the Clarke's gradient for the resolution of this problem and we obtain a necessary condition for minimum.

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